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INTRANSITIVE ACTION OF $< y, t : y^4 = t^4 = 1 >$ ON $Q(\sqrt{n})$

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By using the coset diagrams for the action of $H =< y, t : y^4 = t^4 = 1 >$ on $Q(\sqrt{n})$, we show that if α is of the form $\frac{a+\sqrt{n}}{2c}$ then every element in the orbit αH is also of form $\frac{a'+\sqrt{n}}{2c'}$ and $\alpha H \subset Q^*(\sqrt{n})$. We also show that the action of the group H on $Q(\sqrt{n})$ is intransitive.

1. INTRODUCTION

Let M be a group generated by the linear-fractional transformations x, y satisfying the relations $x^2 = y^m = 1$. If $y : z \to \frac{az+b}{cz+d}$ is to act on real quadratic fields then a, b, c, d must be rational numbers and can be considered as integers. Thus $\frac{(a+d)^2}{ad-bc} - 2 = \omega + \omega^{-1}$, where ω a primitive m-th root of unity, is rational, only if m = 1,2,3,4 or 6. The group M is trivial, D_∞ (an infinite dihedral group), or PSL $(2, Z)$ if m = 1, 2 or 3. The case m = 3 has been discussed in detail in [1] and [3].

An element $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$, where

$$Q^*(\sqrt{n}) = \left\{ \frac{a+\sqrt{n}}{c}, a, c \in Z,\ c \neq 0, b = \frac{a^2-n}{c},\ (a,b,c) = 1 \right\}$$

and its conjugate $\bar{\alpha} = \frac{a-\sqrt{n}}{c}$ may have different signs for a fixed non-square positive integer n, such α is called an ambiguous number. If α and $\bar{\alpha}$ are both positive (negative), then α is called a totally positive (negative) number. Ambiguous numbers play an important role in the study of actions of the groups $M =< x, y : x^2 = y^m = 1 >$, for $m = 1, 2, 3, 4$ or 6, on $Q(\sqrt{n})$. In the action of M on $Q(\sqrt{n})$, $Stab_\alpha(M)$ are the only non-trivial stabilizers and in the orbit αM, there is only one (up to isomorphism) non-trivial stablizer.

In this note we are interested in the subgroup of $G =< x, y : x^2 = y^4 = 1 >$, acting on the real quadratic irrational numbers, where $(z)y = \frac{-1}{2(z+1)}$ are linear-fractional transformations. An action of G on real quadratic irrational numbers has been considered in [2]. It has been shown that the set of ambiguous numbers is finite and that part of the coset diagram containing these numbers form a single closed path and it is the only closed path in the orbit of α.

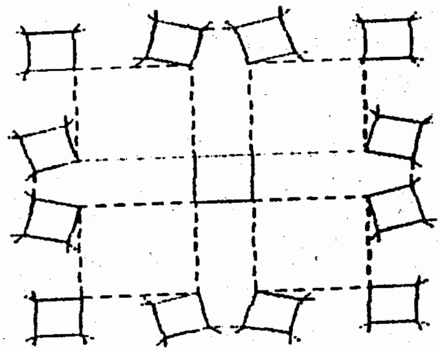
If we let $t = xyx$ then t is the linear-fractional transformation defined by $(z)t = 1 - \frac{1}{2z}$ and $t^4 = 1$. The group $H =< t, y >$ is thus a subgroup of G. Some number-theoretic properties of the ambiguous numbers belonging to the orbit of G when acting on $Q^*(\sqrt{n})$ have been discussed in [2]. In this paper we show that if α is of the form $\frac{a+\sqrt{n}}{2c}$ then every element in the orbit αH is also of the form $\frac{a'+\sqrt{n}}{2c'}$ and $\alpha H \subset Q^*(\sqrt{n})$. We also show that the action of the group H on $Q(\sqrt{n})$ is intransitive.

2. COSET DIAGRAMS

We use coset diagrams for the group H and study its action on the projective line over real quadratic fields. The coset diagrams for the group H are defined as follows. The four cycles of the transformation y are denoted by four edges (unbroken) of a square permuted anti-clockwise by y and the four cycles of the transformation t are denoted by four edges (broken) of a square permuted anti-clockwise by t. Fixed points of y and t, if they exist, are denoted by heavy dots. This graph can be interpreted as a coset diagram, with the vertices identified with the cosets of $Stab_v(H)$, the stablizer of some vertex v of the graph, or as

1-skeleton of the cover of the fundamental complex of the presentation which corresponds to the subgroup $Stab_v(H)$.

A general fragmenta of the coset diagram of the action of H on $Q(\sqrt{n})$ will look as follows.



In [2], it has been observed that if $k \neq 0, \frac{-1}{2}, -1$ or ∞ is one of the four vertices of a square in the coset diagram, then

(i)	$z < -1$	implies that	$(z)y > 0$
(ii)	$z > 0$	implies that	$\frac{-1}{2} < (z)y < 0$
(iii)	$\frac{-1}{2} < z < 0$	implies that	$-1 < (z)y < \frac{-1}{2}$, and
(iv)	$-1 < z < \frac{-1}{2}$	implies that	$(z)y < -1$.

Also if $k \neq 0, \infty, 1$ or $\frac{1}{2}$ is one of the four vertices of a square in the coset diagram, then

(i)	$z < 0$	implies that	$(z)t > 1$
(ii)	$z > 1$	implies that	$\frac{1}{2} < (z)t < 1$
(iii)	$\frac{1}{2} < z < 1$	implies that	$0 < (z)t < \frac{1}{2}$, and
(iv)	$0 < z < \frac{1}{2}$	implies that	$(z)t < 0$

3. MAIN RESULTS

Theorem 3.1

If $\alpha = \frac{a+\sqrt{n}}{2c} \in Q^*(\sqrt{n})$ then every element in the orbit αH is of the form $\frac{a'+\sqrt{n}}{2c'}$ and $\alpha H \subseteq Q^*(\sqrt{n})$.

Proof

Since $(z)y = \frac{-1}{2(z+1)}$, therefore

$$(\alpha)y = \frac{-1}{2(\alpha+1)} = \frac{-a-2c+\sqrt{n}}{2(2a+b+2c)} = \frac{a_1+\sqrt{n}}{c_1}$$

where $a_1 = -a-2c, c_1 = 2(2a+b+2c)$ and $b_1 = \frac{a_1^2-n}{c_1} = c$. Similarly, we can find new values of a, b, c for $(\alpha)y^j$ where $j = 1, 2, 3$ as follows.

α	a	b	$2c$
$(\alpha)y$	$-a-2c$	c	$2(2a+b+2c)$
$(\alpha)y^2$	$-3a-2b-2c$	$2a+2b+2c$	$2(2a+2b+c)$
$(\alpha)y^3$	$-a-2b$	$2a+2b+c$	$2b$
$(\alpha)t$	$-a+2b$	$-2a+2b+c$	$2b$
$(\alpha)t^2$	$-3a+2b+2c$	$-2a+b+2c$	$2(-2a+2b+c)$
$(\alpha)t^3$	$-a+2c$	c	$2(-2a+b+2c)$

Since every word in H is of the form $t^{\delta_1}y^{\eta_1}t^{\epsilon_2}y^{\eta_2}\cdots t^{\delta_n}y^{\eta_n}$, where $\epsilon_1 = 0, 1, 2$ or 3, $\epsilon_2, \epsilon_3, \cdots, \epsilon_n = 1, 2$ or $3, \eta_1, \eta_2, \cdots, \eta_{n-1} = 1, 2$ or 3 and $\eta_n = 0, 1, 2$ or 3, therefore, every element in αH is of the form $\frac{a'+\sqrt{n}}{2c'}$. Also the new value of b for any element of αH is an integer, therefore, $\alpha H \subseteq Q^*(\sqrt{n})$.

Theorem 3.2

If

$$\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$$

where c is an odd integer, then the elements in the orbit αH of the form $\frac{a'+\sqrt{n}}{2c'}$ do not belong to $Q^*(\sqrt{n})$ but the elements of the form $\frac{a'+\sqrt{n}}{2c'+1}$ belong to $Q^*(\sqrt{n})$.

Proof

We can easily tabulate the following information.

α	a	b	c
$(\alpha)y$	$-a-c$	$\frac{c}{2}$	$4a+2b+2c$
$(\alpha)y^2$	$-3a-2b-c$	$2a+b+c$	$4a+4b+c$
$(\alpha)y^3$	$-a-2b$	$\frac{4a+4b+c}{2}$	$2b$
$(\alpha)t$	$-a+2b$	$\frac{-4a+4b+c}{2}$	$2b$
$(\alpha)t^2$	$-3a+2b+c$	$-2a+b+c$	$-4a+4b+c$
$(\alpha)t^3$	$-a+c$	$\frac{c}{2}$	$-4a+2b+2c$

AS we see from the above information that in the orbit αH elements is either of the form $\frac{a'+\sqrt{n}}{2c'}$ or $\frac{a'+\sqrt{n}}{2c'+1}$. Clearly, in the orbit αH all the elements of the form $\frac{a'+\sqrt{n}}{2c'+1}$ are in $Q^*(\sqrt{n})$ and the elements of the form $\frac{a'+\sqrt{n}}{2c'}$ do not belong to $Q^*(\sqrt{n})$.

Theorem 3.3

The action of H on $Q(\sqrt{n})$ is intransitive.

Proof

Since n is a positive integer, so it may be even or odd.

(i) Let n be an even positive integer, that is, $n = 2m$. If we take $\alpha = \frac{\sqrt{n}}{2}$, so here $a = 0, c = 2$ and $b = \frac{a^2-n}{c} = -m$. Hence $\alpha \in Q^*(\sqrt{n})$. Also if we take $\beta = \sqrt{n}$ then $\beta \in Q^*(\sqrt{n})$. Since the numbers of the form $\frac{a+\sqrt{n}}{2c}$ and $\frac{a+\sqrt{n}}{2c+1}$ lie in two different orbits, as shown in theorem 3.1, 3.2, therefore, $\sqrt{n}$ and $\frac{\sqrt{n}}{2}$ lie in two different orbits. Thus there are at least two orbits of $Q(\sqrt{n})$. Hence the action of H on $Q(\sqrt{n})$ is intransitive.

(ii) Let n be an odd positive integer, that is, $n = 2m+1$. If we take $\alpha = \frac{1+\sqrt{n}}{2}$, so here $a = 1$, $c = 2$ and $b = \frac{a^2-n}{c} = -m$. Hence $\alpha \in Q^*(\sqrt{n})$. Also if we take $\beta = \sqrt{n}$ then $\beta \in Q^*(\sqrt{n})$. Thus, there are at least two orbits of $Q(\sqrt{n})$, one containing $\frac{1+\sqrt{n}}{2}$ and some other containing $\sqrt{n}$. Hence the action of H on $Q(\sqrt{n})$ is intransitive.

REFERENCES

[1] Mushtaq, Q. *Modular group acting on real quadratic fields* Bull. Austral. Math. Soc., 37 (1988), 303-306.

[2] Mushtaq, Q. and M. Aslam *Group generated by two elements of order 2 and 4* Acta Math. Sin, New. Ser, (1), 9(1993), 48-54.

[3] Mushtaq, Q. *On word structure of the modular group over finite and real quadratic fields* Disc. Math. 178(1998), 155-164.

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ON RATIONALITY OF CERTAIN GROUPS

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ABSTRACT

An important problem in $Q - group$ theory is to classify particular classes of those groups which are not $Q - groups$. In this paper we completely classify abelian $Q - groups$ and also establish some relation between Q-groups, and its faithful irreducible representation. We also discuss rationality of some transitive groups.

Key Words and Phrases: $\mathrm{Hom}_{FG}(V, V)$, $\mathrm{Aut}(G)$, Frobenius Groups, $Q - groups$, $Q' - groups$.

AMS Subject Classification (1995): 20C15

Notations: Notations will be those of [1] except where mentioned otherwise.

1. INTRODUCTION

A $Q - group$ is a finite group all of whose ordinary complex representations have rationally valued characters, otherwise it is called $Q' - group$.

The interplay between the structure of a finite group and representation has had, and continues to have, deep consequence for both theories. By imposing certain conditions on the group, such as being abelian or nilpotent, one is able to draw conclusions about its representations. Conversely, restrictions on the representations can lead to specific structures. It is in this context that we approach the study of $Q-groups$. It is quite interesting to note that the order of a non-trivial $Q-group$ must be divisible by 2 thus ensuring the existence of an involution.

A transitive permutation group G of degree $n > 2$ which has minimal degree $n-1$ i.e. no non-identity element fixes more than one letter and a subgroup H of G fixing a letter in non trivial, is called a **Frobenius Group**.

A subgroup K of G fixing no letter is called Frobenius Kernal and H is termed as Frobenius Subgroup (also called complement). It is known that G is the semi-direct product of H and K.

Among the transitive permutation groups, an improvement is obtained through the concept of fractional transitivity. A group G is said to be $\frac{1}{2}$- transitive if all orbits of G have the same length > 1. Similarly G is $\left(k+\frac{1}{2}\right)$ - fold transitive $(k = 1, 2, \cdots, n-1)$ if G is transitive on X (ground set) and G, is $\left(k-\frac{1}{2}\right)$ - fold transitive on $X - \{x\}$.

2. DEFINITIONS AND SOME KNOWN RESULTS

Definition 2.1[2]

For any representation ρ of G, we say that the group G acts on V via ρ. If V contains no nontrivial G-invariant subspace, then ρ is called an irreducible representation. Otherwise, ρ is said to be reducible.

Definition 2.2[2]

Let m be an integer greater than 1. The set $\{[n]\}$ of residue classes modulo m, where $l \leq n < m$ and n is coprime to m, is a group and is denoted by G_m. A routine proof using the Euclidean Algorithm, shows that G_m is an abelian group, under the usual multiplication of residue classes. More over $|Gm| \geq 2$ for $m > 2$.

Proposition 2.1[2]

Let G be a cyclic group of order m, then we have

a. If G contains an element of order greater than 2, then $|Aut(G)| \geq 2$.

b. If m is an odd prime, say p, then $Aut(G)$ is cyclic of order $p-1$.

c. $Aut(G) \cong G_m$.

d. If $m = p^n$ for some prime p then $Aut(G)$ is cyclic group of order $p^{n-1}(p-1)$.

Proposition 2.2[1] & [3]

Suppose G is a Frobenius group having kernel K and Frobenius subgroup H. Then:-

a). The slow p-subgroups of H are cyclic for odd p and are cyclic or generalized quaternion for $p = 2$.

b). Let x be non-identity element of H. If $x^y \in H$ then $y \in H$ i.e. H is without fusion in G.

c). If $|H|$ is even, then:-

(i) H contains only one involution which is necessarily contained in $Z(H)$

(ii) K is abelian.

(iii) Let $i \in H$ be involuiton of H, then for $k \in K$, $k^i = k^{-1}$.

Proposition 2.3[6]

Let G be a finite group. Then G is a $Q-group$ if and only if, for every cyclic subgroup H of G, we have

$$N_G(H)/C_G(H) \cong Aut(H)$$

From proposition 2.3 it follows that a finite group G is a $Q'-group$ if and only if there exists a cyclic subgroup H of G such that

$$N_G(H)/C_G(H) \ncong Aut(H)$$

Proposition 2.4[6]

Let G be a $Q-group$. Then G contains an irreducible involution if and only if a sylow 2-subgroup of G is either Z_2 or Q_8.

3. ABELIAN Q-GROUPS

Following beautiful result about abelian Q-groups has already been proved [6]. The proof contains some complicated number theoratical properties. However we give here altogether a new and simple proof.

Theorem 3.1

A finite abelian group G is Q-group if and only if it is elementary abelian 2-group.

Proof

Let G be an abelian Q-group. If G is not an elementary abelian 2-group, then there exists an element $g \in G$ such that $|<g>| > 2$, then $|Aut(<g>)| \geq 2$.

Since G is abelian, therefore for each $g \in G$ we have

$$|N_G(<g>)| = |C_G(<g>)| = |G|$$

Thus $|N_G(<g>)/C_G(<g>)| = 1$, so that

$N_G(<g>)/C_G(<g>) \ncong Aut(<g>)$ which is contradiction to proposition 2.3 proving that for non-identity $g \in G, |<g>| = 2$. Hence G must be an elementary abelian 2-group.

Conversaly let G be an elementary abelian 2-group. Then for each non-identity $g \in G, <g> \cong C_2$, so that $Aut(<g>) \cong E$.

Now $N_g(<g>)/C_G(<g>) \cong E \Rightarrow N_G(<g>) \cong Aut(<g>) \forall g \in G$.

Thus G is a Q-group.

Corollary 3.1

Let Gbe a finite Q-group. Then $Z(G)$ is an elementary abelian 2-group.

Proof

Follows immediately from theorem 3.1.

4. IRREDUCIBILITY AND Q'-GROUPS

Theorem 4.1

If the finite group G possesses a faithful irreducible representation and if order of its centre is greater than 2, then G is a Q'-group.

Proof

Let ρ be a faithful irreducible representation of G on a vector space V. Let Z be the centre of G, then $\rho(Z) \subseteq \mathrm{Hom}_{FG}(V, V)$. Since $\mathrm{Hom}_{FG}(V, V)$ is a division algebra, $\rho(Z)$ generates a field contained in $\mathrm{Hom}_{FG}(V, V)$. But we know that finite multiplicative subgroups of fields are cyclic, therefore $\rho(Z)$ is cyclic. Since ρ is faithful Z is cyclic.

Let

$$Z = < a >, \quad \text{then} \quad | < a > | > 2 \Rightarrow |Aut(< a >)| > 1$$

But $N_G(< a >) \cong C_G(< a >) \cong G$ (Because $< a >$ is in the centre of G).

$\Rightarrow N_G(< a >)/C_G(< a >) \cong E$ which is not isomorphic to $Aut(< a >)$.

Hence by proposition 2.3 G is a Q'-group.

5. RATIONALITY OF TRANSITIVE GROUPS

Theorem 5.1

Let G be a Q-group. If G is a transitive permutational group on a finite set A, $(|A| = n > 2)$ such that G_x is $\frac{1}{2}$ -transitive on $A - \{x\}$ for any $x \in A$, then $G = E_3Z_2$ where E_3 is elementary abelian 3-group and Z_2 inverts all elements of E_3.

Proof

We take $A = \{1, 2, 3 \cdots n\}$ so that $G_1 = H$ is $\frac{1}{2}$- transitive on $A - \{1\}$. Let length of each orbit of G_1 be $m > 1$, then G is impritive and has non-trivial blocks, say, of length k with $1 < k < n$.Then the elements of A may be arranged in a matrix (x_{ij}) in such a way that every row R_i is a block of G of length k and say $x_{11} = 1$.

Now, R_1 is fixed by G_1, therefore $k \equiv 1 \mathrm{mod}(m) \Rightarrow (k, m) = 1$.

We define $(R_i)^H = \{(x_i)\alpha;\ \alpha \in H, x_i \in R_i\}$

Then for $i > 1, R_i^H$ is fixed by H and does not contain 1. Therefore $|R_i^H|$ is divisible by km (as (k,m) = 1).

On the other hand $|x_{ij}^H| = m$ (because each x_{ij}^H is an orbit of x_{ij}). Therefore $|R_i^H| \leq km$. Thus, it follows that $|R_i^H| = m$ and therefore $x_{ij}^H \cap x_{im}^H = \phi$ for $i \neq j$. Thus means that, for $i > 1, x_{ij}^H \cap R_i = x_{ij}$.

Now it follows that:-

If $x \in R_i(i > 1)$, then

$$H_x = G_{1x} \subseteq G_{Ri} \tag{1}$$

Since $\alpha \in H$ we have

$$\begin{aligned} x_{ij}^{\alpha} &= (x_{ij}^H \cap R_i)^{\alpha} \\ &= x_{ij}^H \cap R_i^{\alpha} \\ &= x_{ij} \end{aligned}$$

Since 1 and x may be interchanged, therefore from (1) we have

$$G_{1x} \subseteq G_{R1} \tag{2}$$

Also since $G_{1x} \subseteq G_{R1} \subseteq G_{Rx_{12}}$ and $|G_{1x}| = |G_{1x_{12}}|$, therefore, $|x_{12}^{G_1}| = m$, and $G_{1x} = G_{1x_{12}} = \cdots = G_{1x_{1k}}$.

$\Rightarrow G_{1x_{12}} = E$, the identity subgroup, as x is arbitrary except that $x \notin R_1$.

But then

$G_{1x} = E$ for any $x \neq 1$, even if $x \in R_1$ and if $x \notin R_1$ then, $G_{1x} = E$ by (2).

This means that minimal degree of $G \geq n - 1$.

Since G can not be regular (because $G_x \neq E$ and $x > 2$), therefore minimal degree of $G = n - 1$ i.e. no element of G fixes more than one letter of A.

Thus, G is a Frobenius Group with $G_x = H$ as Frobenius Subgroup. Then H is without fusion in G, therefore H is also a Q-group and has even order. Then, by proposition 2.2, H contains only one involution, say, i_H, so that sylow 2-subgroup of H is isomorphic either to Z_2 (cyclic group of order 2) or to the Generalized Quaternion group [proposition 2.4]. Now H contains an irreducible involution, therefore by proposition 2.4 sylow 2-subgroup of H is either Z_2 or Q_8.

Let K be the Frobenius kernel of G. Then K has no involution and order of K is odd [6]. Since G is semi-direct product of H and K, therefore H contains sylow 2-subgroup of G. Thus sylow 2- subgroup of G is either Z_2 or Q_8.

Now if sylow 2-subgroup of G is Z_2 then G is of the form E_3Z_2[6].

If sylow 2-subgroup of G is Q_8 then, by Glauberman Z^* Theorem [7], $Z(G)$ is non trivial. Hence involutions of H can not invert elements of K which is a contradiction to the fact that involutions of H invert elements of K [proposition 2.2].

Hence G is of the form E_3Z_2.

REFERENCES

[1] Gorenstein, D *Finite groups* Harper and Row, 1968.

[2] Hans Schwerdtfeger *Introduction to group theory* Noordhoff International, Leyden, 1967.

[3] Huppert and Blackburn *Finite groups-H* Springer Verlag, 1982.

[4] Ion Armeanu *Frobenius Q-groups* Istanbul Univ. Fen. Fek. Mat. Dec, 54(1995), 23-24.

[5] J. S. Rose *A course on group theory* Cambridge University Press, 1985.

[6] Kletzing Dennis *Structure and representation of Q-groups* Springer Verlag. New York, 1984.

[7] M. Suzuki *Group theory-H* Springer Verlag, 1985.

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SOME MATHEMATICAL MODELS AND SURVIVAL CURVES FOR GROWTH AND DECAY OF TUMOUR

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ABSTRACT

In this paper some mathematical models for growth and decay of tumour are reviewed and a simple model is developed. This model uses an exponential distribution as a model for the growth of the tumour and instantaneous kill of the tumour cells after irradiation. To avoid non repairable damage to the connective tissue, the upper limit of the dose is described using both the cumulative radiation effect system and the linear quadratic formula.

Key Words: Tumour, Growth, Exponential distribution, Logistic equation, Gompertz curve, Irradiation, Fractionation, Decay, Cumulative radiation effect system, Linear quadratic formula.

1. INTRODUCTION

Previously the temporal variation in incidence rates of cancer of the larynx [1][2] and factors influencing the five-year local control rates and disease free times of patients with cancer of the larynx were studied [3][4][5]. Generalized linear models and proportional hazard models were used in each case to estimate the effect of the explanatory variables on the response rates and times. Such models were extremely useful and allowed the conclusion that the increase in incidence rates was largely birth cohort based. The local control rates were heavily affected by the initial state of the tumour at the begining of the treatment and less so by the treatment variables [6][7][8]. The survival analysis showed that the survival time after treatment is, also, mainly dependent on the tumour characteristics and the effect of dose is not linear [9].

In this paper, detailed mathematical models of the behaviour of the tumour are considered. The previous models mentioned above are purely descriptive analytic tools. Here an attempt has been made to bring the cell kinetics to the fire to try and establish the effects of treatment. Radiotherapy is the main treatment of cancer of the larynx [10]. Also, an attempt has been made to look in detail at the way the tumours grow and decay through radiotherapy using some mathematical models. The aim is to see if optimal schedules can be found by such study so as to improve the treatment. The important restrictions on the dose are discussed in the form of the cumulative radiation effect system and the linear quadratic formula. Finally models are described which could devise optimal treatment schedules.

The problem of treatment of cancer by radiation involves two aspects.

1) Growth of the tumour

2) The killing of the cells by radiation

These aspects are independent of each other. The overall pattern of the treatment of the tumour is set on a biological basis. The total dose of radiation, say, D_T is decided. It has been further approved hat this dose should be given in fractions i.e. a part of the total be given on a number of occasions. The number of fractions, say, F is based on the understanding of the possible behaviour of the tumour growth and the capacity of the patient to bear a certain amount of radiation on one occasion. F is usually taken as between 4 to 60 for any treatment schedule [11]. The total time for the treatment, say, T days should be calculated such that there exists sufficient evidence to believe that no tumour cell could survive after the final dose. Figure 1 gives a picture of the growth and decay of the tumour cells. $P_1, P_2, \cdots, P_F$ denote the properties of cells immediately prior to the administration of dose fractions. Clearly P_1 is equal to one. $P_1', P_2', \cdots, P_F'$ denote the the cell properties immediately after the administration of each dose fraction. The properties are relative to the initial size N_0. After the first dose fraction, the proportion of cell, P_1 falls down to P_1'. During the time of the first fraction and the second fraction, the tumour grows to P_2. After the second fraction the cell proportion falls down to P_2'. The other values of the proportions can be interpreted in the same fashion.

Mathematics has to play its own role in predicting the behaviour of the growth and decay of tumour, both individually and thereafter combined. Apparently it may took improper to express the biological aspects into rigid mathematical terms, but fine similarities may be revealed and an attempt could be made to frame the biological phenomenon into a pre-controlled mathematical term. The problem thus arises to establish a model which justifies statistically the claim that one treatment is better than another.

The next Section 2 describes the development of an exponential distribution for tumour growth. With further practical assumption the logistic equation is stated in Section 3. Some other basis models which could represent the growth of a tumour are described in Section 4. The important aspect of the decay of a tumour is the cell survival. Survival curves from statistical models are described in Section 5. The cell survival after fractionated irradiation is described in Section 6. This survival takes into account both growth between fractions and instantaneous kill by radiation.

The biological effect of the dose on the human body is an important factor for both biologists and mathematicians. This effect is to be calculated keeping in view the connective tissue tolerance and the removal of the tumour in minimum time. The connective tissue tolerance is the upper limit of a dose which could damage the connective tissues up to a limit such that these tissues could be believed to repair themselves before the next fractionation. If a dose greater than the connective tissue tolerance is administered then the non cancerous cells surrounding the tumour would not be able to repair themselves before the next dose. Sections 7 and 8 take into account of this matter and describe the concept of the cumulative radiation effect and the linear quadratic formula. Section 9 describes the development of simple growth-decay models using the cumulative radiation effect system and the linear quadratic formula. These models are deterministic. The stochastic aspects of a simple growth and decay model can be simulated to devise optimal treatment schedules.

2. EXPONENTIAL GROWTH MODEL

If a constant rate of change is working on some population then the size of that population can be determined by an exponential model. The analogy of various mathematical parameters of the exponential distribution with biological terms is described in this Section.

The exponential model assumes that,

i) All cancer cells are alike.

ii) There is no cell death.

iii) Essential nutrients (such as oxygen or necessary substrates i.e. surfaces of plantation of tumour) remain available to the cells.

iv) The cancer cells have sufficient space to exist.

Let N_i be the number of cells in the i^{th} generation. Let the initial number of cells, N_0 (say) be 1. After the ith doubling we have

$$\begin{aligned} N_i &= 2N_{i-1} = 2(2 \times 2 \times \cdots i-1 \text{ terms}). \\ &= 2^i, \qquad i = 0, 1, 2, \cdots \end{aligned} \tag{2.1}$$

The time t for occurrence of i mitoses can be written as

$$t = it_d \tag{2.2}$$

where t_d denotes the tumour doubling time [12]. The tumour doubling time t_d is equal to the constant intermitotic time t_c say. Using equation (2.2) we can also write (2.1) as

$$N_i = 2^{t/t_d} = e^{(Ln2)t/td} \tag{2.3}$$

The variable t can safely be assumed to be continuous. Hence the above equation indicates that N_i can be interpreted as geometric growth or in terms of a type of exponential growth i.e.

$$u = \frac{1}{N}\frac{dN}{dt} = \frac{Ln(2)}{t_d} = \text{constant}, \tag{2.4}$$

where u can be defined as the specific growth rate of tumour. If it is assumed that at some time t_0 the tumour is identified to have a population of cancer cells as N_0, then at some later time t the cell population, N_t will be

$$N_t = N_0 e^{(t-t_0)u} \tag{2.5}$$

If the assumptions stated at the beginning of this Section continue to hold the exponential growth model indicates that the population of the cells will ever increase with a constant rate u. Figure 2 gives a picture of the growth of cell population by exponential growth starting from an initial value of N_0. There exists no finite limit on the final size of the population. Surely this assumption is difficult to hold over more than a limited period of time [13]. A mathematical distribution which takes into account the ever changing circumstances affecting the rate of change is described in the next Section.

3. LOGISTIC EQUATION

The exponential distribution described in the last Section assumes that the specific rate of change is not disturbed throughout some process. This is possible for a short term process. The human body has its own limitations. The tumour can grow exponentially for a short period. Thereafter some constraints must apply. A model which can take account of this limitation is described below.

If by the passage of time the supply of essential nutrients (such as oxygen or necessary substrates) reduces, the cell population experiences an inhibitory effect on its further growth as the rate of growth of the cells is reduced. The population of cells at a particular time has a smallar rate of growth as compared to some population at an earlier time. Thus the rate of increase of the population of cancer cells slows down gradually. The rate of slowing down can be materialized with the assumption that the cell population will ultimately reach a limit beyond which it will not be possible for the cells to proliferate [13]. Figure 3 gives a picture of growth of cells population starting from an initial size N_0 up to maximum of size N_L. The rate of change will now take the form

$$\frac{1}{N}\frac{dN}{dt} = f(N) \tag{3.1}$$

where $f(N)$ is a decreasing function of the cell population at some time t. A linear expression can possibly be the simplest way to represent $f(N)$, i.e.

$$f(N) = a + bN$$

where a and b are constants such that $b < 0$ and $(a + bN) > 0$

Assuming that the specific rate of growth of tumour in the beginning is u, an expression which can reduce the value of u to zero can be written as

$$f(N) = u\frac{N_L - N}{N_L - N_0} \tag{3.2}$$

where N_L is the conceived asymptotic size of the cell population. Using equation (3.2), equation (3.1) can be written as

$$\frac{1}{N}\frac{dN}{dt} = u\frac{N_L - N}{N_L - N_0} \tag{3.3}$$

A solution of the above equation can be written as

$$N = N_0\left[\frac{N_0}{N_L} + \left\{1 + \frac{N_0}{N_L}\right\} e^{\left\{-ut\frac{N_L}{N_L - N_0}\right\}}\right]^{-1} \tag{3.4}$$

This equation is called logistic equation and the curve of population sketched against time t is called logistic curve. Verhulst first used the logistic curve as a population model [14]. By definition other suitable expressions for $f(N)$ various other forms of the logistic curve can be deduced for different situations with suitable interpretations of the parameters [13].

In the above Section the rigidity of the exponential distribution has been moulded gradually to get the logistic equation. In the next Section some more models are described which could possibly represent the growth of tumour when the exponential distribution or logistic equation is not sufficient.

4. SOME OTHER GROWTH MODELS

In practice it has been experimentally found that some times the tumour's growth cannot often be adequately described by exponential model or by logistic equation [15]. Another curve, which can possibly represent the tumour's growth pattern, is the Gompertz curve. This curve is governed by the differential equation.

$$\frac{dN}{dt}\left[\frac{1}{F(N)}\right] = -h\left[\frac{1}{F(N)}\right]$$

where F(N) is a function of N depending upon t, and h is a constant of proportionality [16]. Let us assume that h = u. Defining F(N) as

$$\left[\frac{1}{F(N)}\right] = \log\left[\frac{N}{N_L}\right]$$
$$\frac{1}{N}\frac{dN}{dt} = -u\log\left[\frac{N}{N_L}\right] \qquad (4.1)$$

[15]. From the growth characteristics of the tumour the parameters u and N_L (greater than 0) can be determined. One possible solution of equation (4.1) is

$$N = N_0 \quad e^{\left[\log\left(\frac{N_L}{N_0}\right)\{1-e^{-ut}\}\right]} \qquad (4.2)$$

The Gompertz curve can be called a growth curve since it portrays a process of cumulative expansion to a maximum value. The expansion starts from decreasing relative amounts from the begining stage, but continues to the end without receding. This curve has a value as an empirical representation of certain trend movements [17].

Usher (1980) has developed a general model, where the exponential, logistic and Gompertzian curves can be shown to be special cases [18]. The differential equation for tumour growth can be written as

$$\frac{dN}{dt} = \frac{uN}{c}\left[1 - \left(\frac{N}{N_L}\right)\right]^c \qquad (4.3)$$

where c (greater than 0), u and N_L (greater than 0) are determined from the growth characteristics of the tumour on which the models are being applied.

When c tends to 0 equation (4.3) represents Gompertzian growth rate. When c tends to 1 equation (4.3) represents Verlulst logistic growth rate. When c tends to 1 and N_L tends to infinity, equation (4.3) represents the exponential growth rate.

One of the possible solutions of the equation (4.3) can be written as follows.

$$N = N_0\left[\left(\frac{N_0}{N_L}\right)^c + e^{(-ut)}\left\{1 - \left(\frac{N_0}{N_L}\right)^c\right\}\right]^{-1/c} \qquad (4.4)$$

Sections 2, 3 and 4 described some models which could be used for modelling growth of tumour. The decay of a tumour is not a straight forward process as the growth can be. The decay involves killing of cell by treatment and thereafter the survival of the cell. The survival behaviour of the cell after a radiation insult is considered in the following Section.

5. SURVIVAL CURVES FROM STATISTICAL MODELS

In previous Sections some mathematical models have been described which may represent the growth of the tumour. The determination of the amount of the dose, number of fractions and overall treatment time for a particular cancer patient with the intention to eliminate the tumour is called the treatment scheduling [10]. To ascertain the utility of some particular treatment schedule it is essential to study the behaviour of the survival of the cell after an irradiation insult. In this Section the survival curves are described which may fit the survival of cell after a radiation insult.

Before considering the survival of a cell after an irradiation insult the terms linear energy transfer and relative biological effect are explained. Two of the components of light are short wave length ultraviolet radiation and radiation which is composed of particles. X-rays and gamma rays are examples of particle radiation. The particle radiation can produce ionization in some organisms and can break chemical bonds, which can result in the killing of the target organism. Linear energy transfer can be defined as the rate of energy loss along a track. Various rays have different tracks of linear energy transfer. In this context the relative biological effect can be described as the ratio of radiation dose used by different particles. For example, if a dose of 1 Gray from beta particles is capable of destroying an organism and a dose of 2 Grays of X-rays can destroy the same organism then the beta rays will have twice the relative biological effect as compared to X-rays [11].

The chance of injuring an organism by a radiation insult is proportional to the volume of the organism and has two multiplicative components: The probability that the target gets the hit and the probability that ionization occurs in the organism. The first probability is proportional to the cross sectional area of the organism and the second probability is proportional to the thickness of the organism [11]. In the following Section, two survival models are presented. Section 5.1 describes the model when the cell is considered to be inactivated by a single hit. Section 5.2 takes into account the possibility of more than one nucleus in the cell.

5.1 Single-Hit-to-Kill Model

Let us assume that after each dose a fixed proportion k (say) of the cancerous population is killed. We can write

$$\frac{1}{N}\frac{dN}{dd} = -k \tag{5.1}$$

where N is the size of the tumour and d is the dose per fraction. Integrating both sides of equation (5.1) gives the solution $\log_e(N) = -kd + I$

where I is the constant of integration. Exponentiating the above expression we have

$$N = e^{(-kd+I)} \tag{5.2}$$

At $d = 0$, $N = N_0 = \exp(I)$. Dividing left hand side and right hand side of the equation (5.2) by N_0 and exp(I), respectively, we can define S as a survival fraction as

$$S = \frac{N}{N_0} = e^{(-kd)} \tag{5.3}$$

The above equation can represent survival from exponential distribution with mean $1/k$. Denoting the mean lethal dose by d_0 we can write $d_0 = 1/k$, so that

$$S = e^{(-d/d_0)} \tag{5.4}$$

Of course there is no flexibility in the constancy of the survival rate i.e. there exist no change in the experimental condition before or after the treatment.

In this Section it is assumed that the cell is a single target for irradiation. This means that if a cell is irradiated at least once the cell can be inactivated. In the following Section the situation is considered when the cell is considered to comprise more than one target one of which is required to be hit at least once to inactivate the cell.

5.2 Multi-Target, Single-Hit Survival

The cell may contain more that one nucleus. To inactivate the cell it is necessary to hit each at least one target at least one time. If we assume that a single hit is sufficient to inactivate a target and the probability of a target being hit is independent of the other targets, then the surviving fraction exp (-kd) defined in Section 5.1 can be identically regarded as the probability of a target not being hit. Let us assume that there are n nuclei in the cell. The probability that all targets are hit is, therefore,

$$\left[1 - e^{(-kd)}\right]^n \tag{5.5}$$

The survival S depending upon the parameters k, n, and d can be written as

$$S(k, n, d) = 1 - \left[1 - e^{(-kd)}\right]^n \tag{5.6}$$

Because of its historical importance the above formula is most widely used. For n = 2 and k = 1/96 (per rad). Puck and Marcus (1956) found that above equation (5.6) fitted well to their data [19]. For small radiation fractions or continuous irradiation at low dose rates in mammals or in clinical radiation therapy this formula is inadequate [11].

In previous Section 5.1 and 5.2 the survival fractions after radiation insult are considered. The growth of the tumour has already been discussed in Sections 2, 3 and 4. The growth models and survival fraction models can be combined to study the cell survival after repeated fractionated dose. The next Section presents this combination.

6. CELL SURVIVAL AFTER FRACTIONATED IRRADIATION

The growth behaviour of tumour cells and the survival fraction of a cell population after a single radiation hit and multiple hits has been modelled in Sections 4 and 5. To assess the outcomes of fractionated dose it is necessary to make use of both aspects of the cell population. In its simplest form the idea is presented in this Section.

It is a pleasant fact that normal cells have a superior and rapid capability to repair sublethal (i.e. less than connective tissue tolerance) damage as compared to the tumour cells. The time between successive dose fractions helps the recovery of normal cells. Of course the tumour cells which are not damaged enough to be completely inactivated take benefit of the time between fractions. The study of the growth of cancer cells is an important part of the cell survival after a radiation insult. Some important assumptions involving the process of growth and radiation insult are given in the following.

Let us assume that the dose is irradiated at times 0, t, 2t, and so on. Here t is the number of days between successive fractions. It is further assumed that there is an instantaneous decrease in the level of the tumour cells as result of the radiation dose. How much of the tumour is reduced at the times 0, t, 2t, 3t ... depends upon the dose irradiated. The cancer cells will grow during the open intervals (0,t), (t, 2t), etc. The t is usually one day. As the overall treatment times are not long enough to have to considering limiting tumour size (t is, usually one day), so logistic and Gompertz and other curves are not needed to explain the growth of the tumour and the exponential model would be sufficient for the purpose (Sections 2 and 3) [6]. It can, therefore, be assumed that the growth rate would remain the same in the times between fractionated doses. It means that the growth of tumour can be presented by exponential model and there would no need of assumed that the growth is presented by Logistic equation or Gompertz curve. Of course, another important assumption is that the normal cells recover during the time interval to the next fraction of the dose. With these assumptions the cell survival cure can be conceived as a continuous process of fall and rise. Of course, the rise after a fall should be small enough such that the cell population at the next fraction of dose is considerably less than the previous population. In this way the survival curves for tumour cells have a downward trend. After the last fraction of dose the survival curve is expected to lie on the horizontal axis for a successful treatment.

With the general idea about the survival curve explained in the previous paragraph exponential growth is considered below. The assumptions described are integral part of cell survival study.

Let us assume that the dose is dispensed by equal fractions and equal times between the various fractions of dose. As explained earlier in this Section the first fraction can be assumed to be given on day 0, the second on day t, the third fraction on day 2t and so on. The last fraction F (say) is, clearly, given on day (F - 1)t from the begining of treatment. Let the level of the cell population be $N_1, N_2, \cdots, N_F$ immediately prior to the administration of the first fraction, second fraction and so on the last fraction. Similarly we can define $N'_1, N'_2, \cdots, N'_F$ to be the levels of the cell population immediately after the first fraction, second fraction and so on the last fraction. Assuming the same dose per fraction hence the survival fraction S to be same after each fraction, we can write

$$N_i' = SN_i, \quad i = 1, 2, \cdots, F. \tag{6.1}$$

Here i is the number of the fraction. After the first fraction

$$N_1' = SN_1 \tag{6.2}$$

Assuming that after the first fraction the tumour grows exponentially, from equation (5.2.5), without applying the initial conditions, we have

$$N_t = N_{t-1}e^{(ut)}. \tag{6.3}$$

Hence the cell population immediate prior to second fraction is

$$N_2 = N_1' e^{(ut)}. \tag{6.4}$$

After the second fraction the cell population is

$$N_2' = SN_2 = S^2 N_1 e^{(ut)}. \tag{6.5}$$

Proceeding the same way we have

$$\begin{aligned} N_3' &= SN_3 = SN_2' e^{(ut)} = S^3 N_1 e^{(2ut)} \\ N_4' &= SN_4 = SN_3' e^{(ut)} = S^4 N_1 e^{(3ut)} \\ &\vdots \\ N_F' &= SN_F = SN_{F-1}' e^{(ut)} = S^F N_1 e^{[(F-1)ut]} \end{aligned} \tag{6.6}$$

Dividing both sides of the above equation by N_1 and denoting N_F' by N_F' (for simplicity) the final proportion of survived cell population after F fractions is

$$\frac{N_F}{N_I} = S^F \left[1 - e^{(F-1)ut}\right] \tag{6.7}$$

Practically the above model will never give a value lying on the horizontal line. This would tend to zero when F would tend to infinity (as S is a fraction) and t tends to zero. From equation (5.3), S = exp (- kd). Hence assuming F tends to infinity is equivalent to assuming D, the total dose, tends to infinity. This means that, for practical purposes, he fraction N_F/N_1 could be minimized reduced if the total dose per fraction or the number of fractions is increased and the time between fraction doses is minimized.

In the above Section the cell survival after successive radiation is considered. The growth rate during the intermittent time is assumed to be constant relative to the cell population. It is also mentioned that the normal tissues are to be given a fair chance of recovery before the next fraction. Hence there should be some limit on the dose irradiated. The dose should be chosen such that the biological effect produced should not damage the normal tissues to the extent that they could not sufficiently recover before the next fraction or after the completion of treatment schedule. Such restrictions on the amount of dose are discussed in the following two Sections. Section 7 describes a historical system called the cumulative radiation effect. A comparatively modern approach is to define the dose limit by the linear quadratic formula. This is explained in Section 8.

7. CUMULATIVE RADIATION EFFECT

A historical measure to restrict the dose to some upper limit is discussed in this Section. The cancer therapy is used with the intention of removing all the cancer cells. However, the dose of radiation, used as a treatment for removing the cancer cells, also causes damage to the normal cells. These normal cells, which surround the tumour, cannot bear a heavy dose. This can cause serious effects to the patient such as oedema, radiation necrosis or deterioration in the quality of voice [6]. So the dose used has, surely, some upper limit keeping in view that the normal cells should recover with the help of homeostatic repair mechanisms before the next fraction of dose is dispensed. The homoeostasis is a state of physiological equilibrium produced by a balance of functions and of chemical composition within an organism. Thus the treatment schedule is to be selected which should result in maximum damage to the cancer cells and minimum damage to the normal tissue. It is, therefore, necessary to have some idea of the normal tissue tolerance. A way to measure the damage suffered by the normal tissues described below.

Ellis and coworkers (1969) gave the concept of normal connective tissue tolerance as a practical upper limit of any radiation schedules and defined a term nominal standard dose (NIS) as

$$\text{NSD} = (\text{Total dose})\, F^{-0.24} T^{-0.11}$$

Where, T is the overall treatment time [20]. The constants, -0.24 and -0.11 have been derived by Ellis using the results of Strandqvist (1944) [21]. Strandqvist had derived a slope value of 0.22 for both skin reactions and for the cure of squamous cell carcinomas [22]. Ellis used the value of Strandqvist as representative of tumours to derive his formula for total dose as

$$\text{Total dose} = (\text{constant})\, F^{0.24} T^{0.11}$$

Although the results of Ellis are based on radiation effects on skin, Kirk, Gray and Watson (1978) suggested that these results can also be used to calculate radiation effect on normal tissues [23]. Kirk (1978) and McKenzie (1979) introduced the idea of cumulative radiation effect using the results of Ellis for this purpose [24][25]. Equivalent to the nominal standard dose, the cumulative radiation effect (R_F) is empirically related to total dose D, fractions F and number of treatment days T as

$$R_F = D F^{-0.24} T^{-0.11} \tag{7.1}$$

The cumulative radiation effect can also be written as

$$R_F = d F^{-0.65} t^{-0.11} \tag{7.2}$$

where d (= D/F) is the dose per fraction and t (=T/F) is the time between the fractions. The unit of cumulative radiation effect is rad.day $^{-0.11}$ and can be called reu (radiation equivalent unit).

With the same value of the cumulative radiation effect various treatment schedules can be devised which have the same biological effect. Keeping in view the deterioration of the patient the schedule can be changed to have an equivalent biological effect which is regarded as a target before administration of the treatment schedule. A treatment schedule for total dose of 6000 rads, 30 fractions and total treatment days of 34 (six fractions per week, starting on first working day and no dose on weekend) could produce a value of 1800 reus for the cumulative radiation effect. Various treatment schedules which produce a cumulative radiation effect reus of 1800 can be devised using equation (7.2). For example, a dose of 4750 rads, 4950 rads and 5850 rads dispensed daily for 16 days, 18 days, respectively, will have the same cumulative radiation effect of 1800 reus.

The cumulative radiation effect can be used to compare and assess various treatment schedules. If used for devising optimal treatment schedules, it may suggest some schedules with long overall treatment time. The clinical experience does not favour long treatment times [8]. It means that the cumulative radiation effect can be validated for short term schedules only. Fowler (1989), in his review article criticized the cumulative radiation effect as follows

7.1 The Fraction Number

a) $F^{0.24}$ does not predict the severe late damage that occurs for larger fraction sizes. For large change in the value of F, the change in $F^{0.24}$ will not be large because of the fractional power. The factor $F^{0.24}$ contains only one parameter which predict only acute or early damage. There is no provision for predicting late damages.

b) The graph of total dose against number of fractions is curved, not straight (log-log). This means that if we take logarithm of the formula for total dose, then keeping time as fixed, we get a linear relationship between the logarithm of total dose and logarithm of number of fractions but experimentally it is not true.

c) Fraction sizes, not number, is the important parameter. This means that the dose per fraction is more important than the number of fractions. But the formula for total dose does not take into account the dose per fraction and only considers the number of fraction, F.

7.2 The Time Factor

a) $T^{0.11}$ predicts a large increase of isoeffect dose (dose having the same effect) at first, then increasing more slowly. The biological fact is just the opposite: it shows no increase at first and then a rapid rise of isoeffect dose as proliferation accelerates. The time factor $T^{0.11}$ involves the fractional power. A range of value can be decreased remarkably by taking fractional power of the vlaues in range. The increase in the values of $T^{0.11}$ is much slower than the actual increase in the values of T. Hence the values of the total dose are predicted relatively higher by the smaller values of T than are predicted by greater values of T.

b) The time factor is underestimated for tumours and acutely reacting tissues. This means that the tumours and acutely reacting tissues have more weight than is calculated by the time factor $T^{0.11}$

c) The time factor is overestimated for late-reacting tissues. This means that late-reacting tissues given more weight by the time factor $T^{0.11}$ than the experiments show.

7.3 General

Time dose and fractionation tables (i.e. the treatment schedules determining the total dose, number of fractions and overall treatment time) are too easy to use without thinking about late/early reactions or proliferation rates [26].

Another formula which can take the place of the cumulative radiation effect is the linear quadratic formula. This is described in the next Section.

8. LINEAR QUADRATIC FORMULA

As explained in Section 7, the cumulative radiation effect does not take into account properly the late and early reaction or proliferation rates. The linear quadratic formula takes into account these damages incurred by the radiation insult and is described below.

In conventional radiotherapy the schedules for treatment include dose, fraction and time. The dose per fraction can be defined as dose divided by number of fractions. After the fractioned dose, the tumour cells absorb the energy administered by the dose. The tumour cells are killed at the stage of mitosis i.e. splitting themselves into two daughter cells. The damage by irradiation can be acute i.e. early and or late. These effects are relatively independent of each other. As stated earlier the repair of the normal tissues is an important aspect of the study of tumour decay. The acute or early effects can be reduced, by prolonging treatment. The late effects depend on the total dose administered.

Thames *et al.* (1982) presented a linear quadratic formula for measuring the biological effect of a treatment schedule. Biological effect (B_E) can be written as

$$B_E = F(\alpha d + \beta d^2) \tag{8.1}$$

[27]. Here d is the dose per fraction and F is the total number of fractions for the treatment schedule. The α and β are parameters depending on properties of the tissues being irradiated. This formula is an empirical relationship based on animal studies when radiation is fired at normal tissues and damage measured. Various sets of values of fraction F and d can be found which can give the same biological effect. Conversely with the same biological effect, various values of number of fractions and dose per fraction can be calculated. Equation (8.1) can be written as

$$\frac{B_E}{Fd} = \alpha + \beta d \tag{8.2}$$

For treatment schedules with fixed B_E and F the plot of 1/d against d yields a straight line. Writing equation (8.1) as

$$\frac{1}{Fd} = \frac{\alpha}{B_E} + \frac{\beta}{B_E} d \tag{8.3}$$

the fitting allows $\frac{\alpha}{\beta}$ ratio to be calculated as the ratio of the intercept and slope of the equation (8.3). If B_E is known, α and β can, precisely, be estimated. If total doses D_1 and D_2 give equal biological effect from two schedules with number of fractions as F_1 and F_2, we have

$$B_E = F_1(\alpha d_1 + \beta d_1^2) = F_2(\alpha d_2 + \beta d_2^2)$$

or

$$D_1 \left[\frac{\alpha}{\beta} + d_1\right] = D_2 \left[\frac{\alpha}{\beta} + d_2\right],$$

so that we have

$$\frac{\alpha}{\beta} = \frac{D_2 d_2 - D_1 d_1}{D_1 - D_2}$$

[28]. Here $d_1 \left(= \frac{D_1}{F_1}\right)$ and $d_2 \left(= \frac{D_2}{F_2}\right)$ are the doses per fraction for the total doses D_1 and D_2 respectively.

To estimate $\frac{\alpha}{\beta}$ from an animal experiment an early effect can be estimated from a dose in the range of 6 to 26 Grays. To estimate late effect the range of dose is 2 to 5 Grays (1 Gray = 100 rads).

The important restrictions over some treatment schedules have emerged in the form of the cumulative radiation effect and the linear quadratic formula. These have been described in the above Sections. The two formulae can be incorporated in to survival models described in Section 5 to develop optimization models. Of course the consideration of the growth of the tumour is an integral part of a fractionated schedule. Some growth models have already been discussed in Section 2. Following Section describes the development of optimal models.

9. OPTIMIZATION OF THE MODEL

The dose cannot be bombarded with a dose of radiation without any limitation. The limitations were discussed in the form of the cumulative radiation effect and the linear quadratic formula in Sections 7 and 8 respectively. The fractionation has proved its own advantage over dispensing the dose as a whole. However fractionation does permit the growth of the tumour in the intermittent time. The growth models have been discussed in Sections 2, 3 and 4. All these aspects can be combined to reach some optimal model for fractionated treatment. This Section takes into account of such models. Section 9.1 considers an optimization model consisting of the cumulative radiation effect and Section 9.2 uses the linear quadratic formula.

9.1 Optimization Model Using the Cumulative Radiation Effect

Wheldon and Kirk (1976) have used the cumulative radiation effect system to develop an optimal mathematical model [29]. This model can be used to deduce the treatment schedules which take into account connective tissue tolerance so that the normal tissues get the maximum chance to revive. The details of the Wheldon and Kirk formula are as follows.

In Section 5.2 the survival function after single hit multitarget tumour cells was found to be

$$S(k,n,d) = 1 - \left[1 - e^{(-kd)}\right]^n \tag{9.1}$$

k can be replaced by $1/d_0$ where d_0 is mean lethal dose. The d is the dose per fraction and n is the number of nuclei in the cell. These are the targets each of which is to be hit at least once to inactivate the cell. Let F be the number of fraction for the uniform optimal schedule. From equation (6.7) the proportion of cells remaining after F fractions is

$$\frac{N_F}{N_1} = S^F e^{(F-1)ut} \tag{9.2}$$

where N_F is the final size of the cell population after F fractions, N_1 is the initial size of the cell population, S is the survival function described above in equation (9.1), u is the constant rate of growth per day or regeneration parameter and t is the intermittent time between fractions. In Section 7 the cumulative radiation effect is defined as

$$R_F = dF^a t^{-b} \tag{9.3}$$

The optimization problem is to minimize the proportion of cells defined in equation (9.2) subject to the condition defined by cumulative radiation effect in equation (9.3). This can be accomplished by, putting the value of F in (9.2), after calculating from (9.3) and putting the value of S from (9.1). Denoting the proportion of cells survived after F fractions by Y, the model can take the form, after the value of S is incorporated, as

$$Y = \frac{N_F}{N_1} = \left[1 - \left\{1 - e^{(-d/d_0)^n}\right\}\right]^F e^{(F-1)ut} \tag{9.4}$$

Putting the value of F from equation (7.2), the optimal model is

$$Y_{\mathrm{CRE}}(d,t) = \left[1 - \left\{1 - e^{(-d/d_0)^n}\right\}\right]^{\left(\frac{R_F}{d}\right)^{1/a} t^{b/a}} \times e^{\left[\left\{\left(\frac{R_F}{d}\right)^{1/a} t^{b/a} - t\right\} u\right]} \tag{9.5}$$

The above equation involves two basic treatment variables d and t. The surviving fraction Y is a function of these two parameters. The parameters R_F, a, and b concern with the radiation damage to normal tissues. The parameters d_0 and n refer to radiobiological characteristics of the tumour. u is the specific exponential growth rate of the tumour.

The standard value of the cumulative radiation effect is 1800 reus. As stated earlier, Kirk suggested the values of a and b as 0.65 and 0.11, respectively. The n is usually taken as one or 2. u can be estimated from tumour characteristics. The tumour doubling time could be from 10 to 130 days [11].

The model presented by equation (9.5) is deterministic. This will never become zero. As the value of d i.e. Dose per Fraction tends to infinity the model will represent an asymptote to the horizontal axis. As t i.e. tends to infinity the value of the function tends to infinity. Under certain conditions, however, this function can become asymptotically zero. Keeping t as fixed and increasing the value of d from zero, the derivative of the function Y_{CRE} with respect to d passes through zero. Similar behaviour can be observed keeping d fixed. This means that the pair of values (d, t) can provide minima which would be global one. Some details can be found in [11].

An optimal model is developed in the above Section using the cumulative radiation effect suggested by Wheldon and Kirk (1976) [29]. The cumulative radiation effect system has a place in the history of radiotherapy. The criticism over the system made a way to consider the linear quadratic formula for the purpose of derivation of a mathematical model to get optimum treatment schedules. Such a model is discussed in the next Section.

9.2 Optimization Model Using the Linear Quadratic Formula

In Section 7 the dependency of the lesion was described by the linear quadratic formula proposed by Thames *et al* (1982)[27]. From equation(8.1) the biological effect can be written as

$$B_E = F(\alpha d + \beta d^2) \tag{9.6}$$

Using the above equation for the relative biological effect another optimal model can be derived parallel to the model represented by equation (9.5) using which used the cumulative radiation effect as the relative biological effect.

From equation (5.6), the survival function after single hit to multitarget tumour cells is

$$S(k, n, d) = 1 - \left[1 - e^{(-kd)}\right]^n \tag{9.7}$$

where $k = 1/d_0$, d_0 is mean lethal dose, d is dose per fraction and n is the extrapolation number representing the number of nuclei in the cell. From equation (6.7) the proportion of cells remaining after F fraction is

$$\frac{N_F}{N_1} = S^F e^{(F-1)ut} \tag{9.8}$$

Here N_F and N_1 are the final and initial sizes of the cell population, u is the constant rate of growth per day and t is the time between fractions. The optimal model representing the proportion of cells after F fractions can be had by putting the value of F from equation (9.6) in equation (9.8). From equation (9.6), F can be calculated as

$$F = \frac{B_E}{\alpha d + \beta d^2} \tag{9.9}$$

Putting this value of F in equation (9.8) and denoting the proportion of cells survived after F fractions by Y_{LQ} we have

$$Y_{LQ} = S^{\left(\frac{B_E}{\alpha d+\beta d^2}\right)} e^{\left[\frac{B_E}{\alpha d+\beta d^2}-1\right]ut} \tag{9.10}$$

Inserting the value of S from (9.7) the optimal model using linear quadratic formula for the relative biological effect is

$$Y_{LQ}(d,t) = \left[1 - \left\{1 - e^{(-kd)}\right\}^n\right]^{\left[\frac{B_E}{\alpha d+\beta d^2}\right]} e^{\left[\frac{B_E}{\alpha d+\beta d^2}-1\right]ut} \tag{9.11}$$

Like equation (9.5) the above equation (9.11) represents the surviving fraction Y as a function of these two basic treatment variables d and t. The parameters B_E, a, and b are parameters concerning radiation damage to normal tissues. The d_0 is mean lethal dose and n is the number of nuclei in the cell. These parameters refer to radiobiological characteristics of the tumour. The u is the specific exponential growth rate of the tumour and t is the time between fractions. Keeping the value of d fixed and increasing the value of t, Y_{LQ} would increase. If we keep t fixed and increase the value of d the function would decrease accordingly. This means that we cannot attain a minimum.

10. SUMMARY AND CONCLUSIONS

The information from some previous studies is utilized to develop a model, which could devise optimum treatment schedules. Some mathematical models for growth of tumour have been reviewed in Sections 2, 3 and 4. Survival curves from statistical models are discussed in Section 5.

Combining the growth and survival curves the cell survival after fractionated irradiation is discussed in Section 6. The model described suggests that the fraction of cells remaining after F fractions, N_F/N_0 will tend to zero if the dose per fraction is infinitely large or the number of fractions is infinite. Another condition for the model to tend to zero is that the time between fractions tends to zero. In any case the survival curve will be asymptotic to the horizontal axis and will never touch it. It can be concluded that we cannot get a value of zero for the fraction N_F/N_0 with deterministic models.

Incorporating the cumulative radiation effect system as a connective tissue tolerance, which is the upper limit for any treatment schedules, an optimal model for the cell survival is described in Section 9.1. The cumulative radiation effect system has certain limitations. These are described in Section 7. An alternative method of estimating the biological effect is the linear quadratic formula. The formula is described in Section 8. An optimal model incorporating linear quadratic formula is described in Section 9.2. For both models, the increase in the dose would decrease the value of the survival function and any increase in the time t would increase the survival function.

The main conclusion from this study are that long treatment times and very low doses are bad for cure and models allow investigation of optimum schedules.

11. CRITICISM OF WORK AND SUGGESTIONS FOR FUTURE RESEARCH

The models for growth and decay of tumours reviewed in this paper are deterministic. They are asymptotic to the horizontal line. Practically the dose cannot be too large at one instance or as a whole. The minimization of overall treatment time implies the increase of dose per fraction or overall dose. The number of fractions can only be very limited as they cause the increase in the overall time of treatment. Simulation can be used to examine for their stochastic equivalents. This needs unique estimates (or at least a range of estimated values) of parameters of the model.

The optimal values of the δ and t for the model represented by equation 9.5 can be found with the help of numerical analysis follwoing the lines of Swan, 1981. Swan has given two equations in two parameters. The method of Newton and Raphson for two variables, for example, can be used for this purpose. The growth rate indicated by various types of cancer can be studies and combined together to get some closer estimate. Some estimate or workable guess of the rate of cell kill can be found by an extensive study of both biological and mathematical literature and analysis of clinical data. It would be possible, then, to discriminate rigorously among various schedules for their optimality.

The concept of hyperfractionation (smaller doses with overall treatment time not reduced) and accelerated fractionation (smaller doses with overall treatment time reduced) can be incorporated by the use of multiple fraction per day to suggest more extensions of the growth and deacay model. In this way the prolongation of treatment can be avoided and the rapidly proliferating tumour cells would not be able to escape the treatment. Previously the proliferation between dose fractions has been considered on per day as the dose fraction is dispensed on daily basis. In the context of hyper-fractionation and accelerated fractionation, the time between the dose fraction would be in hours and hence the proliferation rate can be considered as continuous. It is further suggested that suitable biological basis should be found or sound mathematical reasoning should be developed to justify the assumptions involved in the extensions of the growth and decay model. To be sure of the validity of the models a sensitivity analysis should be accomplished.

12. DIAGRAMS

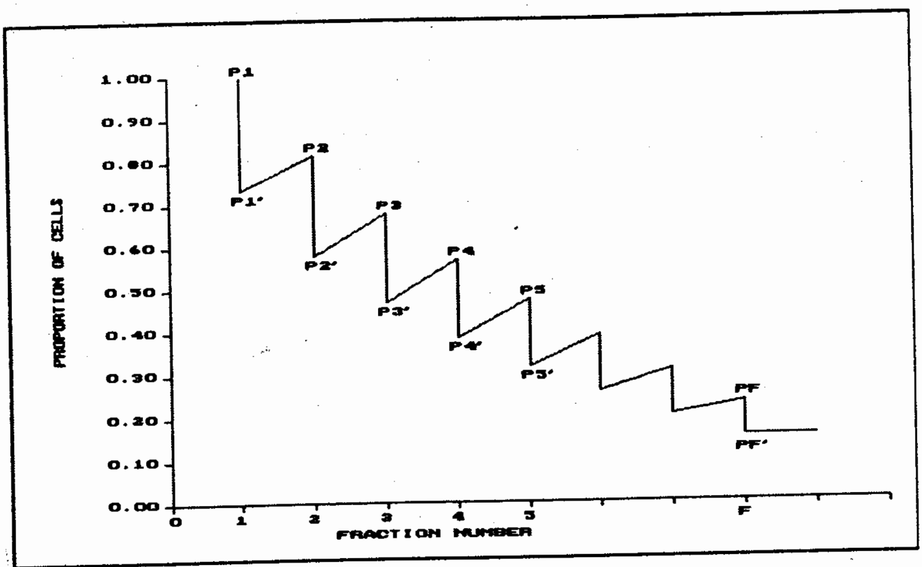

Figure 1

Instantaneous decrease of tumour population (%) as a result of anticancer therapy

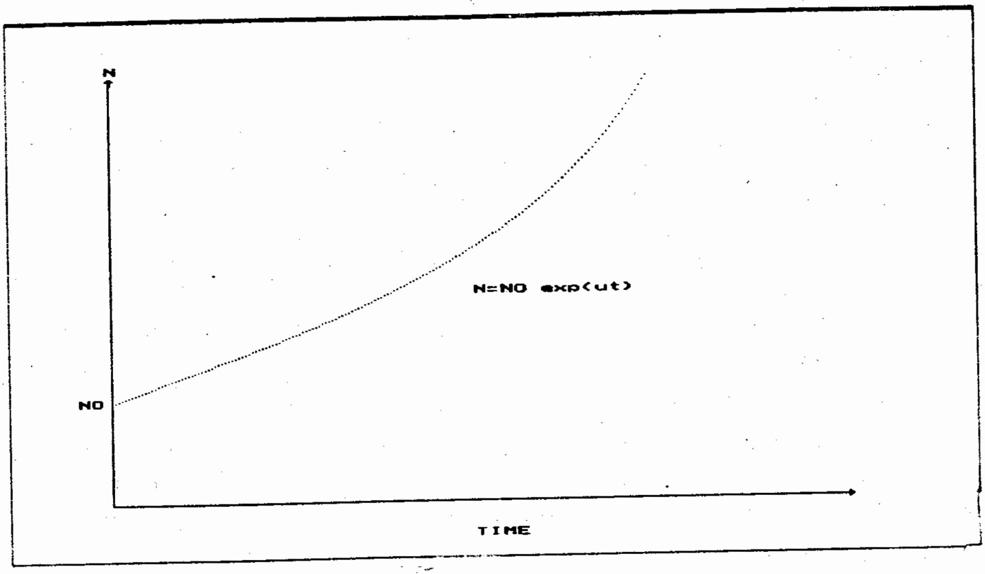

Figure 2

Exponential growth curve

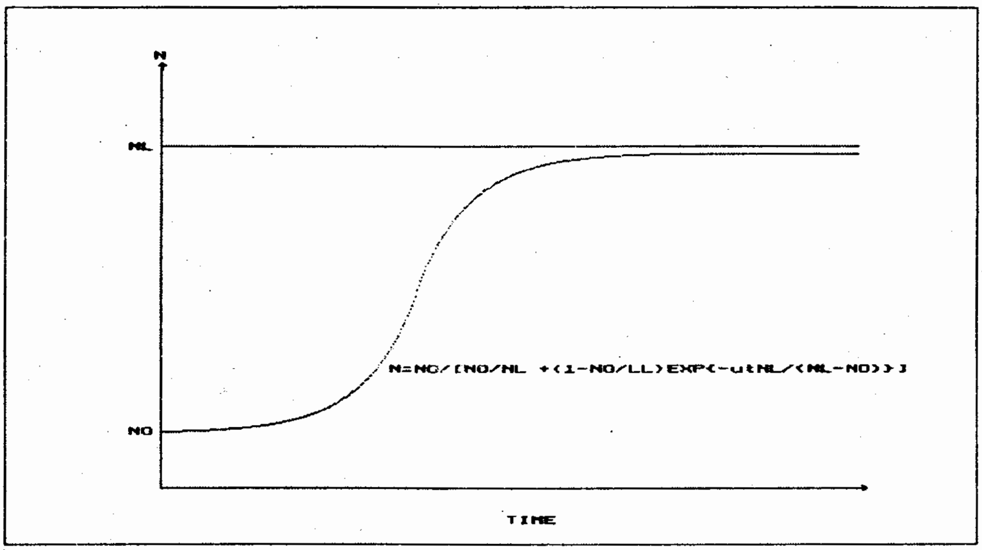

Figure 3

Logistic growth curve

REFERENCES

[1] Khan, Khudadad *A Preliminary Analysis to Study Time Trends in the Incidence of Cancer in the Larynx* Journal of Statistics (Department of Statistics, Government College, Lahore) Vol. 8, No. 1, 23-35 (2001).

[2] Khan, Khudadad *Some Age-Period-Cohort Models to Study Time Trends in the Incidence of Cancer in the Larynx* Journal of Statistics (Department of Statistics, Government College, Lahore) Vol. 8, No. 1, 7-22 (2001).

[3] Khan, Khudadad *Fitting of Age-Period-Cohort Models to Study Time Trends in the Incidence of Cancer in the Larynx* Journal of Statistics (Department of Statistics, Government College, Lahore) Vol. 9, No. 1 (2002) (Accepted for Publication).

[4] Khan, Khudadad *An Age-Period-Cohort Model to Study Time Trends in the Incidence of Cancer in the Larynx* Science International (Lahore), (ISSN 1013-536, CODEN: SINTE 8), Vol. 13, No. 2, 189-194 (2001).

[5] Khan, Khudadad *Simulation of an Age-Period-Cohort Model to Study Time Trends in the Incidence of Cancer in the Larynx* Science International (Lahore), (ISSN 1013-536, CODEN: SINTE 8), Vol. 13, No. 3, 215-220 (2001).

[6] Khan, Khudadad *Description and Preliminary Analysis of a Treatment Data for Cancer of the Larynx* Science International (Lahore), (ISSN 1013-536, CODEN: SINTE 8), Vol. 13, No. 3, 223-230 (2001).

[7] Khan, Khudadad *Survival Curves of a Treatment Data for Cancer of the Larynx* Science International (Lahore), (ISSN 1013-536, CODEN: SINTE 8), Vol. 13, No. 4, 295-300 (2001).

[8] Khan, Khudadad *Logistic Regression Analysis of five year Survival of a Treatment Data for Cancer of the Larynx* Science International (Lahore), (ISSN 1013-536, CODEN: SINTE 8), Vol. 14, No. 1, 9-20 (2002).

[9] Khan, Khudadad *Survival Analysis of a Treatment Data for Cancer of the Larynx* Science International (Lahore), (ISSN 1013-536, CODEN: SINTE 8), Vol. 14, No. 2 (2002).

[10] Khan, Khudadad *Cancer and its Therapy-An Introduction* Journal of Statistics, Department of Statistics, Government College, Lahore, Vol. 9, No. 1 (2002) (Accepted for Publication).

[11] Swan G. W. *Lecture Notes in Biomathematics Optimization of Human Cancer Radiotherapy* Springer-Verlag Berlin-Heidelberg, N.Y. (1981).

[12] Swan, G. W. *Tumour Growth Models and Cancer Chemotherapy* Monograph on Cancer Modelling, edited by Thompson James R. and Brown Barry W., Marcel Dekker, Inc. New York (1987).

[13] Keyfitz, Nathan and Beekman, John A. *Demography Through Problems* Springer-Verlag, New York (1984).

[14] Verhulst, P.F. *Notice sur la loi que la population suit dans son accroissement* Correspondence Mathematique et Physique Publiee par A. Quetelet (Brussels), Vol. 10, pp. 113-121 (1838).

[15] Usher, J. R. and Abercrombie, D. A. *Tumour Growth and the Response of Cells to Irradiation* Case studies in Mathematical Modelling, edited by James, D. J. G. and McDonald, J.J. Stanley Thorns Publishers Ltd., Cheltenham, Great Britain (1981).

[16] Gompertz, B. *On the Nature of Function expressive of the law of Human Mortality* Philosophical Transaction, Royal Society of London (1825).

[17] Mills, Frederick C. *Statistical Methods* Sir Issac Pitman and sons, Ltd., London (1965).

[18] Usher, J. R. *Mathematical Derivation of Optimal Uniform Treatment Schedules for the Fractionated Irradiation of Human Tumours* Mathematical Bioscience 49, 157-184 (1980).

[19] Puck, T.T. and Marcus, P.I. *Action of X-Ray on Mammalian Cell* Journal of Experimental Medicine, 103, 653-666 (1956).

[20] Ellis, F. *Dose and Time Fractionation: A Clinical Hypothesis* Clin. Radiol, 20, 1-7 (1969).

[21] Ellis, F. *Nominal Standard Dose and the Ret* Br. J. Radiol, 44, 101-108 (1971).

[22] Strandqvist, M. *Studien uber die Kumulative Wirkung der Rontgenstrahlen bei Fraktionierung* Acta Radiol. Suppl., 55, 1-293 (1944).

[23] Kirk, J. Gray, W. M. and Watson, E. R. *Cumulative Radiation Effect. Part-I Fractionated Treament Regimes* Clin. Radiol., 22, 145-155 (1978).

[24] Kirk, J. *The Cumulative Radiation Effect* Edinburgh: Churchil Livingstone(1978).

[25] McKenzie, A. L. *Cell Survival Description of the Cumulative Radiation Effect* Acta Radiol. Oncol. Rad. Phys. Bio., 2, 307-317 (1979).

[26] Fowler, J. F. *The Linear-Quadratic Formula and progress in Fractionated Radiotherapy* The British Journal of Radiology, 62, 679-694 (1989).

[27] Thames D. Howard, Jr. Withers, H.R. Peters, L.J. and Fletcher, G.H. *Changes in Early and Late Radiation Response with altered Dose Fractionation: Implication for Dose-Survival Relationship* Int. J. Radiat. Oncol. Bio. Phys. 8:219-226 (1982).

[28] Fowler, J.F. *What next in Fractionated Radiotherapy?* British Journal of Cancer, 49, Suppl. VI, 285-300 (1984).

[29] Wheldon, T.E. and Kirk, J. *Mathematical Derivation of Optimal Treatment Schedules for the Radiotherapy of Human Tumours. Fractionated Irradiation of Exponentially Growing Tumours* The British Journal of Radiology, 49, 441-449 (1976).

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A WELL-POSED PROBLEM FOR THE STOKES-BITSADZE SYSTEM

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ABSTRACT

For certain types of boundary conditions the Stokes-Bitsadze system (stream function Airy stress function formulation of two dimensional Stokes flow), is a well known ill-posed problem. Appropriate boundary conditions of Poincare and prescribed for the Stokes-Bitsadze system and well-posedness is proved.

Key Words: Boundary Conditions of Poincare, Cauchy-Riemann System, Div-Curl Formulation, Stokes-Bitsadze System, Stream Function Stress Function Formulation.

1. INTRODUCTION

In this paper we shall be concerned with the following two-dimensional Stokes flow based on stream function $\psi(x, y)$ and Airy stress functions $\phi(x, y)$ [Coleman, 1981]

$$\begin{aligned} \phi_{xx} - \phi_{yy} + 2\psi_{xy} &= 0 \\ \psi_{xx} - \psi_{yy} - 2\phi_{xy} &= 0 \end{aligned} \tag{1}$$

which is the second order elliptic system. The ellipticity of the system in the sense of Petrovskii, [1946] is proved by Thatcher [1997]. Tahir [1999a] identifies it as Stokes-Bitsadze system (SBS). By introducing the notations $w = \phi + i\psi$, $z = x + iy$ and $\bar{z} = x - iy$, the SBS may be written in the form

$$w_{\bar{z}\bar{z}} = 0 \tag{2}$$

where $2\partial_{\bar{z}} = \partial_x + i\partial_y$. From (2) the regular solution SBS can be represented in the form

$$w = \bar{z}f(z) + g(z) \tag{3}$$

where $f(z)$ and $g(z)$ are arbitrary analytic functions of the complex variable z. On the grounds of (3) Bitsadze [1964] shows that in the circular domain $|z - z_0| < \epsilon$ the homogeneous Dirichlet problem for the SBS (1) has the infinite set of linearly independent solutions given by

$$w = \left\{\epsilon^2 - |z - z_0|^2\right\} g(z) \tag{4}$$

where $g(z)$ is a function which is arbitrary and analytic in the domain $|z - z_0| < \epsilon$ Bitsadze [1964] concludes that the Dirichlet problem for the SBS is neither Fredholmian nor Neotherian. [1] For the Fredholm and Noether theory we refer to [Bitsadze, 1968], [Bitsadze, 1988] and [Mikhlin, 1970]. Bitsadze [1988] shows that Fredholmian character of the Neumann problem is also violated [2] for the SBS Wendland [1979] considers the Dirichlet problem for the system (1) and proves the violation of Lopatinski condition to show the problem to be non-Fredholm. For the details on Lopatinski condition we refer to Wendland [1979].

The researchers have been interested in the SBS but suffering a difficulty concerning the appropriate boundary conditions, see for example [Cassidy, 1996] and [Thatcher, 1997]. Owen and Phillips, [1994] embed the system in biharmonic equations and determine the appropriate boundary conditions for the higher order system. The ill-posedness of Dirichlet and Neumann problem for the SBS (1) has motivated us to further investigate the situation. In the paper [Tahir & Davies, 2000] we have proved the existence of a unique solution for the SBS with velocity boundary conditions ψ, ψ_n along with additional single point conditions. Existence of a unique solution for the periodic boundary conditions is proved in [Tahir, 2001].

[1] The situation contrasts greatly with a system of a single elliptic equation, see for the details [Kuz'min, 1967] and [Bitsadze, 1968].

[2] Similar facts can also be observed when the number of independent variables is more than two. For some multidimensional analogs of Bitsadze systems we refer to [Yanushaukas, 1995], [Treneva, 1985] and [Kuz'min, 1967].

1.1 Boundary-Value Problem of Poincare

The Stokes-Bitsadze system (1) can also be expressed as

$$A\mathcal{X}_{xx} + 2B\mathcal{X}_{xy} + C\mathcal{X}_{yy} = 0 \tag{5}$$

where $\mathcal{X} = (\phi, \psi)^T$ is the required real vector and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6}$$

In the domain $\Omega \subset \mathcal{R}^2$ with boundary Γ the Poincare problem for the SBS is formulated as follows, to seek a solution $\mathcal{X} = (\phi, \psi)^T$ for the system (5) subject to the boundary conditions

$$p^1\mathcal{X}_x + p^2\mathcal{X}_y + q\mathcal{X} = \mathcal{J}(x,y), \quad (x,y) \in \Gamma \tag{7}$$

where p^1, p^2 and q are real 2×2 matrices given on the boundary Γ and $\mathcal{J}$ a real vector given on Γ. For a detailed study on the Pioncare problem, for the second order elliptic systems in the plane, we refer to Bitsadze [1968].

1.2 Cauchy-Riemann System

The Cauchy-Riemann system (or div-curl system) is of special interest to us. The div-curl formulation of SBS plays a key role for the study of boundary value problems for the stream function. Airy stress function formulation of the Stokes flow. The inhomogeneous Cauchy-Riemann which, in planar cartesians, appears as below

$$\begin{aligned} \text{div}(\phi, \psi) &= f_1, \\ \text{curl}(\phi, \psi) &= f_2, \end{aligned} \tag{8}$$

has been of interest for the researchers, see for example [Borzi et al, 1997]. [Chang & Gunzburger, 1990], [Neittaanmaki & Saranen, 1981] and [Tahir, 1999b]. Collectively the Cauchy-Riemann system (8) is elliptic while individually both the partial differential equations are hyperbolic. If ϕ and ψ are twice continuously differentiable and $f_1 = f_2 = 0$ then ϕ and ψ are harmonic

Let $f_1, f_2 \in L_2(\Omega)$. In a square domain $\Omega = (0,1) \times (0,1)$ with boundary Γ, Chang & Gunzburger [1990], Neittaanmaki & Saranen [1981] and Vanmaele et al [1994] discuss the div-curl system (8) with boundary conditions

$$(\phi, \psi) \times \mathbf{n} = 0 \quad \text{on} \quad \Gamma \tag{9}$$

where $\mathbf{n}$ is the outer unit normal and $(\phi, \psi) \times \mathbf{n} = \phi n_2 + \psi n_1$. The well-posedness of (8), (9) is proved in $H^1(\Omega) \times H^1(\Omega)$ subject to the compatibility condition

$$\int_\Omega f_2 d\Omega = 0 \tag{10}$$

Subject of the compability condition

$$\int_\Omega f_1 d\Omega = 0 \tag{11}$$

the problem

$$(\phi, \psi) \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma \tag{12}$$

considered with the system (8) is well-posed in $H^1(\Omega) \times H^1(\Omega)$, see [Vanmaele et al, [1994]. [Chang & Gunzburger, [1990] and [Wendland, 1979].

1.3 The Div-Curl Formulation of SBS [Tahir, 1999a]

The SBS (1) can be written as

$$\begin{aligned} \partial_x(\psi_y + \phi_x) + \partial_y(\psi_x - \phi_y) &= 0, \\ \partial_x(\psi_x - \phi_y) + \partial_y(\psi_y + \phi_x) &= 0, \end{aligned} \tag{13}$$

Introducing $\Phi(x, y)$ and $\Psi(x, y)$ as follows

$$\begin{aligned} \Phi(x, y) &\equiv \mathrm{div}(\phi, \psi) = \psi_y + \phi_x, \\ \Psi(x, y) &\equiv \mathrm{curl}(\phi, \psi) = \psi_x - \phi_y, \end{aligned} \tag{14}$$

the div-curl formulation

$$\begin{aligned} \mathrm{div}(\Phi, \Psi) &= 0, \\ \mathrm{curl}(\Phi, \Psi) &= 0, \end{aligned} \tag{15}$$

is then immediately obtained. It is easy to verify that SBS remains unchanges either (ϕ, ψ) is replaced by $(-\psi, \phi)$ or (Φ, Ψ) is replaced by $(-\Psi, \Phi)$.

In this paper we prescribe boundary conditions of Poincare for the Stokes-Bitsadze system. The above results, for the Cauchy-Riemann system, are used to prove the well-posedness of the Stokes Bitsadze problem.

2. WELL-POSED PROBLEM FOR THE STOKES-BITSADZE SYSTEM

We consider the SBS in a square domain $\Omega = (0,1) \times (0,1)$ with boundary Γ and prescribe the boundary conditions of Poincare (see figure 1).

$$\left.\begin{array}{rcl} (\Phi, \Psi) \cdot \mathbf{n} & = & f \\ (\phi, \psi) \cdot \mathbf{n} & = & g \end{array}\right\} \quad \text{on} \quad \Gamma \tag{16}$$

where $\mathbf{n}$ is the outward unit normal.

Theorem 2.1

For $f, g \in H^{\frac{1}{2}}(\Gamma)$, the Stokes-Bitsadze problem (1), (16) is a well-posed problem in $H^1(\Omega) \times H^1(\Omega)$ subject to the compatibility condition

$$\int_\Gamma f ds = 0 \tag{17}$$

Proof

For $f \in H^{\frac{1}{2}}(\Gamma)$ the div-curl system (15) with the boundary (16a) is a well-posed problem subject to the compatibility condition (17) which determines $(\Phi, \Psi) \in H^1(\Omega) \times H^1(\Omega)$ uniquely. It obviously implies that $\Phi, \Psi \in L_2(\Omega)$. Now for $g \in H^{\frac{1}{2}}(\Gamma)$ the Cauchy-Riemann system

$$\left.\begin{array}{rcl} \operatorname{div}(\phi, \psi) & = & \Phi \\ \operatorname{curl}(\phi, \psi) & = & \Psi \end{array}\right\} \quad \text{in} \quad \Omega \tag{18}$$

with the boundary conditions (16b) is also a well-posed problem in $H^1(\Omega) \times H^1(\Omega)$ satisfying already the compability condition $\int\int_\Omega \Phi dx\ dy = \int_\Gamma g\ ds$. Hence the Stokes-Bitsadze problem (1), (16) is well-posed in $H^1(\Omega) \times H^1(\Omega)$ subject to compability condition (17) and the proof is complete.

We can also prescribe the boundary conditions

$$\left.\begin{array}{rcl} (\Phi, \Psi) \times \mathbf{n} & = & f \\ (\phi, \psi) \times \mathbf{n} & = & g \end{array}\right\} \quad \text{on} \quad \Gamma \tag{19}$$

which after replacing (Φ, Ψ) by $(-\Psi, \Phi)$ and (ϕ, ψ) by $(-\psi, \phi)$, may be considered equivalent to (16). Therefore we present the Stokes-Bitsadze problem with boundary conditions (19) as a corollary which can be proved similarly as in theorem 2.1.

Corollary 2.2

For $f, g \in H^{\frac{1}{2}}(\Gamma)$, the Stokes-Bitsadze problem (1), (19), see figure 2, is well-posed problem in $H^1(\Omega) \times H^1(\Omega)$ subject to the compatibility condition $\int_\Gamma f \, ds = 0$

Corollary 2.3

For $f, g \in H^{\frac{1}{2}}(\Gamma)$, the Stokes-Bitsadze problem for the Poincare conditions (see also figure 3)

$$\left.\begin{array}{lcl} (\Phi, \Psi) \times \mathbf{n} & = & f \\ (\phi, \psi) \cdot \mathbf{n} & = & g \end{array}\right\} \quad on \;\; \Gamma \tag{20}$$

is a well-posed problem in $H^1(\Omega) \times H^1(\Omega)$ subject to the compatibility condition $\int_\Gamma f \, ds = 0$.

Corollary 2.4

For $f, g \in H^{\frac{1}{2}}(\Gamma)$, the Stokes-Bitsadze problem for the Poincare conditions (see also figure 4).

$$\left.\begin{array}{lcl} (\Phi, \Psi) \cdot \mathbf{n} & = & f \\ (\phi, \psi) \times \mathbf{n} & = & g \end{array}\right\} \quad on \;\; \Gamma \tag{21}$$

is a well-posed problem in $H^1(\Omega) \times H^1(\Omega)$ subject to the compatibility condition $\int_\Gamma f \, ds = 0$.

3. CONCLUSION

Appropriate boundary conditions of Poincare are prescribed for the two-dimensional Stokes flow based on stream function and stress function and the well-posedness of the problem is proved.

REFERENCES

[1] Bitsadze, A.V. (1964) *Equations of mixed type* Pergamon Press Ltd., London.

[2] Bitsadze, A.V. (1968) *Boundary value problems for second order elliptic equations* North-Holland Publishing Company Amsterdam.

[3] Bitsadze, A.V. (1988) *Some classes of partial differential equations* Gordon & Breach Science Publishers London.

[4] Borzi, A. et al, (1997) *Multilevel solution of cell vertex Cauchy-Riemann Equations* SIAM J. Sci. Comput., 18(2), 441-459.

[5] Cassidy, M. (1996) *A spectral element method for viscoelastic extrudate swell* PhD thesis, University of Wales, Aberystwyth.

[6] Chang, C-L and Gunzberger, M.D. (1990) *A subdomain-Galerkin/least-squares method for first-order elliptic systems in the plane* Siam Journal on Numerical Analysis 27(5), 1197-1211.

[7] Coleman, C.J. (1981) *A countour integral formulation of plane creeping Newtonian flow* Q.J. Mech. Appl. Math. XXXIV, 453-464.

[8] Jyz'min, E.N. (1967) *On the Dirichlet problem for elliptic systems in space* Differentsial'nye Uravneniya, 3(1), 155-157 (English version pp 78-79).

[9] Mikhlin, S.G. (1970) *Mathematical Physics, An advance course* North-Holland Publishing Company, Amsterdam, London.

[10] Neitaanmäki, P. and Saranen, J. (1981) *On finite element approximation of the gradient for solution of Poisson equation* Numer, Math. 37, 333-337.

[11] Owens, R.G. and Phillips, T.N. (1994) *Mass and momentum conserving spectral methods for Stokes flow* Journal of Computational and Applied Mathematics, 53, 185-206.

[12] Petrovsii, I.G. (1946) *On Some problems of the theory of partial differential equations* Usp. Matem. Nauk. 1 (3-4), 44-70 (in Russian), Amer. Math. Soc. Translation No. 12, 373-414.

[13] Tahir, M. and Davies, A.R. (2000) *Stokes-Bitsadze problem for the velocity boundary conditions* JOPAS, 19(2), 95-103.

[14] Tahir, M. (1999a) *Boundary value problems for the Cauchy- Riemann System* PUJM, XXXII, 79-88.

[15] Tahir, M. (199b) *The Stokes-Bitsadze System* PUJM, XXXII, 173-180.

[16] Tahir, M. (2001) *Stokes-Bitsadze problem for the periodic boundary conditions* To appear in JOPAS.

[17] Thatcher, R.W. (1997) *A least squares method for Stokes flow based on stress and stream functions* Report No. 97-8, Department of Mathematics, UMIST.

[18] Treneva, T.V. (1985) *Multidimensional analogs of a Bitsadze system* Differential Equations, 21, No. 1, 114-117.

[19] Vanmaele, M., Morton, K.W., Suli, E. and Borzi, A. (1994) *Analysis of the cell vertex finite volume method for the Cauchy- Riemann equations* Report No. 94/5, Oxford University Computing Laboratory, Oxford.

[20] Wendland, W.L. (1979) *Elliptic systems in the plane* Pitman London.

[21] Yanushauskas, A. (1995) *A class of systems homotopic to a multidimensional analog of A.V. Bitsadze system* Differential Equations, 31, No. 1, 131-136.

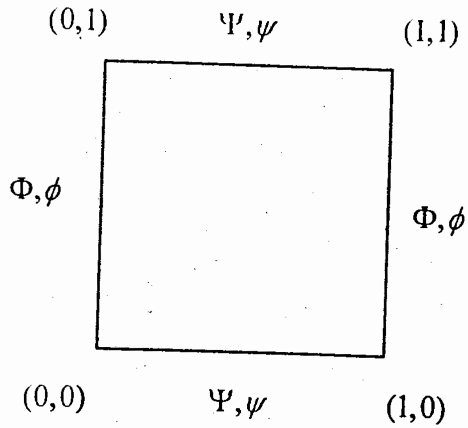

Figure 1

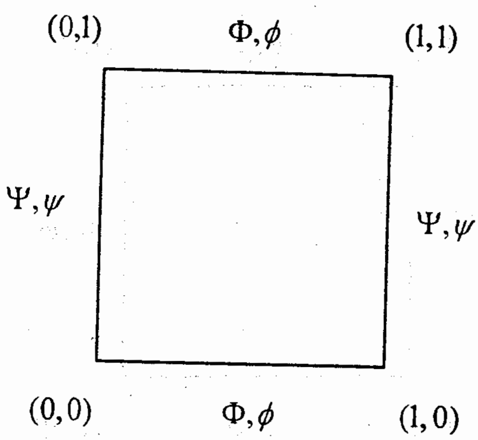

Figure 2

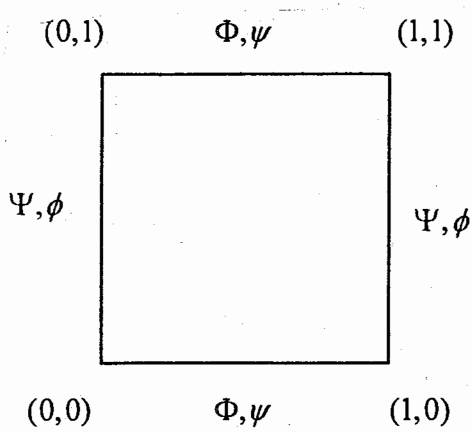

Figure 3

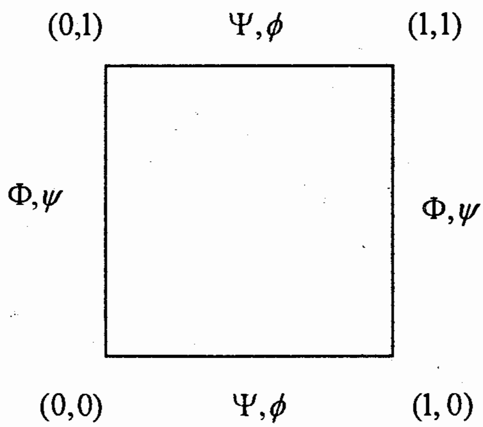

Figure 4

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BITRANSFORMATION SEMIGROUPS AND a-MINIMAL SETS

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ABSTRACT

In the development of a-minimal sets discussion ([2], [3] and [4]) we want to have a short view on bitransformation semigroups.

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Key Words: a-minimal set, Bitransformation semigroup, Transformation Semigroup.

PRELIMINARIES

By a right transformation semigroup (X, S, ρ) (or simply (X, S)) we mean a compact Hausdorff topological space X, a discrete topological semigroup S with identity e_S and a continuous map $\rho : X \times S \to X(\pi(x, s) = xs(\forall x \in X, \forall s \in S))$ such that:

- $\forall x \in X \quad xe_S = x$,
- $\forall x \in X \quad \forall s, t \in S \quad x(st) = (xs)t$,

By a left transformation group (G, X, λ) (or simply (G, X)) we mean a compact Hausdorff topological space X, a discrete topological group G with identity e_G and a continuous map $\lambda : G \times X \to X((g, x) = gx(\forall g \in G, \ \forall x \in X))$ such that:

- $\forall x \in X \quad e_G x = x$,
- $\forall x \in X \quad \forall g, h \in G \quad (gh)x = g(hx)$,

In a right transformation semigroup (X, S) we have the following definitions:

1. For each $s \in S$, define the continuous map $\pi^s : X \to X$ by $x\pi^s = xs \quad (\forall x \in X)$, then $E(X, S)$ is the closure of $\{\pi^s | s \in S\}$ in X^X with point-wise convergence, moreover it is called the enveloping semigroup (or Ellis semigroup) of (X, S). $E(X, S)$ has a semigroup structure [1, Chapter 3], a nonempty subset K of $E(X, S)$ is called a right ideal if $KE(X, S) \subseteq K$, and it is called a minimal right ideal if none of the right ideals of $E(X, S)$ be a proper subset of K.

2. A nonempty subset Z of X is called invariant if $ZS \subseteq Z$, moreover it is called minimal if it is closed and none of the closed invariant subsets of X be a proper subset of Z. The element $a \in X$ is called almost periodic if $aE(X, S)$ be a minimal subset of X.

3. Let $a \in X$, A be a nonempty subset of X and C be a nonempty subset of $E(X, S)$, then we introduce the following sets:

$$\begin{aligned} F(a, C) &= \{p \in C | ap = p\}, \\ F(A, C) &= \{p \in C | \forall b \in A \quad bp = p\}, \\ \bar{F}(A, C) &= \{p \in C | Ap = p\}, \\ J(C) &= \{p \in C | p^2 = p\}. \end{aligned}$$

4. Let (Y, S) be a transformation semigroup, the continuous map

$\psi : (X, S) \to (Y, S)$ is a homomorphism if $\psi(xs) = \psi(x)s \; (\forall x \in X, \; \forall s \in S)$. If $\psi : (X, S) \to (Y, S)$ is an onto homomorphism, then there exists an onto homomorphism $\hat{\psi} : (E(X, S), S) \to (E(Y, S), S)$ (which is also a semigroup homomorphism) such that for each $x \in X$ and $p \in E(X, S)$ we have $\psi(xp) = \psi(x)\hat{\psi}(p)$ [1, Proposition 3.8].

5. Let $a \in X$, A be a nonempty subset of X and K be a closed right ideal of $E(X, S)$, then [2, Definition1]:

- $K \in M_{(X,S)}(a)$ if :

  $-aK = aE(X,S)$,

  $-K$ does not have any proper subset like L, such that L be a closed right ideal of $E(X,S)$ such that $aL = aE(X,S)$,

- $K \in \bar{M}_{(X,S)}(A)$ if:

  $-\forall b \in A \quad bK = bE(X,S)$,

  $-K$ does not have any proper subset like L, such that L be a closed right ideal of $E(X,S)$ such that $bL = bE(X,S)$ for all $b \in A$,

- $K \in \bar{\bar{M}}_{(X,S)}(A)$ if:

  $-AK = AE(X,S)$,

  $-K$ does not have any proper subset like L, such that L be a closed right ideal of $E(X,S)$ such that $AL = AE(X,S)$,

$\bar{M}_{(X,S)}(A)$ and $M_{(X,S)}(a)$ are nonempty.

6. Define $\bar{\mathcal{M}}(X,S) = \{\emptyset \neq A \subseteq X | \forall K \in \bar{M}_{(X,S)}(A) \quad J(F(A,K)) \neq \emptyset\}$ and $\bar{\bar{\mathcal{M}}}(X,S) = \{\emptyset \neq A \subseteq X | \forall K \in \bar{\bar{M}}_{(X,S)}(A) \quad J(\bar{F}(A,K)) \neq \emptyset\}$.

7. Let $a \in X$ and A be a nonempty subset of X, then [2, Definition 13]:
   - (X,S) is called a - distal if $E(X,S) \in M_{(X,S)}(a)$,
   - (X,S) is called A-distal if (X,S) be b-distal for each $b \in A$,
   - (X,S) is called $\underline{A^{(\bar{M})}}$ distal if $E(X,S) \in \bar{M}_{(X,S)}(A)$,
   - (X,S) is called $\underline{A^{(\bar{\bar{M}})}}$ distal if $E(X,S) \in \bar{\bar{M}}_{(X,S)}(A)$.

8. Let A, B be nonempty subsets of X and $R, Q \in \{\bar{M}_{(X,S)}, \bar{\bar{M}}_{(X,S)}\}$, then [2, Definition 13]:

   - B is called A-almost periodic if:

$$\forall a \in A \quad \forall K \in M_{(X,S)}(a) \quad \forall b \in B \quad \exists L \in M_{(X,S)}(b) \quad L \subseteq K,$$

   - B is called $\underline{A^{(R,-)}}$ almost periodic if:

$$\forall a \in A \ \ \forall K \in M_{(X,S)}(a) \ \ \exists L \in R(B) \quad L \subseteq K,$$

- B is called $A^{\underline{(-,Q)}}$ almost periodic if $Q(A) \neq \emptyset$ and:

$$\forall K \in Q(A) \quad \forall b \in B \quad \exists L \in M_{(X,S)}(b) \quad L \subseteq K$$

- B is called $A^{\underline{(R,Q)}}$ almost periodic if $Q(A) \neq \emptyset$ and:

$$\forall K \in Q(A) \quad \exists L \in R(B) \quad L \subseteq K.$$

9. Let A be a nonempty set of X, we introduce the following sets:

$$P(X,S) = \{(x,y) \in X \times X | \exists p \in E(X,S) \quad xp = yp\},$$

$$P_A(X,S) = \{(x,y) \in X \times X | \exists a \in A \ \exists I \in M_{(X,S)}(a) \quad \forall p \in I \quad xp = yp\},$$

$$\bar{P}_A(X,S) = \{(x,y) \in X \times X | \exists I \in \bar{M}_{(X,S)}(A) \quad \forall p \in I \quad xp = yp\},$$

$$\bar{\bar{P}}_A(X,S) = \{(x,y) \in X \times X | \exists I \in \bar{\bar{M}}_{(X,S)}(A) \quad \forall p \in I \quad xp = yp\}.$$

10. Let A be a nonempty subset of X and Z be a closed invariant subset of X, we introduce the following sets:

$$h_{(X,S)}(Z) = \{n \in N \cup \{0\} | \exists Z_0, \cdots, Z_n \quad \ni$$

$$((Z_0 \subseteq Z_1 \subseteq \cdots \subset Z_n) \wedge (\forall i \in \{0, \cdots, n\} \quad \forall j \in \{0, \cdots, n\} - \{i\} \quad Z_i \neq Z_j)$$

$$\wedge(\forall i \in \{0, \cdots, n\}, \quad Z_i \text{ is a closed invariant subset of } Z))\}$$

For each $a \in A$ define:

- $\mathrm{Dim}^T(a) = \sup\{h_{(E(X,S),S)}(I) | I \in M_{(X,S)}(a)\}$,
- $\mathrm{Dim}^T(A) = \sup\{\mathrm{Dim}^T(x) | x \in A\}$,
- $\mathrm{Dim}^{T(\bar{M})}(A) = \sup\{h_{(E(X,S),S)}(I) | I \in \bar{M}_{(X,S)}(A)\}$,
- $\mathrm{Dim}^{T(\bar{\bar{M}})}(A) = \sup\{h_{(E(X,S),S)}(I) | I \in \bar{\bar{M}}_{(X,S)}(A)\}$.

By a bitransformation semigroup (G, X, S) we mean (G, X) is a left transformation group and (X, S) is a right transformation semigroup such that:

$$g(xs) = (gx)s \quad (\forall g \in G, \ \ \forall x \in X, \ \ \forall s \in S).$$

CONVENTION 1

Let (G, X, S) be a bitransformation semigroup such that $\frac{X}{\Re}$ where $\Re = \{(x, y) \in X \times X | \exists g \in G \quad gx = y\}$, is Hausdorff, then $\frac{X}{\Re}$ is denoted by $\frac{X}{G}$ (the natural quotient map is denoted by $\pi_G : X \to \frac{X}{G}(\pi_G(x) = [x]_G \quad \forall x \in X))$. Moreover let e_S be the identity of S and e_G be the identity of G. Let $\Re' = \{((x, x'), (y, y')) \in (X \times X) \times (X \times X) | ([x]_G, [x']_G) = ([y]_G, [y']_G\}$, then $\frac{X \times X}{\Re'}$ will be denoted by $\frac{X \times X}{G \times G}$.

Lemma 2

Let H, H' be nonempty subsets of $G, g, g' \in G, A, A'$ be nonempty subsets of X, and $x, x' \in X$, then we have:

(1) (a) $F(HA, E(X, S)) = F(A, E(X, S))$,

(b) $\bar{F}(gA, E(X, S)) = \bar{F}(A, E(X, S))$,

(c) $\bar{M}_{(X,S)}(HA) = \bar{M}_{(X,S)}(A)$,

(d) $\bar{\bar{M}}_{(X,S)}(gA) = \bar{M}_{(X,S)}(A)$.

(2) (a) $(x, x') \in P(X, S) \Leftrightarrow (gx, gx') \in P(X, S)$,

(b) $(x, x') \in P_A(X, S) \Leftrightarrow (gx, gx') \in P_{HA}(X, S)$,

(c) $(x, x') \in \bar{P}_A(X, S) \Leftrightarrow (gx, gx') \in \bar{P}_{HA}(X, S)$,

(d) $(x, x') \in \bar{\bar{P}}_A(X, S) \Leftrightarrow (gx, gx') \in \bar{\bar{P}}_{g'A}(X, S)$.

(3) In the right transformation semigroup (X, S) we have:

(a) A is A' - almost periodic if and only if HA is $H'A'$ - almost periodic,

(b) A is $A' \ \underline{(-, \bar{M})}$ almost if and only if HA is $H'A' \ \underline{(-, \bar{M})}$ almost periodic,

(c) A is $A' \ \underline{(-, \bar{\bar{M}})}$ almost periodic if and only if HA is $g'A' \ \underline{(-, \bar{\bar{M}})}$ almost periodic,

(d) A is $A' \xrightarrow{(\dot{\bar{M}},-)}$ almost periodic if and only if gA is $H'A' \xrightarrow{(\dot{\bar{M}},-)}$ almost periodic,

(e) A is $A' \xrightarrow{(\dot{\bar{M}},\bar{M})}$ almost periodic if and only if gA is $H'A' \xrightarrow{(\dot{\bar{M}},\bar{M})}$ almost periodic,

(f) A is $A' \xrightarrow{(\dot{\bar{M}},\dot{\bar{M}})}$ almost periodic if and only if gA is $g'A' \xrightarrow{(\dot{\bar{M}},\dot{\bar{M}})}$ almost periodic.

(4) (a) (X,S) is A - distal if and only if (X,S) is HA-distal,

(b) (X,S) is $A \xrightarrow{(\bar{M})}$ distal if and only if (X,S) is $HA \xrightarrow{(\bar{M})}$ distal,

(c)(X,S) is $A \xrightarrow{(\dot{\bar{M}})}$ distal if and only if (X,S) is $gA \xrightarrow{(\dot{\bar{M}})}$ distal.

(5) $\quad \mathrm{Dim}^{T(\bar{M})}_{(X,S)}(A) = \mathrm{Dim}^{T(\bar{M})}_{(X,S)}(HA)$ and $\mathrm{Dim}^{T(\dot{\bar{M}})}_{(X,S)}(A) = \mathrm{Dim}^{T(\dot{\bar{M}}}_{(X,S)}(HA)$.

Proof:

In order to prove (1) let K be a closed right ideal of $E(X,S)$ and use the following facts:

- $\forall p \in E(X,S) \quad \forall a \in A \quad \forall h \in H \quad (ap = a \Leftrightarrow (ha)p = ha)$,
- $\forall p \in E(X,S) \quad (AP = A \Leftrightarrow (gA)p = gA$,
- $(\forall h \in H \quad \forall a \in A \quad haK = haE(X,S)) \Leftrightarrow (\forall a \in A) \quad aK = aE(X,S))$,
- $gAK = gAE(X,S) \Leftrightarrow AK = AE(X,S)$.

The other items are proved by (1).

Theorem 3:

Let $\hat{\pi}_G : (E(X,S),S) \to \left(E\left(\frac{X}{G},S\right),S\right)$ be the homomorphism induced by $\pi_G : (X,S) \to \left(\frac{X}{G},S\right)$, such that for each $x \in X$ and $p \in E(X,S)$, $[xp]_G = [x]_G\hat{\pi}_G(p)$. Let also A and B be nonempty subsets of X, then:

(1)(a) $\quad \bar{M}_{(\frac{X}{G},S)}([A]_G) \subseteq \hat{\pi}_G(\bar{M}_{(X,S)}(A))$,

(b) $A \in \bar{\mathcal{M}}(X,S)$ if and only if $[A]_G \in \bar{M}\left(\frac{X}{G},S\right)$,

(c) if $A \in \bar{\mathcal{M}}(X,S)$, then $\bar{M}_{\left(\frac{X}{G},S\right)}([A]_G) = \hat{\pi}_G(\bar{M}_{(X,S)}(A))$.

(2)(a) $\left(\frac{X}{G},S\right)$ is distal if and only if (X,S) is distal,

(b) $\left(\frac{X}{G},S\right)$ is $[A]_G$ -distal if and only if (X,S) is A - distal,

(c) $\left(\frac{X}{G},S\right)$ is $[A]_G \underline{(\bar{M})}$ distal if and only if (X,S) is $A \underline{(\bar{M})}$ distal.

(d) if $\left(\frac{X}{G},S\right)$ is $[A]_G \underline{(\bar{M})}$ distal then (X,S) is $A \underline{(\bar{M})}$ distal.

(3)(a) B is A-almost periodic if and only if $[B]_G$ is $[A]_G$ -almost periodic,

(b) B is $A \underline{(-,\bar{M})}$ almost periodic if and only if $[B]_G$ is $[A]_G \underline{(-,\bar{M})}$ almost periodic.

Proof:

(1)

(a) Let $K \in \bar{M}_{\left(\frac{X}{G},S\right)}([A]_G)$, so for each $a \in A$, there exists $u \in J(F([a]_G,K))$, and $\{p \in E(X,S) | \hat{\pi}_G(p) = u\}$ is nonempty, closed and has a semigroup structure, thus there exists $v \in J(E(X,S))$ such that $\hat{\pi}_G(v) = u$, moreover $[a]_G = [a]_G u$ and $[a]_G \hat{\pi}_G(v) = [av]_G$, therefore there exists $g \in G$ such that $ga = av$, since $ga = av = av^2 = (av)v = (ga)v = g(av) = g(ga) = g^2 a$, so $a = ga = av$, thus $aE(X,S) = avE(X,S)$, but $v \in \pi_G^{-1}(K)$ and $vE(X,S) \subseteq \pi_G^{-1}(K)$, so $aE(X,S) = a\hat{\pi}_G^{-1}(K)(\forall a \in A)$, thus there exists $L \in \bar{M}_{(X,S)}(A)$ such that $L \subseteq \pi_G^{-1}(K)$ [2, Corollary 3], as for each $a \in A$, $aL = aE(X,S)$ and $[a]_G \hat{\pi}_G(L) = [a]_G E\left(\frac{X}{G},S\right)$, by $\hat{\pi}_G(L) \subseteq \hat{\pi}_G(\pi_G^{-1}(K)) = K$ and $K \in \bar{M}_{\left(\frac{X}{G},S\right)}([A]_G)$, we have $K = \hat{\pi}_G(L) \in \hat{\pi}_G(\bar{M}_{(X,S)}(A))$.

(b) Let $A \in \bar{\mathcal{M}}(X,S)$ and let $K \in \bar{M}_{\left(\frac{X}{G},S\right)}([A]_G)$, by (a), there exists $K \in \bar{M}_{(X,S)}(A)$ such that $K = \hat{\pi}_G(L)$, let $u \in J(F(A,L))$ so $\hat{\pi}_G(u) \in J(F([A]_G,K))$ and $[A]_G \in \bar{\mathcal{M}}\left(\frac{X}{G},S\right)$. On the other hand if $[A]_G \in \bar{\mathcal{M}}\left(\frac{X}{G},S\right)$, choose $L \in \bar{\mathcal{M}}_{(X,S)}(A)$ such that $\hat{\pi}_G(L) \in \bar{M}_{\left(\frac{X}{G},S\right)}([A]_G)$ (by (a) there exists such an L). Let $u \in J(F([A]_G, \hat{\pi}_G(L))$, $\{p \in L | \hat{\pi}_G(p) = u\}$ is a closed nonempty subset of L and has a semigroup structure, so there exists $v \in J(L)$ such that $\hat{\pi}_G(v) = u$

and for each $a \in A$, $[a]_G = [a]_G u = [a]_G \hat{\pi}_G(v) = [av]_G$ so there exists $g \in G$ such that $ga = av$ so $v \in J(F(A, L))$ and $J(F(A, L)) \neq \emptyset$, therefore $A \in \bar{\mathcal{M}}(X, S)$ [2, Corollary 9].

(c) Let $A \in \bar{\mathcal{M}}(X, S)$ and let $L \in \bar{M}_{(X,S)}(A)$, thus for each $a \in A$ we have $[a]_G \hat{\pi}_G(L) = [aL]_G = [aE(X, S)]_G = [a]_G E\left(\frac{X}{G}, S\right)$, so there exists $K \in \bar{M}_{(\frac{X}{G},S)}([A])_G)$ such that $K \subseteq \hat{\pi}_G(L)$ [2, Corollary 3], by (b) there exists $u \in J(F([A]_G, K)$. Using a similar argument as in (b) there exists $v \in J(F(A, L))$ such that $\hat{\pi}_G(v) = u$, moreover $K = uE\left(\frac{X}{G}, S\right) = \hat{\pi}_G(v)E\left(\frac{X}{G}, S\right) = \hat{\pi}_G(vE(X, S)) = \hat{\pi}_G(L)$, therefore $\hat{\pi}_G(L) \in \bar{M}_{(\frac{X}{G},S)}([A])_G)$. By (a) we have $\hat{\pi}_G(\bar{\bar{M}}_{(X,S)}(A)) \subseteq \bar{M}_{(\frac{X}{G},S)}([A]_G)$.

(2)

(c) Let (X, S) be $A \ \underline{\bar{M}}$ distal, then $\bar{M}_{(X,S)}(A) = \{E(X, S)\}$, by (1) we have $E\left(\frac{X}{G}, S\right) = \hat{\pi}_G(E(X, S)) \in \bar{M}_{(\frac{X}{G},S)}([A]_G)$ and $\left(\frac{X}{G}, S\right)$ is $[A]_G \ \underline{(\bar{M})}$ distal. On the other hand if $\left(\frac{X}{G}, S\right)$ is $[A]_G \ \underline{(\bar{M})}$ distal, then $E\left(\frac{X}{G}, S\right)$ belongs to $\bar{M}_{(\frac{X}{G},S)}([A]_G)$ and by (1) there exists $L \in \bar{M}_{(X,S)}(A)$ such that $\hat{\pi}_G(L) = E\left(\frac{X}{G}, S\right)$, using a similar argument as in (1), there exists $v \in J(L)$ such that $\hat{\pi}_G(v) = e_S$, therefore for each $x \in X$ we have $[x]_G = [x]_G e_S = [x]_G \hat{\pi}_G(v) = [xv]_G$, so there exists $g \in G$ such that $gx = xv$ now as it was verified in (1) $x = xv$ $(\forall x \in X)$, and $e = v \in L$ thus $L = E(X, S)$ and (X, S) is $A \ \underline{(\bar{M})}$ distal.

(d) Let $\left(\frac{X}{G}, S\right)$ is $[A]_G \ \underline{(\bar{\bar{M}})}$ distal and let L be a closed right ideal of $E(X, S)$ such that $AL = AE(X, S)$, thus $[A]_G = \hat{\pi}_G(L) = [A]_G E\left(\frac{X}{G}, S\right)$, since $E\left(\frac{X}{G}, S\right) \in \bar{M}_{(\frac{X}{G},S)}([A]_G)$ so $\hat{\pi}_G(L) = E\left(\frac{X}{G}, S\right)$, using a similar method described in (c) we have $L = E(X, S))$, therefore $E(X, S) \in \bar{\bar{M}}_{(X,S)}(A)$ and (X, S) is $A \ \underline{(\bar{\bar{M}})}$ distal.

(3)

(b) Let B be $A \xrightarrow{(-,\bar{M})}$ almost periodic, $b \in B$ and $K \in \bar{M}_{(\frac{X}{G},S)}([A]_G)$, by (1) there exists $L \in \bar{M}_{(X,S)}(A)$ such that $\hat{\pi}_G(L) = K$, choose $N \in M_{(X,S)}(b)$ such that $N \subseteq L$, using (1) (part (c) (note that $\{b\} \in \bar{\mathcal{M}}(X,S)$)) we have $\hat{\pi}_G(N) \in M_{(\frac{X}{G},S)}([b]_G)$ moreover $\hat{\pi}_G(N) \subseteq \hat{\pi}_G(L) = K$, thus $[B]_G$ is $[A]_G \xrightarrow{(-,\bar{M})}$ almost periodic. Conversely let $[B]_G$ be $[A]_G \underline{(-,\bar{M})}$ almost periodic, $b \in B$ and $L \in \bar{M}_{(X,S)}(A)$, then for each $a \in A$ we have $[a]_G\pi_G(L) = [aL]_G = [aE(X,S)]_G = [a]_G E\left(\frac{X}{G},S\right)$ thus there exists $K \in \bar{M}_{(\frac{X}{G},S)}([A]_G)$ such that $K \subseteq \hat{\pi}_G(L)$ [2, Corollary 3], since $[B]_G$ is $[A]_G \xrightarrow{(-,\bar{M})}$ almost periodic, $[b]_G K = [b]_G E\left(\frac{X}{G},S\right)$ [2, Lemma 15], so $[bL]_G = [b]_G\hat{\pi}_G(L) = [b]_G E\left(\frac{X}{G},S\right) = [bE(X,S)]_G$ thus $\{p \in L | [bp]_G = [b]_G\}$ is a nonempty closed subsemigroup of L, thus there exists $v \in J(L)$ such that $[bv]_G = [b]_G$. By a similar method described in (1) we have $bv = b$, so $bE(X,S) = bvE(X,S)$. Since $vE(X,S) \subseteq L$, we have $bE(X,S) = bL \quad (\forall b \in B, \ \forall L \in \bar{M}_{(X,S)}(A))$. Therefore B is $A \xrightarrow{(-,\bar{M})}$ almost periodic [2, Lemma 15].

Theorem 4:

If A be a nonempty subset of X and $x, y \in X$, then:

1. The following statements are equivalent:
   (a) $([x]_G, [y]_G) \in P\left(\frac{X}{G},S\right)$,
   (b) $\exists g \in G \quad (gx, y) \in P(X,S)$,
   (c) $[x,y]_{G\times G} \in [P(X,S)]_{G\times G}$.

2. The following statements are equivalnet:
   (a) $([x]_G, [y]_G) \in P_{[A]_G}\left(\frac{X}{G},S\right)$,
   (b) $\exists g \in G \quad (gx, y) \in P_A(X,S)$,
   (c) $[x,y]_{G\times G} \in [P_A(X,S)]_{G\times G}$.

3. If $A \in \bar{\mathcal{M}}(X,S)$, then the following statements are equivalent:
   (a) $([x]_G, [y]_G) \in \bar{P}_{[A]_G}\left(\frac{X}{G},S\right)$,

(b) $\exists g \in G \quad (gx, y) \in \bar{P}_A(X, S)$,

(c) $[x, y]_{G\times G} \in [\bar{P}_A(X, S)]_{G\times G}$.

Proof:(3)

(a) $\Rightarrow$ (b):

If $([x]_G, [y]_G) \in \bar{P}_{[A]_G}\left(\frac{X}{G}, S\right)$, then there exists $K \in \bar{M}_{\left(\frac{X}{G}, S\right)}([A]_G)$ such that for each $p \in K$, $[x]_G p = [y]_G p$. By Theorem 3 (1), there exists $L \in \bar{M}_{(X,S)}(A)$ such that $\hat{\pi}_G(L) = K$. For $p \in L$, $[xp]_G = [x]_G \hat{\pi}_G(p) = [y]\hat{\pi}_G(p) = [yp]_G$, so there exists $g \in G$ such that $gxu = yu$, thus for each $p \in uE(X, S)(= L)$ we have $gxp = yp$ and $(gx, y) \in \bar{P}_A(X, S)$.

(c) $\Rightarrow$ (a):

If $[x, y]_{G\times G} \in [\bar{P}_A(X, S)]_{G\times G}$, then there exist $g, g' \in G$ such that $(gx, g'y) \in \bar{P}_A(X, S)$, let $L \in \bar{M}_{(X,S)}(A)$ be such that $gxp = g'yp$ $(\forall p \in L)$ thus $[x]_G p = [y]_G p$ $(\forall p \in \hat{\pi}_G(L))$, but there exists $K \in \bar{M}_{\left(\frac{X}{S}, S\right)}([A]_G)$ such that $K \subseteq \hat{\pi}_G(L)$, so for each $p \in K$ we have $[x]_G p = [y]_G p$. Thus $([x]_G, [y]_G) \in \bar{P}_{[A]_G}\left(\frac{X}{G}, S\right)$.

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REFERENCES

[1] R. Ellis *Lecturers on Topological Dynamics* W.A. Benjamin, New York, 1969.

[2] M. Sabbaghan and F. Ayatollah Zadeh Shirazi *a-minimal sets and related topics in transformation semigroups (I)* International Journal of Mathematics and Mathematical Sciences, to appear.

[3] M. Sabbaghan and F. Ayatollah Zahed Shirazi *a - minimal sets and related topics in transformation semigroups (II)* International Journal of Mathematics and Mathematical Sciences, to appear.

[4] M. Sabbaghan, F. Ayatollah Zadeh Shirazi and T. S. Wu *a- minimal sets* Journal of Science of the University of Tehran, 2 (1997), 1-12.

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ON CHARACTERIZATION OF THE CHEVALLEY GROUP BY THE SMALLER CENTRALISER OF A CENTRAL INVOLUTION II

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ABSTRACT

In this paper we obtain further results on the structure of a finite group G having a subgroup isomorphic to the smaller centraliser of an involution in $F_4(2)$.

1. INTRODUCTION

Let $F_4(2)$ denote the Chevalley group of type F_4 over the field $T = \{0, 1\}$.

The center of a Sylow$_2$-subgroup S of F_4 is a four group. The elements of order two in this subgroup of S lie in three distinct conjugacy classes in $F_4(2)$. Let t_1, t_2 and $t_3 = t_1t_2$ be these involutions in the center of S. Now in $F_4(2)$:

$$\begin{aligned} C(t_1) &\cong C(t_2) \qquad \text{and} \\ C(t_1) \cap C(t_2) &= C(t_3) \end{aligned}$$

The root system $\sum$ of type (F_4) consists of 48 roots: $\pm\xi_i \pm \xi_i \pm \xi_j, \frac{1}{2}(\pm\xi_i \pm \xi_j \pm \xi_m \pm \xi_n)$, where $i, j, m, n = 1, 2, 3, 4$ and i, j, m, n are all distinct.

For each $i, 1 \leq i \leq 24$ and each $s \in \sum$ let $\tilde{w}_i(s) = s - s(r_i)r_i$. Then $\tilde{w}_i$ is a permutation of $\sum$. The pemutation group $\tilde{W}$ generated by $\{\tilde{w}_i | 1 \leq i \leq 24\}$ is the Weyl group of $\sum$.

$\tilde{W}$ is of order $2^7 3^2$ and is generated by $\tilde{w}_1, \tilde{w}_2, \tilde{w}_5$ and $\tilde{w_{10}}$. If $a_{ij} = |\tilde{w}_i \tilde{w}_j|$, then generators $\tilde{w}_1, \tilde{w}_2, \tilde{w}_5$ and $\tilde{w_{10}}$ together with the relations $(\tilde{w}_i \tilde{w}_j)^{a_{ij}} = 1$, $\{i, j\} \subseteq \{1, 2, 5, 10\}$, form a presentation of $\tilde{W}$.

It will be convenient to think of the elements $\tilde{w} \in \tilde{W}$ as permutations of $\{\pm i | 1 \leq i \leq 24\}$ defined as follows:

$$\tilde{w}(i) = \begin{cases} j \text{ if } \tilde{w}(r_i) & = r_j; \quad \tilde{w}(-i) = -\tilde{w}(i) \\ -j \text{ if } \tilde{w}(r_i) & = -r_j \end{cases}$$

The values $\tilde{w}_i(j)$ for $i = 1, 2, 5, 10$ and $1 \leq i \leq 24$ are also included in Table-1.

For each $i, 1 \leq i \leq 24$, let $S_i = \{x_i(\alpha) \mid \alpha \in \Gamma\}$. Then each S_i is a group of order 2. The elements of S_i multiply according to the rule $x_i(\alpha)x_i(\beta) = x_i(\alpha + \beta), \alpha, \beta \in \Gamma$.

Let $S = < S_i | 1 \leq i \leq 24 >$. Then S is a Sylow$_2$-subgroup of F.

An element $x \in S$ can be expressed uniquely in the form $x = \Pi_{i=1}^{24} x_i(\alpha_i)$

Which we shall abbreviate as $x = \Pi x_i(\alpha_i)$. Hence S has order 2^{24}. The product of any two elements of S may be obtained by use of the commutators $[x_i(1), x_j(1)]$, $1 \leq i \leq 24$. The nontrivial commutators are listed in Table-2.

Let $W_1 = < w_1, w_2, w_5 >$ and $C_1 = \{sws' | s \in S, s' \in Sw, w \in W_1\}$

Then $C_1 = C_F(x_{21}(1))$ $\quad (ii) \quad$ $Z(C_1) = S_{21}$

[4]. Let $W_2 = < w_1, w_2, w_{10} >$ and $C_2 = \{sws' | s \in S, w \in W_2, s' \in S_w\}$.

Then $C_2 = C_F(x_{24}(1))$; $\quad Z(C_2) = S_{24}$

Let $W_3 = < w_1, w_2 >$ and $C_3 = \{sws' | s \in S, w \in W_3, s' \in S_w\}$

Then $C_3 = C_F(x_{21}(1)x_{24}(1))$

From now onwards, since the only involution in any root subgroup S_i of $F_4(2)$ is $x_i(1)$, we will write x_i for $x_i(1)$ except where there is ambiguity.

Table - 1

i	$\tilde{w}_1(i)$	$\tilde{w}_2$ (i)	$\tilde{w}_5(i)$	$\tilde{w}_{10}(i)$	$r_i(r_1)$	$r_i(r_2)$	$r_i(r_5)$	$r_i(r_{10})$	$\lambda(r_i)$	$\bar{i}$
1	-1	3	1	11	2	- 1	0	-1	1	2
2	4	- 2	6	2	- 2	2	- 1	0	2	1
3	3	1	7	12	0	1	- 1	- 1	1	4
4	2	4	8	18	2	0	- 1	- 2	2	3
5	5	6	- 5	5	0	- 1	2	0	2	10
6	8	5	2	6	- 2	1	1	0	2	11
7	7	7	3	14	0	0	1	- 1	1	18
8	6	9	4	19	2	- 1	1	- 2	2	12
9	9	8	9	20	0	1	0	- 2	2	13
10	11	10	10	- 10	- 1	0	0	2	1	5
11	10	12	11	1	1	- 1	0	1	1	6
12	13	11	14	3	- 1	1	- 1	1	1	8
13	12	13	15	13	1	0	- 1	0	1	9
14	15	14	12	7	- 1	0	1	1	1	19
15	14	16	13	15	1	- 1	1	0	1	20
16	17	15	16	16	- 1	1	0	0	1	22
17	16	17	17	21	1	0	0	- 1	1	23
18	18	18	19	4	0	0	- 1	2	2	7
19	19	20	18	8	0	- 1	1	2	2	14
20	22	19	20	9	- 2	1	0	2	2	15
21	21	21	21	17	0	0	0	1	1	24
22	20	23	22	22	2	- 1	0	0	2	16
23	23	22	24	23	0	1	- 1	0	2	17
24	24	24	23	24	0	0	1	0	2	21

Table-2

Values $(i, j : m)$ for which $[x_i(1), x_j(1)] = x_m(1)$			
(1, 10 : 11)	(1, 12 : 13)	(1, 14 : 15)	(1, 16 : 17)
(2, 5 : 6)	(2, 8 : 9)	(2, 19 : 20)	(2, 22 : 23)
(3, 10 : 12)	(3, 11 : 13)	(3, 14 : 16)	(3, 15 : 17)
(4, 5 : 8)	(4, 6 : 9)	(4, 19 : 22)	4, 20 : 23)
(5, 18 : 19)	(5, 23 : 24)	(6, 18 : 20)	(6, 22 : 24)
(7, 10 : 14)	(7, 11 : 15)	(7, 12 : 16)	(7, 13 : 17)
(8, 18 : 22)	(8, 20 : 24)	9, 18 : 23)	(9, 19 : 24)
(10, 17 : 21)	(11, 16 : 21)	(12, 15 : 21)	(13, 14 : 21)
Values $(i, j : m, n)$ for which $[x_i(1), x_j(1)] = x_m(1)x_n(1)$			
(1, 2 : 3, 4)	(1, 6 : 7, 8)	(1, 20 : 21, 22)	
(3, 5 : 7, 9)	(3, 19 : 21, 23)	(7, 18 : 21, 24)	
(2, 11 : 12, 18)	(2, 15 : 16, 24)	(4, 10 : 13, 18)	
(4, 14 : 17, 24)	(5, 12 : 14, 20)	(5, 13 : 15, 22)	
(6, 11 : 14, 19)	(6, 13 : 16, 23)	(8, 10 : 15, 19)	
(8, 12 : 17, 23)	(9, 10 : 16, 20)	(9, 11 : 17, 22)	

Then $D_{10} = C(x_{17})$ and $D_5 = C_5(x_{23})$. We write M for D_{10}. Then M and D_5 and subgroups S of order 2^{23} with centers of order 2^3. For our convenience we write D for D_5.

We identify C with C_3. We shall refer tables 3 and 4 of [1].

For necessary details about the group $F_4(2)$, we refer the reader to [9].

It is easily observed that $Z(S) = S_{21}S_{24}$.

The Chevalley group $F_4(2^n)$ of type F_4 over the field of 2^n elements have been characterized by Guterman [3] in terms of the centralisers of 2-central involutions and this characterization is given by the following theorem.

Theorem 1

Let G be a finite group. Suppose the center of a Sylow$_2$-subgroup of G contains elements y_1, y_2 and $y_3 = y_1y_2$ of order two such that $C_G(y_i) \cong C(t_i)$ $i = 1, 2, 3$. Then $G \cong F_4(2^n)$

In [9] Thomas has given an improved characterization of the Chevalley group $F_4(2^n)$ in terms of only the centraliser $C(t_1)$ for all $n > 2$.

Later Husnine [5] treated the case for $n = 1$.

In [1] we proposed the following conjecture to improve the above results.

Conjecture

Let G be a finite simple group with an involution y_3 lying in the center of a Sylow$_2$-subgroup. Suppose $C = C_G(y_3)$ is isomorphic to $C(t_3)$, the centraliser of t_3 in $F_4(2)$. Then G is isomorphic to $F_4(2)$.

However, we proved the following result in [1].

Theorem A

There exists an involution $u \epsilon N_G(D)$ which acts upon $Z_3(D) = S_{16}S_{20}Z_2(D)$ such that $x_{24}^u = x_{23}$ and u centralises $x_{16}, x_{17}, x_{21}, x_{22}$.

In this paper we prove the following results:

Theorem B

There is an involution u in $N_G(D)$ so that

$$\begin{aligned} x_{23}^u &= x_{24}, \quad x_{14}^u = x_{12}, \quad x_{15}^u = x_{13}, \quad x_{18}^u = x_{19}x_{21}(\epsilon) \\ x_8^u &= x_{24} \text{ and } u \text{ centralizes } S_9S_{20}S_{16}S_{17}S_{21} \end{aligned}$$

1.1 Lemma

$$\begin{aligned} Z_4(D) &= S_9S_{13}S_{15}S_{18}S_{19}Z_3(S) \\ Z_5(D) &= S_4S_8S_{12}S_{14}Z_4(S) \end{aligned}$$

Proof

This is verified by direct calculation from table No. 2.

1.2 Lemma

There is an involution u in $N_G(D)$ so that $X_9^u = X_9$, $X_{18}^u = X_{19}X_{21}(\epsilon)$, $X_{15}^u = x_{13}x_{21}(\alpha)x_{16}(\beta)x_{17}(\gamma)$ and u acts upon $Z_3(D)$ as described in Theorem A in [1].

Proof

$Z_4(D) = S_9S_{13}S_{15}S_{18}S_{19}Z_3(D)$ and $S_9S_{19}S_{15}$ is the center of S and $Z_3(D)$, where

$$\bar{S} = S/Z_3(D)$$

Structure of S gives $\bar{S}_{13}^{g_1} = \bar{S}_{15}$, $\bar{S}_{18}^{g_1} = \bar{S}_{19}$
and $\bar{S}_{19}^{g_1} = \bar{S}_9$

We have $x_{19}^{g_1} = x_{18}z$, $z \in Z_3(D) = x_{16}x_{20}x_{22}xx_{21}x_{24}$

We discover the part of Z in $(x_{19})^{g_1}$, by the use of contradiction method.

Let x_{16} appear in $x_{19}^{g_1}[x_{19}^{g_1}, x_{11}] = [x_{16}, x_{21}] = x_{21}$

Then by the commutators relation.

$$\lfloor x_{19}^{g_1}, x_{11} \rfloor = [x_{16}, x_{21}] = x_{21}$$

$$\begin{aligned}
&\Rightarrow \lfloor x_{19}^{g_1}, x_{11} \rfloor = x_{21} \\
&\Rightarrow \lfloor x_{19}, x_{11}^{g_1} \rfloor = x_{21} \\
&\Rightarrow \lfloor x_{19}, x_{11}^{g_1} \rfloor = x_{21} \\
&\Rightarrow (x_{19})^{x_{11}^{g_1}} = x_{19}x_{21}
\end{aligned}$$

This shows that x_{19} is conjugate to $x_{19}x_{21}$ which provides contradiction to the fact in S. Thus x_{16} is not involved in Z.

By the use of similar arguments, it can be shown that x_{20} is not involved in Z.

Thus from structure of S, we have

$$x_{19}^{u_1} \in x_{18}S_{21}S_{23}S_{24}$$

(a) If $x_{19}^{g_1} = x_{18}x_{23}x_{21}x_{24}$ we take $u_1 = g_1^{x_7}$ then $x_{19}^{u_1} = x_{18}$

(b) If $x_{19}^{u_1} = x_{18}x_{23}x_{21}x_{24}$ we write $u_1 = x_5x_7g_1x_5g_1x_7x_5$

(c) If $x_{19}^{g} = x_{18}x_{23}$ we write $u_1 = x_5g_1x_5g_1x_5$ then $x_{19}^{u_1} = x_{18}$

(d) If $x_{19}^{g_1} = x_{18}x_{24}$ we write $u_1 = g_1^{x_7}$ then $x_{19}^{u_1} = x_{18}x_{21}$

(e) If $x_{19}^{g_1} = x_{18}x_{23}x_{24}$ we take $u_1 = x_5x_7g_1x_7x_5$ then $x_{19}^{u_1} = x_{18}x_{21}$

(f) Let $x_{19}^{g_1} = x_{18}x_{21}x_{23}$ writing $u_1 = x_5g_1x_5g_1x_5$ then $x_{19}^{u_1} = x_{18}x_{21}$

Then in above three cases (d), (e), (f)

$$x_{19}^{u_1} = x_{18}x_{21}$$

and u_1 takes x_{24} to x_{23} and centralizes $x_{16}, x_{17}, x_{20}, x_{21}$ and x_{22}.

In order to go further and determine the involvement from Z_3 in the conjugate of x_{15} i.e.

$$x_{15}^{u_1} = x_{13}z, z \in Z_3$$

One finds that x_{21}, x_{22} and x_{23} cannot appear in Z and there exist $v_1 = x_{14}u_1x_{14}$ such that

$x_{18}^{v_1} = x_{19}x_{21}$ and $(x_{15})^{v_1} = x_{13}x_{16}x_{17}x_{24}$ and acts on Z_3 as g_1.

If x_{21} appear in $x_{15}^{u_1}$, we take $v_1 = x_{14}(\delta)u_1x_{14}(\delta)$

now $x_{15}^{v_1} = x_{15}^{x_{14}u_1x_{14}}$

next $x_9^{v_1} = x_9z, \quad z \in Z_3$

And we found that

$$x_9^{v_1} = x_9x_{23}(\eta)x_{24}(\eta)$$

In this case we write

$$u = v_1x_{19}(\eta)x_{18}(\eta)$$

Thus u is the involution in $N_G(D)$ which has all the properties stated in the lemma.

1.3 Lemma

There is an involution u in $N_G(D)$ so that

$$\begin{aligned} x_{23}^u &= x_{24}, \quad x_{14}^u = x_{12}, \quad x_{15}^u = x_{15}, \quad x_{18}^u = x_{19}x_{21}(\in) \\ x_{28}^u &= x_4 \quad \text{and } u \text{ centralizes } \quad S_9S_{20}S_{16}S_{17}S_{21}. \end{aligned}$$

Proof

Let the action of g_1 on Z_4 is same as in 1.2. Since $Z_5(D) = S_4S_8S_{12}S_{14}S_4(D)$ and $Z(S) = S_8S_{14}$ mod Z_4. Structure of S implies that

$$\begin{aligned} \bar{S}_{14}^{g_1} &= \bar{S}_{12} \\ \bar{S}_8^{g_1} &= \bar{S}_4 \end{aligned}$$

Let x_{15} appear in $x_{14}^{g_1}$ then

$$\lfloor x_{14}^{g_1}, x_2 \rfloor = [x_{15}, x_2] = x_{16}x_{24}$$

$$\begin{aligned} &\Rightarrow \lfloor x_{14}^{g_1}, x_2 \rfloor = x_{16}x_{24} \\ &\Rightarrow \lfloor x_2, x_2^{g_1} \rfloor = x_{16}x_{23} \\ &\Rightarrow x_2^{g_1}x_{14}x_2^{g_1} = x_{14}x_{16}x_{23} \end{aligned}$$

This shows that $x_{14} \sim x_{14}x_{16}x_{23}$ but $x_{14}x_{16}x_{23} \sim x_{14}x_{23}$

$\Rightarrow x_{14} \sim x_{14}x_{23}$

but under graph automorphism

put $\phi(x_{14}) = x_{19}$

$\Rightarrow \phi(x_{23}) = x_{17}$

$\Rightarrow$ x_{19} is conjugate to $x_{19}x_{17}$ but this contradicts table 3.

Hence x_{19} is not conjugate $x_{19}x_{17}$

So x_{15} will not appear in $x_{14}^{g_1}$

By the use of similar argument, it can be shown that $x_9, x_{13}, x_{18}, x_{19}, x_{20}, x_{22}, x_{23}$ cannot appear in Z.

Hence from structure of S we found that

$$x_{14}^{g_1} \in x_{12}S_{16}S_{17}S_{21}S_{24}$$

If x_{21} appear in $x_{14}^{g_1}$, we take $g_4^{x_{15}}$ for g_1 and call it u_1. This u_1 satisfies all the properties stated in lemma 1.2.

Hence, we have seen that $x_{14}^{u_1} \in x_{12}S_{16}S_{17}S_{24}$ and u_1 acts on Z_4 as g_1.

next since $x_g^{u_1} \in x_4z, \quad z \in Z_4$

Let x_9 appear in $x_g^{u_1}$ then from table 2.

$$\lfloor x_8^{u_1}, x_{19} \rfloor = [x_9, x_{19}] = x_{24}$$

$$\lfloor x_8^{u_1}, x_{19} \rfloor = x_{24}$$

$$\lfloor x_8, x_{19}^{u_1} \rfloor = x_{23}$$

This implies that $x_8 \sim x_8 x_{23} \sim x_8 x_{17}. \Rightarrow x_8 \sim x_8 x_{17}$. This contradicts table 3.

Hence x_9 cannot appear in $x_8^{u_1}$

Next, to find the involvement from z_4 in the conjugate of x_8 i.e. $x_8^4 = x_4 x_{13}$, where $x_{13} \in z_4$.

One finds that $x_8, x_{13}, x_{15}, x_{16}, x_{18}, x_{22}$ cannot appear in Z.

Finally, we found that there is an involution u_1 which takes x_8 to x_4 and acts on elements of Z_4 as in previous lemma.

u_1 takes x_{15} to $x_{13}x_{16}(\beta)x_{16}(\gamma)(\alpha)$ and u_1 takes x_{14} to $x_{12}x_{16}(\gamma_1)x_{17}(\beta)x_{24}(\alpha)$

Let

$$\lfloor x_{14}^{u_1}, x_1 \rfloor = [x_{12}x_{16}(\gamma_1)x_{17}(\beta)x_{24}(\alpha), x_1]$$

$$= [x_{12}x_{16}(\gamma_1), x_1]^{x_{17}(\beta)x_{24}(\alpha)}$$

Therefore

$$[x_{14}^{u_1}, x_1]^{u_1} = x_{13}^{u_1} x_{17}(\gamma_1)$$

$$\Rightarrow \lfloor x_{14}, x_1^{u_1} \rfloor = x_{15}x_{16}(\beta)x_{17}(\gamma)x_{24}(\alpha)x_{17}(\alpha)$$

$$\Rightarrow \quad \alpha = 0, \qquad \beta = \gamma + \gamma_1$$

Thus

$$x_{14}^{u_1} = x_{12}x_{16}(\gamma_1)x_{17}(\beta)$$

next, Since $x_{19}^{u_1} = x_{18}x_{21}(\epsilon)$ and $x_8^{u_1} \in x_4$

$$\lfloor x_8^{u_1}, x_{10} \rfloor = [x_4, x_{10}] = x_{13}x_{18}$$

$$[x_8^{u_1}, x_{10}] = [x_{13}, x_{18}]^u = x_{15}x_{19}x_{16}(\beta)x_{17}(\gamma)x_{21}(\gamma_1)x_{24}(\gamma_2)$$

$$\Rightarrow x_{10}^u x_8 x_{10}^u = x_8 x_{19} x_{15} x_{16}(\beta)x_{17}(\gamma)x_{21}(\gamma_1)x_{24}(\gamma_2)$$

$$\Rightarrow \beta = \gamma = o, \qquad \text{and} \qquad \gamma_1 = \gamma_2 = o$$

Thus u is the involution satisfying the lemma 1.3 and there by theorem B is established.

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REFERENCES

[1] Rahila and Husnine, S.M. *On characterization of the Chevalley Group by the Smaller Centraliser of a Central Involution, ...1*, Punjab University, Journal of Mathematics, Vol. xxxiv (2001) pp. 17-42.

[2] R.W. Cater *Simple group of Lie type* Wiley, New York, 1972.

[3] D. Gorenstein *Finite groups* Harper and Row, New York, 1968.

[4] M. Guterman *A characterization of* $F_4(2^n)$ J. Algebra 20 (1972) 1-23.

[5] S.M. Husnine *A characterization of the Chevalley group* $F_2(2)$ J. Algebra 44 (1977) 539-549.

[6] S. M. Husnine *On a characterization of* $\hat{F_4}(2)$ *the Ree-extension of the Chavellery group* $F_4(2)\cdots$ *I* Punjab University, Journal of Mathematics vol. xxvi (1993) pp. 101-110.

[7] S. M. Husnine On a characterization of $\hat{F_4}(2)$, the Ree-extension of the Chevalley group $F_4(2)\cdots$ II Punjab University, Journal of Mathematics Vol. xxviii (1995) pp. 30-38.

[8] S. M. Husnine *On a characterization of* $\hat{F_4}(2)$, *the Ree-extension of the Chevalley group* $F_4(2)\cdots$ *III* Punjab University, Journal of Mathematics vol. xxviii(1995) pp. 38-46.

[9] G. Thomas *An improved characterization of the group* $F_4(2^n), n > 2$ J. Algebra 34 (1975), 1-13.

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SOME PROPERTIES OF HADAMARD HYPERNETS WITH CLASS SIZE 3

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ABSTRACT

The designs considered are such that the design and its dual are symmetric affine resolvable, with each parallel class consisting of three blocks and any two non-parallel blocks meeting in μ points. We call an un-ordered triple of blocks a D-triple if all three blocks in the triple contain the same subset of μ points. Some geometric properties of these D-triples are then established for combinatorial classification of the designs.

1. INTRODUCTION

Affine resolvable desings whose duals are also affine have been studies in many equivalent forms: Hadamard systems (Rajkundlia [6]), symmetric nets (Jungnickel [3]) and Hadamard hypernets (McFeat and Neumaier [5]). They can also be regarded as the semi-regular group divisible designs (SRGDD) with $\lambda_1 = 0$ whose duals are also SRGDD with the same parameters. The designs we consider here are those which have three blocks in each parallel class. We adopt the notation of [5] and denote these designs by $H_3(\mu)$ in this paper; μ being the number of points common to any two non-parallel blocks.

We establish that there cannot exist more than three blocks containing the same μ-tuple of points in an $H_3(\mu)$ and call a subset of three non-parallel blocks a D-triple if all three blocks in the triple contain the same μ-tuple. The number of such D-triples occurring in an $H_3(\mu)$ is a characteristic of the design and we call it the characteristic number of the $H_3(\mu)$. Two $H_3(\mu)'s$ are obviously non-isomorphic if they have different characteristic number. D-triples can therefore be used for combinatorially classifying the desings. We also establish some properties of D-triples in the article.

2. BACKGROUND

A *t-design* Π with parameters $t-(v,k,\lambda)$ where $v>k>0$ and $t\leq 1$, is an arrangement of v objects, called points, into subsets called blocks, so that each block consists of k points and any subset of t points is contained in exactly λ blocks.

Normally, the total number of blocks is denoted by b. A design Π is said to be symmetric if $b=v$.

Two designs with the same parameters are *isomorphic* if there exists a bijection between their points, called *isomorphism*, which maps blocks onto blocks.

The *dual design* Π^* of Π has the blocks of Π as its points and then a block of Π^* is defined for each point of Π to consist of all the subset of blocks containing that point.

Design Π is *resolvable* if its blocks can be partitioned into subsets, called parallel clases, such that each parallel class partitions its set of points. In this case two blocks are said to be parallel if they are in the some parallel class and non-parallel otherwise. We call two points to be parallel when they are so as blocks in Π^*. If Π is resolvable so that any two of its non-parallel blocks meet in a constant number of points, say μ,Π is said to be *affine resolvable.*

We call an affine $1-(v,k,r)$ design a *hypernet* (m,r,μ), where $m=v/k$ is the number of blocks in a parallel class and $\mu=k/m$ is a constant such that any two non-parallel blocks intersect in μ points. Note that hypernets are equivalent to the orthogonal arrays of strength two of Bose and Bush [1] and that the dual of a hypernet (m,r,μ) is a transversal design $T[r,\mu,m]$ in the sense of Hanani [2]. If the dual of the latter is a 2-design. Hanani calls the transversal design complete.

Theorem 1[4]

Let Π be a hypernet (m, r, μ) and suppose that Π^* is resolvable then $r \leq m\mu$ with equality if and only if Π^* is affine resolvable.

It follows from the above theorem that if a hypernet (m, r, μ) is symmetric then all its parameters are completely determined by the integers m and μ.

Definition 2

A hypernet (m, r, μ) is a Hadamard hypernet $H_m(\mu)$ if its dual design is also a hypernet with the same parameters.

It is clear from the above discussion that if Π is an $H_m(\mu)$, then Π has μm^2 points (blocks), μm points (blocks) on each block (point) and any two non-parallel blocks (points) are together on exactly μ points. Furthermore, there are exactly m parallel classes of blocks (points) each consisting of exactly m blocks (points).

3. D-TRIPLES

Lemma 3

If c and d are non-parallel blocks of an $H_m(\mu)$, with $\mu > 1$, then $c \cap d$ is contained in at most m blocks.

Proof

Since a pair of non-parallel points is on exactly μ blocks, the number, say η of blocks containing all the μ points of $c \cap d$ is at most μ. The number of points not in $c \cap d$ but on some block containing $c \cap d$ is then $\eta(\mu m - \mu)$. Also the number of points which are not parallel to any of the μ point in $c \cap d$ is $\mu m^2 - \mu m$. It follows that $\eta(\mu m - \mu) \leq \mu m^2 - \mu m$. Hence $\eta \leq m$, with equality if and only if for any point P not parallel to a point in $c \cap d$ there exists a unique block containing P and $c \cap d$.

Definition 4

We call an unordered triple (a, b, c) of non-parallel blocks of an $H_3(\mu)$ a D-triple if all the three blocks contain the same μ-tuple of points. The number of these D-triples in an $H_3(\mu)\Pi$ is called the D-characteristic of Π.

Definition 5

Let $E = \{c_0, c_1, c_2\}$ and $F = \{f_0, f_1, f_2\}$ be two parallel classes of blocks of an $H_3(\mu)$ Π

	e_0	e_1	e_2
f_0	$e_0 \cap f_0$	$e_1 \cap f_0$	$e_2 \cap f_0$
f_1	$e_0 \cap f_1$	$e_1 \cap f_1$	$e_2 \cap f_1$
f_2	$e_0 \cap f_2$	$e_1 \cap f_2$	$e_2 \cap f_2$

The following six subsets, each consisting of 3μ points of Π, are called the diagonals of E and F.

$$(e_0 \cap f_0) \cup (e_1 \cap f_1) \cup (e_2 \cap f_2),$$

$$(e_0 \cap f_0) \cup (e_1 \cap f_2) \cup (e_2 \cap f_1),$$

$$\dots$$

$$(e_0 \cap f_2) \cup (e_1 \cap f_2) \cup (e_2 \cap f_0).$$

If any of the diagonals is a block of Π then it is called a D-block of Π. The D-number of a block d of Π is then defined as the number of unordered pairs, of distinct parallel classes in which d appears as a D-block.

The D-number of Π is essentially the sum of the D-numbers of its blocks.

Lemma 6

Let $\{e_0, e_1, e_2\}, \{f_0, f_1, f_2\}, \{g_0, g_1, g_2\}$ be distinct parallel classes of blocks of an $H_3(\mu)$. If (e_0, f_0, g_0) and (e_1, f_1, g_0) are D-triples then so is (e_2, f_2, g_0).

Proof

If (e_0, f_0, g_0) and (e_1, f_1, g_0) are D-triples then $e_0 \cap f_0 \subseteq g_0$ and $e_1 \cap f_1 \subseteq g_0$. But then g_0 cannot contain any other point from the blocks e_0, f_0, e_1, f_1. Hence $g_0 = (e_0 \cap f_0) \cup (e_1 \cap f_1) \cup (e_2 \cap f_2)$. It follows that (e_2, f_2, g_0) is a D-triple.

Lemma 7

Let $\{e_0, e_1, e_2\}$, $\{f_0, f_1, f_2\}$, $\{g_0, g_1, g_2\}$ be distinct parallel classes of blocks of an $H_2(\mu)$. If (e_0, f_0, g_0) and (e_1, f_1, g_1) are D-triples then so is (c_2, f_2, g_2).

Proof

Since (e_0, f_0, g_0) is a D-triple, g_0 contains the μ points of the set $e_0 \cap f_0$ and no other points from the blocks e_0 and f_0. The remaining 2μ points of g_0 are then clearly from the set $(e_1 \cap f_1) \cup (e_1 \cap f_1) \cup (e_2 \cap f_2)$.

	e_0	e_1	e_1
f_0			
f_1		$e_1 \cap f_1$	$e_2 \cap f_1$
f_2		$e_1 \cap f_2$	$e_2 \cap f_2$

Also, since (e_1, f_1, g_1) is a D-triples, $(e_1 \cap f_1) \subseteq g_1$, and as g_0 is parallel to $g_1, g_0 \cap e_1 = e_1 \cap f_2$ and $g_0 \cap f_1 = e_2 \cap f_1$. It follows that $g_0 \cap e_2 = e_2 \cap f_1$. Thus $g_0 = (e_0 \cap f_0) \cup (e_1 \cap f_2) \cup (e_2 \cap f_1)$. We can similarly prove that $g_1 = (e_0 \cap f_2) \cup (e_1 \cap f_1) \cup (e_2 \cap f_0)$. It follows that $g_2 = (e_0 \cap f_1) \cup (e_1 \cap f_2)$ and (e_2, f_2, g_2) is a D-triple.

REFERENCES

[1] R.C. Bose & K. A. Bush *Orthogonal arrays of strength two and three* Ann. Math. Statist. 23 (1952), 508-524.

[2] H. Hanani *On transversal designs in Combinatorics* Proc. of the Advanced Study Institute. Pat I. Math. Centrum Tracts 55, Amsterdum, 1974, pp. 42-52.

[3] D. Jungnickel *On difference matrices, resolvable transversal designs and generalized Hadamard matrices* Math. Z. 167 (1979), 49-60.

[4] V.C. Mavron *Translations and parallel classes of lines in affine designs* J. Comb. Theory 22 (1977), 322-330.

[5] A. B. McFeat & A. Neumaier *Some tuple system constructions with applications to resolvable desings* J. Combin. Theory A 33 (1) (1984), 92-100.

[6] D. Rajkundlia *Some techniques for constructing infinite families of BIBD's* Discrete Maths. 44 (1983), 61-96.

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RECONSTRUCTION OF 3D-OBJECTS USING CROSS-SECTIONAL DATA

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Dedicated to the memory of Dr. M. Rafique

ABSTRACT

A scheme for designing/reconstruction of 3D objects, using cross-sectional data, is discussed. Literature, discussing the schemes, managing the situations when the cross-sectional data is given in monotone order, is available. A new scheme is introduced to manage the situations when the cross- sectional data is not given in monotone order.

Keywords: Rational splines, C^1-surface, Cross-sectional data.

1. INTRODUCTION

A Scheme is discussed by T.N.T. Goodman *et al.* [4] to construct a C^1 closed surface from (horizontal) cross-sectional data. While the heights, where the cross-sectional data is provided, must be monotonic. At each height a single contour (closed curve) is constructed from the given information. However, in this paper, it is assumed that at some heights, the horizontal plane may intersect the given surface in more than one contours with the proviso that a contour is labelled a height number and are rearranged in the form of a sequence so that the surface can be traced out. The sequence of heights in this case, may not be monotonic. This problem is resolved by modifying the tangent scheme presented by T.N.T. Goodman *et al.* [4].

In section 2, the new tangent scheme is presented. In section 2.1, construction of the base is discussed. In section 3, two examples are given. It is to be noted that if three consecutive heights are in monotone order then the definition of tangent at the middle of the three heights, coincides with that given by T.N.T. Goodman *et al.* [4].

2. CONSTRUCTION OF A SURFACE

Define a surface S as follows

$$S = \{\mathbf{S}^i(t,u) : \ i = 0, 1, 2, \ldots, n; \ 0 \le t, u \le 1\} \tag{1}$$

The surface parts $\mathbf{S}^0(t,u)$, $\mathbf{S}^n(t,u)$ will be discussed in the next section, while $\mathbf{S}^i(t,u)$ for $i = 1, 2, 3, \ldots, n-1$ are given by

$$\mathbf{S}^i(t,u) = \mathbf{P}^i(t)L_0(u) + \mathbf{P}^{i+1}(t)L_1(u) + |z^{i+1} - z^i|(\mathbf{G}^i(t)H_0(u) + \mathbf{G}^{i+1}(t)H_1(u)), \tag{2}$$

where $L_0(u)$, $L_1(u)$, $H_0(u)$ and $H_1(u)$ are cubic Hermite polynomials, may be written as

$$\left.\begin{aligned} L_0(u) &= 1 - 3u^2 + 2u^3, \\ L_1(u) &= 3u^2 - 2u^3, \\ H_0(u) &= u - 2u^2 + u^3, \\ H_1(u) &= -u^2 + u^3, \end{aligned}\right\}$$

Equation (1) determines the surface between the first and the last heights using the provided data. To define $\mathbf{P}^i(t)$ and $\mathbf{G}^i(t), i = 1, 2, 3, \ldots, n$ in (2), suppose $\{\mathbf{I}_j^i : 0 \le j \le N^i, N^i \in N\}$ is the set of cross-sectional data at the height z^i. From data $\mathbf{I}_j^i$, obtain the contour $\mathbf{r}^i$, $0 \le t \le 1$, using the scheme discussed by T.N.T. Goodman [2] or T.N.T. Goodman and K. Unsworth [3]. Then using the techniques *viz.* normalization and matching parameters as explained in T.N.T. Goodman *et al.* [4], $\mathbf{r}^i(t)$ becomes $\mathbf{R}^i(t)$. Now denote $\mathbf{P}^i(t) = (\mathbf{R}^i(t), z^i)$ and $\mathbf{G}^i(t) = (\mathbf{g}_R^i(t), \ g_z^i(t))$,representing the tangent vector at $\mathbf{P}^i(t)$, where

$$\mathbf{g}_R^i(t) = \frac{|\mathbf{R}^{i+1}(t) - \mathbf{R}^i(t)|(\mathbf{R}^i(t) - \mathbf{R}^{i-1}(t)) + |\mathbf{R}^i(t) - \mathbf{R}^{i-1}(t)|(\mathbf{R}^{i+1}(t) - \mathbf{R}^i(t}{|\mathbf{R}^{i+1}(t) - \mathbf{R}^i(t)||z^i - z^{i-1}| + |\mathbf{R}^i(t) - \mathbf{R}^{i-1}(t)||z^{i+1} - z^i|}$$

$$2 \le i \le n-1$$

$$g_z^i(t) = \frac{|\mathbf{R}^{i+1}(t) - \mathbf{R}^i(t)|(z^i - z^{i-1}) + |\mathbf{R}^i(t) - \mathbf{R}^{i-1}(t)|(z^{i+1} - z^i)}{|\mathbf{R}^{i+1}(t) - \mathbf{R}^i(t)||z^i - z^{i-1}| + |\mathbf{R}^i(t) - \mathbf{R}^{i-1}(t)||z^{i+1} - z^i|}, 2 \leq i \leq n-1$$

$$\mathbf{g}_R^1(t) = \frac{2|\mathbf{R}^2(t) - \mathbf{R}^1(t)|(\mathbf{R}^1(t) - \mathbf{R}^0) + |\mathbf{R}^1(t) - \mathbf{R}^0|(\mathbf{R}^2(t) - \mathbf{R}^1(t))}{2|\mathbf{R}^2(t) - \mathbf{R}^1(t)||z^1 - B| + |\mathbf{R}^1(t) - \mathbf{R}^0||z^2 - z^1|},$$

$$g_z^1(t) = \frac{2|\mathbf{R}^2(t) - \mathbf{R}^1(t)|(z^1 - B) + |\mathbf{R}^1(t) - \mathbf{R}^0|(z^2 - z^1)}{2|\mathbf{R}^2(t) - \mathbf{R}^1(t)||z^1 - B| + |\mathbf{R}^1(t) - \mathbf{R}^0||z^2 - z^1|}.$$

While $\mathbf{g}_R^n(t)$, $g_z^n(t)$ are defined similarly.
The points $\mathbf{P}^0 = (\mathbf{R}^0, B)$ and $\mathbf{P}^{n+1} = (\mathbf{R}^{n+1}, T)$, are referred to as base and crown points respectively, which can be provided by the user or determined algorithmically as in T.N.T.Goodman *et al.* [4]. In case $\mathbf{R}^{i-1}(t) = \mathbf{R}^i(t) = \mathbf{R}^{i+1}(t)$, $\mathbf{g}_R^i(t)$ is taken as $(0,0)$ and $g_z^i(t)$ is 1 if z^{i-1}, z^i, and z^{i+1} are monotonically increasing otherwise -1.
Obviously $\mathbf{G}^i(t)$ is a linear combination of vectors $\mathbf{P}^i(t)-\mathbf{P}^{i-1}(t)$ and $\mathbf{P}^{i+1}(t)-\mathbf{P}^i(t)$, thus lies in the plane of vectors $\mathbf{P}^{i-1}(t)$, $\mathbf{P}^i(t)$ and $\mathbf{P}^{i+1}(t)$. In particular if these three points lie in the vertical plane, then the vectors $\mathbf{R}^i(t) - \mathbf{R}^{i-1}(t)$ and $\mathbf{R}^{i+1}(t) - \mathbf{R}^i(t)$ becomes parallel and hence can be written as

$$\mathbf{R}^i(t) - \mathbf{R}^{i-1}(t) = \lambda \mathbf{v}(t),$$

$$\mathbf{R}^{i+1}(t) - \mathbf{R}^i(t) = \mu \mathbf{v}(t), \text{ for some unit vector } \mathbf{v}(t).$$

Hence

$$\mathbf{g}_R^i(t) = \left[\frac{|\mu|\lambda + |\lambda|\mu}{|\mu||z^i - z^{i-1}| + |\lambda||z^{i+1} - z^i|}\right]\mathbf{v}(t).$$

Now two cases are possible either both $\mathbf{R}^i(t) - \mathbf{R}^{i-1}(t)$ and $\mathbf{R}^{i+1}(t) - \mathbf{R}^i(t)$ are in the same direction or both in the opposite direction.
When $\lambda\mu > 0$,

$$\mathbf{g}_R^i(t) = \left[\frac{2\lambda\mu}{\mu|z^i - z^{i-1}| + \lambda|z^{i+1} - z^i|}\right]\mathbf{v}(t)$$

$$= \left[\frac{2\frac{\lambda}{|z^i - z^{i-1}|}\frac{\mu}{|z^{i+1} - z^i|}}{\frac{\lambda}{|z^i - z^{i-1}|} + \frac{\mu}{|z^{i+1} - z^i|}}\right]\mathbf{v}(t).$$

It follows that

$$|\mathbf{g}_R^i(t)| \leq \frac{2\lambda}{|z^i - z^{i-1}|}, \frac{2\mu}{|z^{i+1} - z^i|}.$$

While $\lambda\mu < 0$,

$$\mathbf{g}_R^i(t) = (0, 0).$$

For $\mathbf{g}_R^1(t)$, it can be written as

$$\mathbf{R}^1(t) - \mathbf{R}^0 = \lambda \mathbf{v}(t),$$

$$\mathbf{R}^2(t) - \mathbf{R}^1(t) = \mu \mathbf{v}(t), \text{ for some unit vector } \mathbf{v}(t).$$

It follows that if $\lambda\mu > 0$, then

$$\mathbf{g}_R^1(t) = \left[\frac{3 \frac{\lambda}{|z^1 - B|} \frac{\mu}{|z^2 - z^1|}}{\frac{\lambda}{|z^1 - B|} + \frac{2\mu}{|z^2 - z^1|}} \right] \mathbf{v}(t),$$

which shows that

$$|\mathbf{g}_R^1(t)| \leq \frac{3\lambda}{2|z^1 - B|}, \frac{3\mu}{|z^2 - z^1|}.$$

Thus when the three points $\mathbf{P}^{i-1}(t)$, $\mathbf{P}^i(t)$ and $\mathbf{P}^{i+1}(t)$ lie in a vertical plane the gradient used in (1), satisfies the Fritsch and Carlson criteria [1]. This shows that the curve (in scalar case for a fixed t), generated by (1) in the direction of $\mathbf{v}(t)$, preserves monotonicity.

Finally the scheme produces a C^1 surface. It is to be noted that sharp turns or even edges can occur at extreme heights in case numerators of $\mathbf{g}_R^i(t)$ and $g_z^i(t$ approach to zero.

3. CONSTRUCTION OF BASE

In this section the construction of base is discussed. The construction of the crown can be followed similarly.

If one may desire to close the surface at one end, the aim is gained by constructing the base part of the surface referred to as $\mathbf{S}^0(t, u)$, otherwise it is null (open). Depending on the object under reconstruction, two types of base with C^0 or C^1 continuity at the base point can be constructed.

To obtain the base of C^0 type, $\mathbf{G}^0(t)$ is taken to be same as $\mathbf{G}^1(t)$ and $\mathbf{P}^0 = (\mathbf{R}^0, B)$. $\mathbf{S}^0(t, u)$ is obtained from the equation (1).

The base $\mathbf{S}^0(t, u)$ of C^1 type, is the same as given by T.N.T. Goodman *et a*

[4] *i.e.*,

$$\mathbf{S}^0(t,u) = \mathbf{P}^0(t)L_0(u) + \mathbf{P}^1(t)L_1(u) + f(t)\mathbf{G}^0(t)H_0(u) + 2|z^1 - B|\mathbf{G}^1(t)H_1(u)),$$

where $\mathbf{P}^0 = (\mathbf{R}^0, B)$ and

$$f(t) = \frac{\sqrt{2}|\mathbf{R}^1(t) - \mathbf{R}^0|}{\sqrt{1 + \frac{|z^1 - B||\mathbf{g}_R^1(t)|}{|\mathbf{R}^1(t) - \mathbf{R}^0|}}}.$$

However, to calculate $\mathbf{G}^0(t)$, a procedure is given.
$\mathbf{G}^0(t) = (\mathbf{g}_R^0(t), 0)$. Clearly $\alpha(\mathbf{P}^1(t) - \mathbf{P}^0) + \beta(\mathbf{P}^2(t) - \mathbf{P}^1(t))$ where $\alpha, \beta \in R$, represents a point in the plane of $\mathbf{P}^0, \mathbf{P}^1(t)$ and $\mathbf{P}^2(t)$. Then the pairs α_1, β_1 and α_2, β_2 corresponding to the points $(\mathbf{R}_{L1}(t), B)$ and $(\mathbf{R}_{L2}(t), B)$ respectively lying in this plane, can be calculated as follows:
Choosing arbitrary non-zero α_1, then β_1 and $\mathbf{R}_{L1}(t)$ can be determined from the relation

$$\alpha_1(z^1 - B) + \beta_1(z^2 - z^1) = B.$$
$$\mathbf{R}_{L1}(t) = \alpha_1(\mathbf{R}^1(t) - \mathbf{R}^0) + \beta_1(\mathbf{R}^2(t) - \mathbf{R}^1(t)).$$

Now α_1 is replaced by its additive inverse and name it α_2. β_2 and $\mathbf{R}_{L2}(t)$ are determined as above with the new choice.
If θ denotes the angle between $\mathbf{R}^1(t) - \mathbf{R}^0$ and $\mathbf{R}_{L1}(t) - \mathbf{R}_{L2}(t)$. Define $\mathbf{g}_R^0(t)$ as follows:

$$\mathbf{g}_R^0(t) = \begin{cases} \frac{\mathbf{R}^1(t) - \mathbf{R}^0}{|\mathbf{R}^1(t) - \mathbf{R}^0|} & \text{if } \theta = 0, \pi, \\ \frac{\mathbf{R}_{L1}(t) - \mathbf{R}_{L2}(t)}{|\mathbf{R}_{L1}(t) - \mathbf{R}_{L2}(t)|} & \text{if } 0 < \theta < \frac{\pi}{2}, \\ \frac{\mathbf{R}_{L2(t)} - \mathbf{R}_{L1}(t)}{|\mathbf{R}_{L2}(t) - \mathbf{R}_{L1}(t)|} & \text{if } \frac{\pi}{2} < \theta < \pi. \end{cases}$$

4. EXAMPLES

In order to demonstrate the algorithm, two examples are presented.

EXAMPLE 1

To reconstruct an apple, the following data is considered.

Height	Cross-sectional data
3.5	Null _ Height
2.3	(0.2 0), (0 0.2), (-0.2 0), (0 -0.2)
0.7	(0.7 0), (0 0.7), (-0.7 0), (0 -0.7)
0	(3 0), (0 3), (-3 0), (0 -3)
2	(6 0), (0 6), (-6 0), (0 -6)
7	(7.5 0), (0 7.5), (-7.5 0), (0 -7.5)
9	(7 0), (0 7), (-7 0), (0 -7)
11	(4 0), (0 4), (-4 0), (0 -4)
10.5	(1.3 0), (0 1.3), (-1.3 0), (0 -1.3)
9.5	(0.7 0), (0 0.7), (-0.7 0), (0 -0.7)
7.3	Null _ Height

Note:- Null_ Height indicates that the data is not provided at this height.

Figure 1 gives the top and bottom views of the apple.
Figure 2 gives the whole surface.

EXAMPLE 2

To reconstruct a vase, the following data is considered.

Height	Cross-sectional data
1.7	Null_ Height
1.5	(1 0), (0 1), (-1 0), (0 -1)
0.75	(3 0), (0 3), (- 3 0), (0 -3)
0	(3.75 0), (0 3.75), (-3.75 0), (0 -3.75)
0.75	(3.5 0), (0 3.5), (-3.5 0), (0 -3.5)
1.5	(3.25 0), (0 3.25), (-3.25 0), (0 -3.25)
5	(6 0), (0 6), (-6 0), (0 -6)
8	(3.5 0), (0 3.5), (-3.5 0), (0 -3.5)
15	(1.75 0), (0 1.75), (-1.75 0), (0 -1.75)
21.5	(3 0), (0 3), (-3 0), (0 -3)
21	(4 0), (0 4), (-4 0), (0 -4)
20	Null_Height

To obtain the required surface, the crown and the second last surface part are ignored, as shown in figure 3.

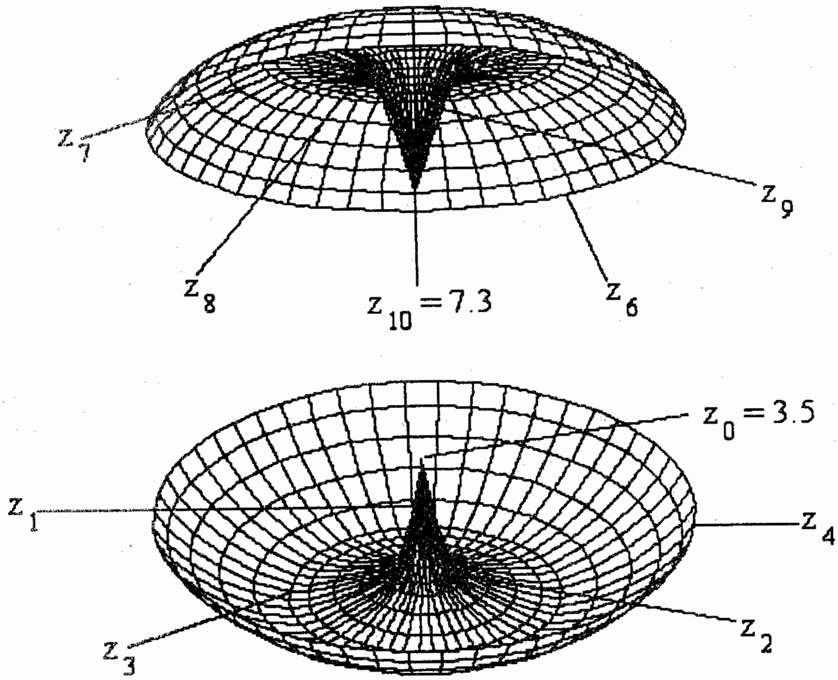

Figure 1.

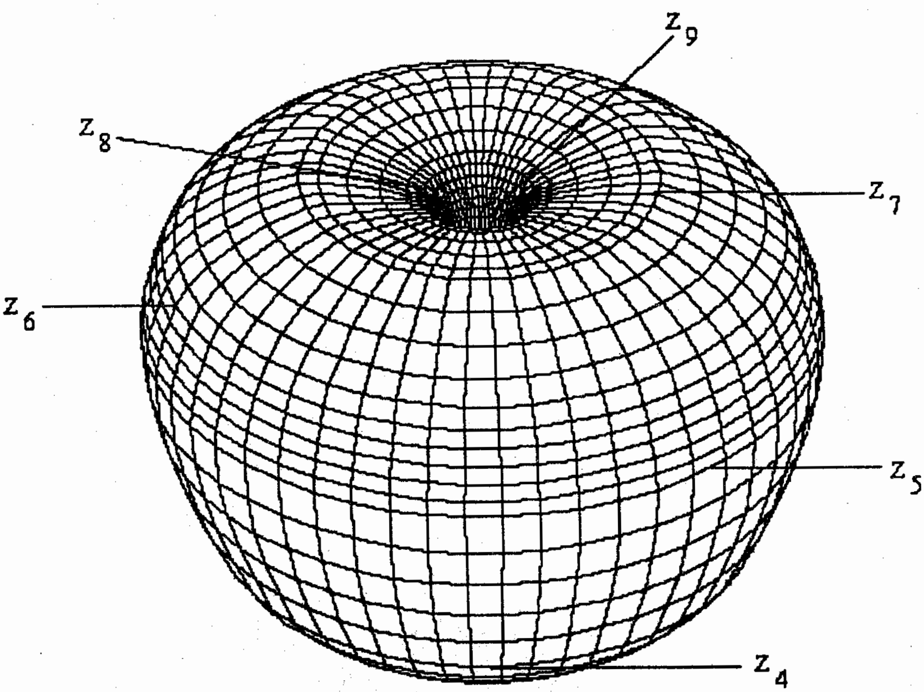

Figure 2.

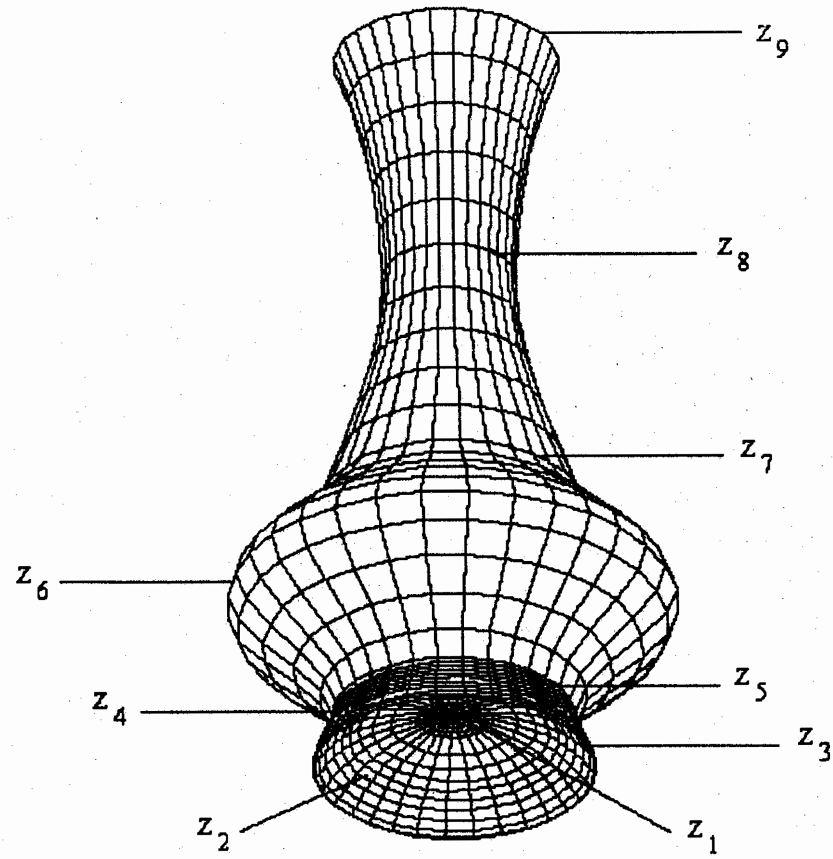

Figure 3.

REFERENCES

[1] Fritsch F.N. and Carlson R.E. *Monotone piecewise cubic interpolation* SIAM J. Numer. Anal. 17(1980), 238-246.

[2] Goodman T.N.T. *Shape preserving interpolation by parametric rational cubic splines* Intnl. Series of Numerical Mathematics, Vol. 86, Birkhauser Verlag, Basel, 149-158, 1988.

[3] Goodman T.N.T. and Unsworth K. *An algorithm for generating shape preserving interpolating curves using rational cubic splines* University of Dundee Comp. Sci. Report 89/01, 1989.

[4] Goodman T.N.T., Ong B.H. and Unsworth K. *Reconstruction of C^1 closed surfaces from cross-sectional data* University of Dundee Comp. Sci. Report 91/12, 1991.

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FIXED POINTS AND BEST APPROXIMATION IN CONVEX METRIC SPACES

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ABSTRACT

A fixed point theorem in convex metric spaces is proved. As an application, a result in best approximation is also derived.

Let X be a metric space with metric d and I = [0, 1]. A continuous mapping W: $X \times X \times I \to X$ is said to be a convex structure if for each $(x, q, \lambda) \in X \times X \times 1$ and $u \in X$,

$$d(u, W(x, q, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, q)$$

The metric space X together with a convex structure W is called a convex metric space. A subset C of X is called (1) convex if $W(x, q, \lambda) \in C$ for all $(x, q, \lambda) \in C \times C \times 1$; (2) q-starshaped if there exists $q \in C$ such that $W(x, q, \lambda) \in C$ for all $x \in C$ and all $\lambda \in I$. A convex metric space X is said to satify the property (I) ([2]) if

$$d(W(x, q, \lambda), W(y, q, \lambda)) \leq \lambda d(x, y)$$

for all $x, y \in X$ and all $\lambda \in 1$.

Obviously, all normed spaces are convex metric spaces satisfying the property (I). However, there are many examples of convex metric spaces which are not embedded in any normed space (ef. [6]).

Let T and S be selfmaps of X. Then T is called S nonexpansive on $C \subset X$ if $d(Tx, Ty) \leq d(Sx, Sy)$ for all $x, y \in C$. The maps T and S are said to be commuting on $C \subset X$ if $STx = TSx$ for all $x \in C$. Suppose $\hat{x} \in X$. Then the set $P_c(\hat{x}) = \{ x \in C: d(x, \hat{x}) = d(\hat{x}, C) \}$ is called the set of all best approximants to $\hat{x}$, where $d(\hat{x}, C) = \inf \{ d(y, \hat{x}): y \in C \}$. A point $x \in X$ is a fixed point of T (resp. S) if $Tx = x$ (resp. $Sx = x$). The set of all fixed points of T (resp. S) is denoted by F(T) (resp. F(S)). If T is a selfmap of X, then $\mathcal{G}_T$ is the set of all selfmaps S of X such that $ST = TS$.

Recently, I.Beg, N. Shahzad and M. Iqbal [2] proved the following result on best approximation, which extends Theorem 3 of Sahab, Khan and Sessa [5].

Theorem 1

Let X be a convex metrix space satisfying condition (I), T and S selfmaps of X with $\hat{x} \in F(T) \cap F(S)$, and $C \subset X$ with $T(\partial C) \subset C$, where ∂C denotes the boundary of C. Suppose S is continuous and affine on $P_c(\hat{x})$. T and S are commuting on $P_c(\hat{x})$ and T is S-nonexpansive on $P_c(\hat{x}) \cup \{\hat{x}\}$. If $P_c(\hat{x})$ is non-empty compact, and q-starshaped with $q \in F(S)$, and if $S(P_c(\hat{x})) = P_c(\hat{x})$, then $P_c(\hat{x}) \cap F(T) \cap F(S) \neq \phi$.

In this paper, we first establish a fixed point theorem in convex metric spaces and then use it to obtain a generalization of Theorem 1.

The following fixed point theorem is an immediate consequence of Theorem 4.5 of [3].

Theorem 2

Let X be a compact metric space and T a continuous selfmap of X. Suppose $Tx \neq Ty$ implies $d(Tx, Ty) < d(Sx, Ry)$ for some $S, R \in \mathcal{G}_T$. Then $F(T) \cap F(S)$ is singleton for all $S \in \mathcal{G}_T$.

Now, we prove our main theorem, which generalizes Theorem 3 of [4].

Theorem 3

Let X be a convex metric space satisfying condition (I), $C \subset X$ a compact set, and T a continuous selfmap of C. Suppose $\mathcal{F} \subset \mathcal{G}_T$ is a family of affine selfmaps S of C with $q \in F(S)$ and for each pair $(x, y) \in C^2$, there exists $S = S(x,y)$, $R = R(x,y) \in \mathcal{F}$ such that

$$d(Tx, Ty) \leq d(Sx, Ry). \tag{1}$$

If C is q-starshaped, then $F(T) \neq \phi$ and $F(T) \cap F(S) \neq \phi$ for all continuous $S \in \mathcal{F}$.

Proof

Let $\{\lambda_n\}$ be a sequence of real numbers such that $0 \leq \lambda_n < 1$, which converges to 1. For each n, define a sequence of maps T_n by $T_n X = W(Tx, q, \lambda_n)$ for all $x \in C$. Clearly, each T_n is a continuous selfmap of C because C is q-starshaped, W is continuous, and T is continuous selfmap. For any $S \in \mathcal{F}$ and $n \in \mathcal{N}$, we have

$$\begin{aligned} T_n Sx &= W(TSx, q, \lambda_n) \\ &= W(STx, Sq, \lambda_n) \\ &= S(W(Tx, q, \lambda_n)) \\ &= ST_n x \end{aligned}$$

for all $x \in C$. Thus $\mathcal{F} \subset \mathcal{G}_{T_n}$

Let n be fixed. Then, for each pair $(x, y) \in C^2$, there exist $S, R \in \mathcal{F}$ such that

$$\begin{aligned} d(T_n n, T_n y) &= d(W(Tx, q, \lambda_n), W(Ty, q, \lambda_n)) \\ &\leq \lambda_n d(Tx, Ty) \\ &\leq \lambda_n d(Sx, Ry) \end{aligned}$$

Therefore, for all $x, y \in C$, we get

$$d(T_n x, T_n y) < d(Sx, Ry)$$

whenever $T_n x \neq T_n y$ for some $S, R \in \mathcal{F}$. It follows, by Theorem 2, that there exists $x_n \in C$ such that

$$F(T_n) \cap F(S) = \{x_n\}$$

for all $S \in \mathcal{F}$. By the compactness of C, there exists a subsequence $\{x_m\}$ of $\{x_n\}$ with $x_m \to y \in C$. Now,

$$\begin{aligned} d(Tx_m, x_m) &= d(Tx_m, W(Tx_m, q, \lambda_n)) \\ &\leq (1-\lambda)d(Tx_m, q) \end{aligned}$$

for all m. The continuity of T further implies that $y = Ty$, that is, $y \in F(T)$. Moreover, for all continuous $S \in \mathcal{F}$, we have $y = Sy$ because $x_m = Sx_m$ for all m. Hence $F(T) \cap F(S) \neq \phi$ for all continuous $S \in \mathcal{F}$.

The following Theorem is an extension of Theorem 1. It also includes Theorem 4 of [4] as a special case.

Theorem 4

Let X be a convex metric space satisfying condition (I), T and S selfmaps of X with $\hat{x} \in F(T) \cap F(S)$, and $C \subset X$ with $T\ (\partial C \cap C) \subset C$. Suppose T is continuous, I affine on $P_c(\hat{x}), T$ and S are commuting on $P_c(\hat{x})$, and for $x, y \in P_c(\hat{x}) \cup \{\hat{x}\}$, there exist $n = n\ (x, y)$, $m = m(x, y)$ in $\mathcal{N} \cup \{0\}$ such that

$$d(Tx, Ty) \leq d(S^n x, S^m y). \qquad (2)$$

If $P_c(\hat{x})$ is nonempty, compact, and q-starshaped with $q \in F(S)$, and if $S(P_c(\hat{x}) \subset P_c(\hat{x})$, then $P_c(\hat{x}) \cap F(T) \neq \phi$. If, in addition, S is continuous, then $P_c(\hat{x}) \cap F(T) \cap F(S)$

Proof

Let $y \in P_c(\hat{x})$. Since $S(P_c(\hat{x})) \subset P_c(\hat{x})$, we have $S^n y \in P_c(\hat{x})$ for $n \in \mathcal{N} \cup \{0\}$ and so S^n is a selfmap of $P_c(\hat{x})$. Also, as in Lemma 3.2 of [1], $y \in \partial C \cap C$. But $T(\partial C \cap C \subset C)$. Therefore, $Ty \in P_c(\hat{x})$. It follows from (2) that

$$\begin{aligned} d(Ty, \hat{x}) &= d(Ty, T\hat{x}) \\ &\leq d(S^n y, S^m \hat{x}) \\ &= d(S^n y, \hat{x}) = d(\hat{x}, s). \end{aligned}$$

This is possible since $\hat{x} \in F(T) \cap F(S)$ and $S^n y \in P_c(\hat{x})$. Thus $Ty \in P_c(\hat{x})$ and so T is a selfmap of $P_c(\hat{x})$. Note that, for each n, S^n is affine and $q \in F(S^n)$. Set

$$\mathcal{F} = \{S^n : n \in \mathcal{N} \cup \{0\}\}.$$

Then $\mathcal{F} \subset \mathcal{G}_T$ because T and S are commuting on $P_c(\hat{x})$. Further, (2) implies (1). Hence the result follows from Theorem 3.

REFERENCES

[1] M.A. Al-Thagafi, *Best approximation and fixed points in strong M-starshaped metric spaces*, Internat. J. Math. and Math. Sci. 18 (1995), 613-616.

[2] I. Beg, N. Shahzad and M. Iqbal *Fixed point theorems and best approximation in convex metric spaces*, Approx. Theory and its Appl., 8 (1992), 97-105.

[3] G. Jungck, *Common fixed points for commuting and compatible maps on compacta*, Proc. Amer. Math. Soc., 103(1988), 977-983.

[4] G. Jungck and S. Sessa, *Fixed point theorems in best approximation theory*, Math. Japonica, 42(1995), 249-252.

[5] S.A. Sahab, M.S. Khan, and S. Sessa, *A result in best approximation theory*, J. Approx. Theory, 55 (1988), 349-351.

[6] W. Takahashi, *A convexity in metric spaces and non-expansive mappings I*, Kodai Math. Sem. Rep., 22(1970), 142-149.

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