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**THE ORBITS OF $Q^*(\sqrt{P})$, $P \equiv 3 \pmod{4}$, UNDER THE
ACTION OF THE MODULAR GROUP $G = \langle x, y : x^2 = y^3 = 1 \rangle$**

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Abstract:

In this paper we determine the number of orbits of $Q^*(\sqrt{p})$ under the action of the modular group $G = \langle x, y : x^2 = y^3 = 1 \rangle$, where p is a prime and $p \equiv 3 \pmod{4}$.

1. INTRODUCTION

Throughout this paper for any two rational integers a and b , (a, b) denotes

the greatest common divisor of a and b and n denotes a non square positive rational integer. Let

$$Q^*(\sqrt{n}) = \left\{ \frac{a + \sqrt{n}}{c} : \frac{a^2 - n}{c^2} \text{ is a rational integer and } \left(a, \frac{a^2 - n}{c}, c \right) = 1 \right\}$$

For $\alpha = \frac{a + \sqrt{n}}{c} \in Q^*(\sqrt{n})$; its conjugate $\bar{\alpha} = \frac{a - \sqrt{n}}{c}$ may or may not have the same sign as that of α . If α and $\bar{\alpha}$ have different signs, then α is called an ambiguous number [5].

If $\alpha = \frac{a + \sqrt{n}}{c}$, then $N(\alpha) = \alpha\bar{\alpha} = \frac{a^2 - n}{c^2}$ is called the norm of α .

Clearly an $\alpha \in Q^*(\sqrt{n})$ is an ambiguous number if $N(\alpha) = -1$. In such a case $n = a^2 + c^2$. Also if $\alpha = \frac{a + \sqrt{n}}{c}$, then we write $-\alpha = \frac{a + \sqrt{n}}{-c}$ so that $-\bar{\alpha} = \frac{-a + \sqrt{n}}{c}$.

A coset diagram is just a graphical representation of a permutation action of a finitely generated group on a non empty set.

In this paper we study the cost diagrams of the modular group

$G = \langle x, y : x^2 = y^3 = 1 \rangle$ under its action on $Q^*(\sqrt{n})$. Thus in our case the diagram consists of a set of small triangles representing the action of $C_3 = \langle y : y^3 = 1 \rangle$ and a set of edges representing the action of $C_2 = \langle x : x^2 = 1 \rangle$.

In our diagrams, there are only two generators, namely x and y . In the case of y , which has order 3, there is a need to distinguish y from y^{-1} . The 3-cycles of y are therefore represented by small triangles, with the convention that y permutes their vertices counter - clockwise, while the fixed points of y are denoted by heavy dots.

Also to make the diagram slightly less complicated, we omit the loops corresponding to fixed points of x , because then the geometry of the figure makes the distinction between x -edges and y -edges obvious.

Let $C' = C \cup \{\infty\}$ be the extended complex field. Mushtaq [5] has proved that $Q^*(\sqrt{n})$ is invariant under the action of $G = \langle x, y : x^2 = y^3 = 1 \rangle$ where $x : C' \rightarrow C'$ and $y : C' \rightarrow C'$ are the Mobius transformations defined by:

$$x(z) = \frac{-1}{z}, y(z) = \frac{z-1}{z}$$

He has also shown that $Q^*(\sqrt{n})$ contains only a finite number ambiguous numbers and those occurring in a particular orbit of $Q^*(\sqrt{n})$ form a unique closed path in the coset diagram under the action of G on $Q^*(\sqrt{n})$.

The actual number of ambiguous numbers in $Q^*(\sqrt{n})$ has been determined in [2] as a function of n .

In [3], the integers, units and primes of $Q^*(\sqrt{n})$ have been investigated. The exact number of ambiguous integers, ambiguous units and ambiguous primes in $Q^*(\sqrt{n})$ have also been determined there.

The actual number of orbits of $Q^*(\sqrt{p})$, p a rational prime, under the action of the modular group $G = \langle x, y : x^2 = y^3 = 1 \rangle$ in the cases $p \equiv 2 \pmod{4}$ and $p \equiv 1 \pmod{4}$ have been determined in [4].

The number of ambiguous numbers in the orbit

$\alpha^G = \{\alpha^g = g(\alpha) : g \in G\}$, $\alpha \in Q^*(\sqrt{n})$, is called the ambiguous length of α with respect to G .

In this paper we determine the number of distinct closed paths formed by the ambiguous numbers of $Q^*(\sqrt{p})$, where $p \equiv 3 \pmod{4}$ is a rational prime, under the modular group action on it.

The notation is standard and we follow [1], [2], [3], [4] and [5]. For a real number x , $[x]$ denotes the largest rational integer not greater than x .

2. PRELIMINARIES

The results that follow will be used later in this paper.

Lemma 2.1 [1]

Let p be a rational prime and $p \equiv 3 \pmod{4}$. Then p can not be written as a sum of two squares.

Theorem 2.2 [5]

The ambiguous numbers in the orbit α^G of $\alpha \in Q^*(\sqrt{n})$ form a single closed path in the coset diagram for the orbit α^G .

Note: Closed path in the coset diagram for the orbit α^G is unique except for the triangles.

The following simple remark is useful to determine the number of orbits of $Q^*(\sqrt{n})$ under the action of G .

Remark 2.3

The number of disjoint orbits α^G , $\alpha \in Q^*(\sqrt{n})$, is equal to the number of closed paths in the coset diagram under the action of G on $Q^*(\sqrt{n})$.

The next lemma shows that image of conjugate of element of $Q^*(\sqrt{n})$ under an element of G is the conjugate of image.

Lemma 2.4 [4]

Let $\alpha \in Q^*(\sqrt{n})$. Then $g(\bar{\alpha}) = \overline{g(\alpha)}$, $\forall g \in G$.

Corollary 2.5 [4]

For a real quadratic irrational number β in α^G , $\alpha \in Q^*(\sqrt{n})$ we have:

- i) $x(-\bar{\beta}) = \overline{x(-\beta)} = -\overline{x(\beta)} = -x(\bar{\beta})$
- ii) $y(-\bar{\beta}) = \overline{y(-\beta)} = 2 - \overline{y(\beta)} = 2 - y(\bar{\beta})$
- iii) $xy^2(-\bar{\beta}) = \overline{xy^2(-\beta)} = -\overline{[yx(\beta)]} = -[yx(\bar{\beta})]$
- iv) $yx(-\bar{\beta}) = \overline{yx(-\beta)} = -\overline{[xy^2(\beta)]} = -[xy^2(\bar{\beta})]$
- v) $y^2x(-\bar{\beta}) = \overline{y^2x(-\beta)} = -\overline{[xy(\beta)]} = -[xy(\bar{\beta})]$ and
- vi) $xy(-\bar{\beta}) = \overline{xy(-\beta)} = -\overline{[y^2x(\beta)]} = -[y^2x(\bar{\beta})]$

Remark 2.6 [4]

i) Using lemma 2.4, it is easy to see that

for $\alpha \in Q^*(\sqrt{n})$, if $\bar{\alpha} \in \alpha^G$ then, for all $\beta \in \alpha^G$, $\bar{\beta} \in \alpha^G$.

ii) Similarly for $\alpha \in Q^*(\sqrt{n})$, if $-\alpha \in \alpha^G$ then, for all $\beta \in \alpha^G$, $-\beta \in \alpha^G$.

iii) It then follows, by corollary 2.5 that

for all $\beta \in \alpha^G$, $-\bar{\beta} \in \alpha^G$.

Lemma 2.7 [4]

For each $\alpha \in Q^*(\sqrt{n})$ the ambiguous length of α and $\bar{\alpha}$ is the same.

Lemma 2.8

Let G be the modular group. Then for each $\alpha \in Q^*(\sqrt{n})$ we have:

$$x(\alpha) \neq \pm\alpha, \quad y(\alpha) \neq \pm\alpha, \quad y^2(\alpha) \neq \pm\alpha, \quad yx(\alpha) \neq \pm\alpha, \quad y^2x(\alpha) \neq \pm\alpha$$

$$x(\alpha) \neq -\bar{\alpha}, \quad y(\alpha) \neq \pm\bar{\alpha}, \quad y^2(\alpha) \neq \pm\bar{\alpha}, \quad yx(\alpha) \neq \bar{\alpha}, \quad y^2x(\alpha) \neq \bar{\alpha}$$

and

$$x(\alpha) = \bar{\alpha} \Leftrightarrow N(\alpha) = \frac{a^2 - n}{c^2} = -1 \Leftrightarrow n = a^2 + c^2$$

Lemma 2.9 [4]

For $\alpha \in Q^*(\sqrt{n})$ let $N(\alpha) = \alpha\bar{\alpha} = -1$, then $\alpha^G = (\bar{\alpha})^G$

The converse of lemma 2.9 is false. That is, if $\alpha^G = (\bar{\alpha})^G$, then $N(\alpha)$ may or may not be -1. For example $(\sqrt{2})^G = (-\sqrt{2})^G$, but $N(\sqrt{2}) = -2 \neq -1$.

However, the following theorem gives a necessary and sufficient condition for the orbits α^G and $(\bar{\alpha})^G$ to be identical or disjoint.

Lemma 2.10

Let $\alpha \in Q^*(\sqrt{n})$. Then $\alpha^G = (\bar{\alpha})^G$ if and only if there exists an element β in α^G such that $\beta\bar{\beta} = -1$.

Proof

Let $\beta \in \alpha^G$ such that $\beta\bar{\beta} = -1$. Then $\bar{\beta} = \frac{-1}{\beta} = x(\beta)$ and $x(\bar{\beta}) = \beta$. So $\beta \in (\bar{\beta})^G$ and $\bar{\beta} \in \beta^G$. As $\beta \in \alpha^G$ so α^G and β^G are not disjoint. Also $\beta^G = (\bar{\beta})^G$ and hence, by lemma 2.4, $\alpha^G = (\bar{\alpha})^G$.

Conversely suppose that $\alpha^G = (\bar{\alpha})^G$. We have to prove that there exists β in α^G such that $\beta\bar{\beta} = -1$. That is, $x(\beta) = \frac{-1}{\beta} = \bar{\beta}$

As $\alpha^G = (\bar{\alpha})^G$ so α^G contains the conjugate of each of its element. Hence, by lemma 2.4 and theorem 2.2, there exists β in α^G such that at least one of the

numbers

$$x(\beta), y(\beta), y^2(\beta), yx(\beta) \quad \text{and} \quad y^2x(\beta)$$

is equal to $\bar{\beta}$

But we know that none of

$$y(\beta), y^2(\beta), yx(\beta) \quad \text{and} \quad y^2x(\beta)$$

is equal to $\bar{\beta}$ for all $\beta \in Q^*(\sqrt{n})$.

Hence $x(\beta) = \bar{\beta}$. That is, $\beta\bar{\beta} = -1$.

Note: For any $\alpha \in Q^*(\sqrt{n})$, the following statements are obviously equivalent.

1. α^G and $(\bar{\alpha})^G$ are disjoint orbits of $Q^*(\sqrt{n})$, if and only if there does not exist an element β in α^G such that $\beta\bar{\beta} = -1$.
2. $\alpha^G = (\bar{\alpha})^G$ if and only if there exists an element β in α^G such that $\beta\bar{\beta} = -1$.
3. α^G and $(\bar{\alpha})^G$ are disjoint orbits of $Q^*(\sqrt{n})$ if and only if $\beta\bar{\beta} \neq -1$ for all $\beta \in \alpha^G$.
4. $\alpha^G = (\bar{\alpha})^G$ if and only if $\beta\bar{\beta} = -1$ for some $\beta \in \alpha^G$.

Lemma 2.11

Let p be an odd rational prime and $\alpha = \frac{a+\sqrt{p}}{c}$ be an ambiguous number in $Q^*(\sqrt{p})$. Then:

1. $y^2x(\alpha) = -\bar{\alpha} \Leftrightarrow \alpha = \frac{1+\sqrt{p}}{c}$, or $\frac{-1+\sqrt{p}}{-c}$, where $p = 1 + 2c$.
2. $yx(\alpha) = -\bar{\alpha} \Leftrightarrow \alpha = \frac{1+\sqrt{p}}{-2}$, or $\frac{-1+\sqrt{p}}{2}$.

Proof

1. Let $\alpha = \frac{a+\sqrt{p}}{c} \in Q^*(\sqrt{p})$. Then α is an ambiguous number $\Leftrightarrow a^2 < p$. Now

$$y^2x(\alpha) = \bar{\alpha} \Leftrightarrow \frac{a+b+\sqrt{p}}{2a+b+c} = \frac{-a+\sqrt{p}}{c}, \quad b = \frac{a^2-p}{c}$$

$$\begin{aligned} \Leftrightarrow a + b &= -a, & 2a + b + c &= c \\ \Leftrightarrow b &= -2a \end{aligned} \tag{1}$$

We know that $\frac{a+\sqrt{p}}{c} \in Q^*(\sqrt{p}) \Leftrightarrow \frac{a+\sqrt{p}}{b} \in Q^*(\sqrt{p})$. But this is possible only if $-2a|(a^2 - p)$.

i.e. if $(a^2 - p)$ is even and $a|(a^2 - p)$.

As p is odd a must be odd and so $a|p$ (because $a|(a^2 - p), a|a^2$)

This is $a = \pm 1$ (because $a^2 < p$ and p is a rational prime)

Hence by equation (1) $b = \mp 2$

Now $\frac{1-p}{c} = \mp 2 \Leftrightarrow p = 1 \pm 2c$

$a^2 - n = bc < 0$ (because α is ambiguous number) $\Leftrightarrow b$ and c have opposite signs $\Leftrightarrow a$ and c have same signs (because a, b have opposite signs, by equation (1)).

So c is positive or negative according as $a = 1$ or -1 .

Moreover c is even or odd according as $p \equiv 1$ or $3 \pmod{4}$. Hence

$$\alpha = \frac{1 + \sqrt{p}}{c} \quad \text{or} \quad \frac{-1 + \sqrt{p}}{-c} \quad \text{where} \quad p = 1 + 2c$$

This completes the first part of the lemma.

2. Proof is analogous to the proof of 1 of lemma 2.11.

Remark 2.12

Let

$$\alpha = \frac{a + \sqrt{n}}{c} \in Q^*(\sqrt{n}).$$

Then

$$\alpha = -\bar{\alpha} \Leftrightarrow \frac{a + \sqrt{n}}{c} = \frac{-a + \sqrt{n}}{c}$$

$$\Leftrightarrow \alpha = \frac{\sqrt{n}}{c}$$

Obviously such numbers of $Q^*(\sqrt{n})$ are all ambiguous. Moreover the number of such ambiguous numbers is $2\pi(n)$. In particular, if $n = p$, then the elements $\pm\sqrt{p}, \frac{\pm\sqrt{p}}{p}$ of $Q^*(\sqrt{p})$ are the only such numbers.

Also there is no α in $Q^*(\sqrt{n})$ such that $\alpha = \bar{\alpha}$ or $\alpha = -\alpha$. In particular, there is no α in $Q^*(\sqrt{p})$ such that $\alpha = \bar{\alpha}$ or $\alpha = -\alpha$.

3. THE ORBITS OF $q^*(\sqrt{p}), P \equiv 3 \pmod{4}$ UNDER THE ACTION OF THE MODULAR GROUP $G = \langle x, y : x^2 = y^3 = 1 \rangle$

This section is concerned with the determination of number of orbits of $Q^*(\sqrt{p}), p \equiv 3 \pmod{4}$, under the action of G . In contrast with the action of G on $Q^*(\sqrt{2})$ we prove that G does not act transitively on $Q^*(\sqrt{p}), p \equiv 3 \pmod{4}$.

Theorem 3.1

Let $p \equiv 3 \pmod{4}$ be a rational prime. Then $Q^*(\sqrt{p})$ splits into exactly two disjoint orbits under the action of G . These are precisely $(\sqrt{p})^G$ and $(-\sqrt{p})^G$.

Proof

Since $p \equiv 3 \pmod{4}$, so $p \neq a^2 + c^2$, $a, c \in \mathbb{Z}$, by theorem 2.1. This implies that $\frac{a^2 - p}{c^2} \neq -1, \forall a, c \in \mathbb{Z}$. Hence if $\beta = \frac{a + \sqrt{p}}{c} \in Q^*(\sqrt{p})$ then $\beta\bar{\beta} \neq 1$. This shows that $\bar{\beta} \neq \frac{-1}{\beta}x(\beta)$ for all $\beta \in Q^*(\sqrt{p})$. Therefore, by lemma 2.10, α^G and $\bar{\alpha}^G$ are disjoint orbits of $Q^*(\sqrt{p})$ under the action of G .

We prove that there are exactly two distinct closed paths of ambiguous numbers of $Q^*(\sqrt{p})$ in the coset diagram under the action of G .

By the remark 2.12, if γ is one of the numbers $\pm\sqrt{p}$ and $\frac{\pm\sqrt{p}}{p}$ then $-\bar{\gamma} = \gamma$ and no other element of $Q^*(\sqrt{p})$ satisfies this condition. Also there is no α in $Q^*(\sqrt{p})$ such that α is equal to $\bar{\alpha}$ or $-\alpha$.

As

$$x\left(\frac{\pm 1 + \sqrt{p}}{c}\right) = \frac{\mp 1 + \sqrt{p}}{-2}, \quad x\left(\frac{\pm 1 + \sqrt{p}}{-c}\right) = \frac{\mp 1 + \sqrt{p}}{2}$$

and

$$\frac{-1 + \sqrt{p}}{2}, \frac{1 + \sqrt{p}}{-2}$$

are conjugates of one another, similarly

$$\frac{1 + \sqrt{p}}{c}$$

and

$$\frac{-1 + \sqrt{p}}{-c}$$

are conjugate of one another, so exactly one of the sets

$$\left\{ \frac{\pm 1 + \sqrt{p}}{-c}, \frac{\pm 1 + \sqrt{p}}{2} \right\}$$

and

$$\left\{ \frac{\pm 1 + \sqrt{p}}{c}, \frac{\pm 1 + \sqrt{p}}{-2} \right\}$$

is contained in $(\sqrt{p})^G$ and the other is contained in $(-\sqrt{p})^G$.

Hence $(\sqrt{p})^G$ and $(-\sqrt{p})^G$ have unique closed paths of ambiguous numbers. These paths are shown in figure 3.1 and 3.2.

Furthermore if $\alpha = \frac{a + \sqrt{p}}{c}$ is an ambiguous number of $Q^*(\sqrt{p})$ then we apply

$$yx(\alpha) = \frac{(a + c) + \sqrt{p}}{c}$$

or

$$xy^2(\alpha) = \alpha - 1 = \frac{(a + c) + \sqrt{p}}{c}$$

according as $(a + c)$ or $(a - c)$ is $\leq [\sqrt{p}]$.

As we have exactly four elements of $Q^*(\sqrt{p})$ satisfying $-\bar{\alpha} = \alpha$, exactly two elements of $Q^*(\sqrt{p})$ satisfy $yx(\alpha) = -\bar{\alpha}$ and exactly two elements of $Q^*(\sqrt{p})$ satisfy $y^2x(\alpha) = -\bar{\alpha}$ for $\alpha \in Q^*(\sqrt{p})$. So there are exactly two distinct closed paths in the coset diagram under the action of G on $Q^*(\sqrt{p})$ and hence, by remark 2.3, $Q^*(\sqrt{p})$ splits into exactly two disjoint orbits under the action of G . They are precisely $(\sqrt{p})^G$ and $(-\sqrt{p})^G$. the figures 3.1 and 3.2 show the paths of these disjoint orbits.

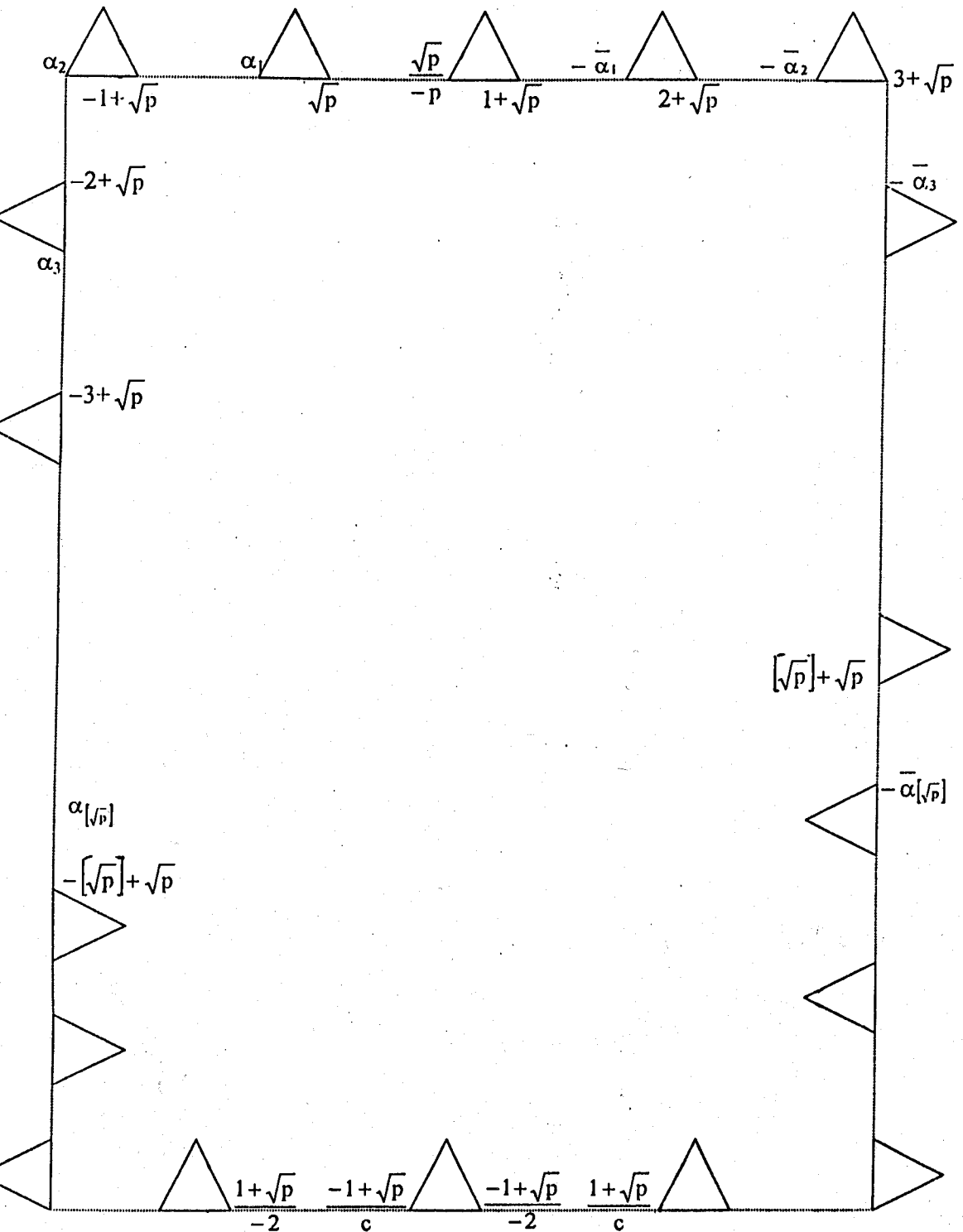


Figure 3.1: Closed path in the coset diagram for the orbit $(\sqrt{p})^G$

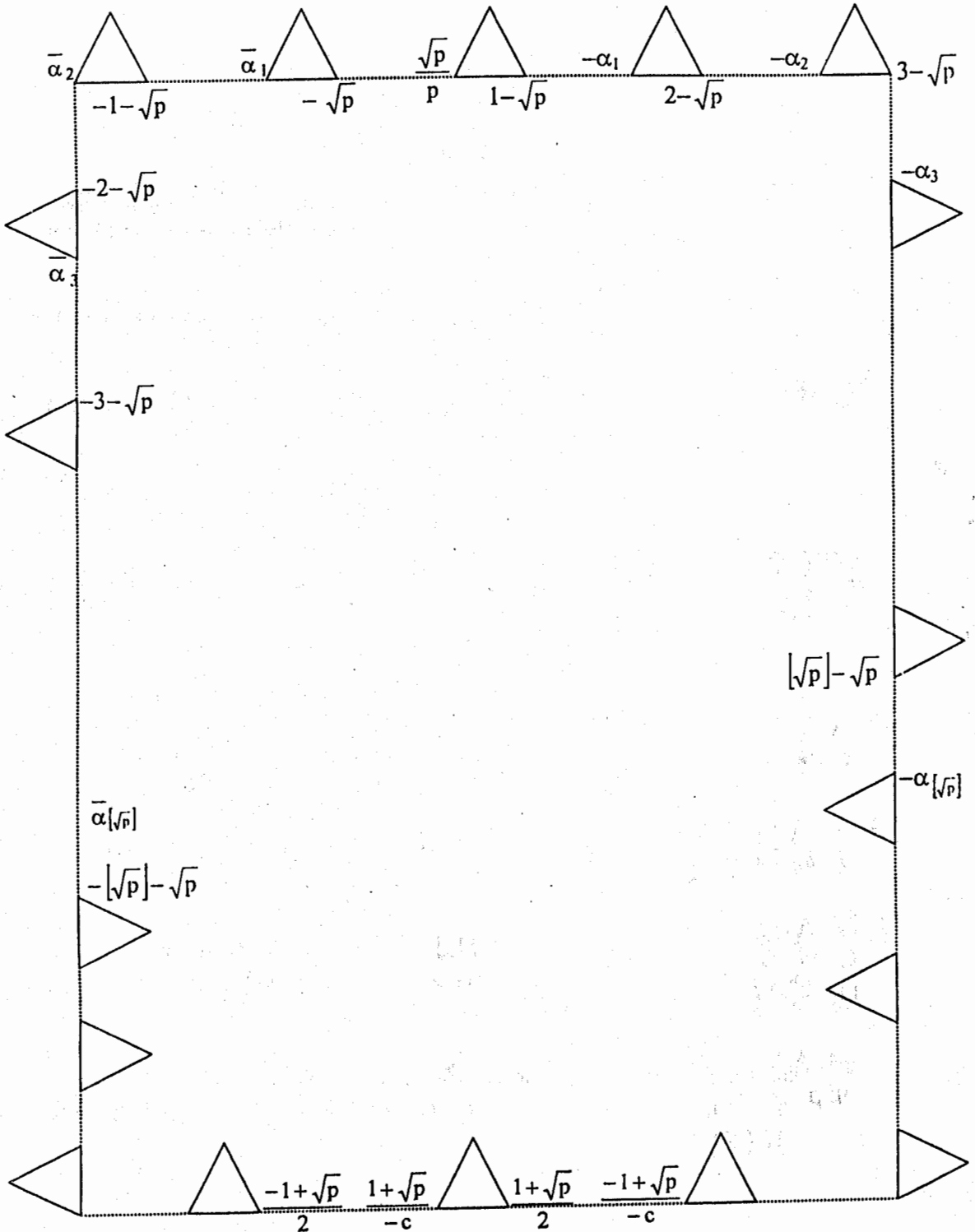


Figure 32: Closed path in the coset diagram for the orbit $(-\sqrt{p})^G$

Remarks 3.2

1. It was proved in [2] that the number of ambiguous numbers in $Q^*(\sqrt{p})$ is $\pi^*(p)$ so, by theorem 3.1, for $p \equiv 3 \pmod{4}$, $(\sqrt{p})^G$ and $(-\sqrt{p})^G$ both have ambiguous length $\frac{1}{2}\pi^* = \pi(p) + \sum_{a=1}^{[\sqrt{p}]} \pi(p - a^2)$

2. In [3] we showed that number of ambiguous integers of $Q^*(\sqrt{p})$ is $2 + 4[\sqrt{p}]$. So by theorem 3.1, for $p \equiv 3 \pmod{4}$, $(\sqrt{p})^G$ have exactly $1 + 2[\sqrt{p}]$ ambiguous integers.

3. The action of G on $Q^*(\sqrt{2})$ is transitive, whereas it is not so on $Q^*(\sqrt{p}), p$ an odd rational prime.

4. $Q^*(\sqrt{p}), p \equiv 1 \pmod{4}$, splits into exactly two disjoint orbits namely $(\sqrt{p})^G = (-\sqrt{p})^G$ and

$$\left[\frac{1 - \sqrt{p}}{2} \right]^G = \left[\frac{1 + \sqrt{p}}{2} \right]^G$$

where $Q^*(\sqrt{p}), p \equiv 3 \pmod{4}$, splits into exactly two disjoint orbits namely $(\sqrt{p})^G$ and $(-\sqrt{p})^G$.

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VARIANCE OF THE SAMPLE MEAN DUE TO CODERS

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Abstract

Total Variance of the sample mean for simple random sample based on the model for coder's effects has been worked out.

1. INTRODUCTION

A commonly used method for obtaining data, to be used for statistical purposes, is a sample survey. It is, however, true that all surveys are subjected to some kind of error and some are quite misleading. The errors present in surveys can be divided into two main groups: sampling errors and non-sampling errors. Sampling errors decrease as the sample size increases. On the other hand, non-sampling errors are likely to increase as the sample size increases. In some surveys non-sampling errors are very large and interpretation of the results without taking these errors into account may be dangerously misleading. A list of errors in surveys is given by Deming[1].

Although statisticians had realized, very early in the development of sampling theory, the importance of non-sampling errors, not enough efforts were made, until recently, to try to find out the extent to which these errors may impair the result of a survey based on a good sample or even the result of some complete enumerations. Hansen and Waksberg [2] have pointed out this problem. Martin Collins and Graham Kalton [3] pointed out that the coder's reliability is affected by their work load. Martin Collin and Gill Courtenay [4] have suggested that field coding has some advantage over office coding. Durbin and Stuart [5] compared the reliabilities of different coders. Crittenden and Hill [6] studied the effect open question on coding. Martin Collins [7] found that the coder's reliability is affected by the type of question.

2. MODEL FOR CODERS EFFECTS

One of the sources of errors, apart from the respondent, is coder's effects. We shall consider the model for coder's effects only i.e. it is assumed that there are no interviewer's effects. Examples are mail surveys in which the respondent is himself an enumerator or a survey in which only one interviewer is used to interview all the respondents. In mail survey there are no interviewer's effects but in the case of one interviewer, the interviewer's effects cannot be estimated. The model for coder's effects only is given as

$$Y_{hir} = Y_i + \beta_h + \eta_{hir} \quad (2.1)$$

$$(h = 1, 2, \dots, t; \quad i = 1, 2, \dots, n'; \quad r = 1, 2, \dots, R)$$

where Y_{hir} denotes the value coded for the i -th form for a hypothetical response variable, by the h -th coder on r -th occasion. Further let the reported response Y_i for the i -th unit be the true response (i.e. no response error), β_h is the h -th coder's effect; and η_{hir} is a purely random component which shows the effects of unexplained factors.

3. VARIANCE OF THE SAMPLE MEAN

The variance of the estimator depends on the sample design and the coder's allocation. The sampling scheme considered here is simple random sampling.

Assume that n units are drawn, with equal probability, from a population of N units and the values recorded for the selected units are true values. Further let

t coders be randomly selected from an infinitely large population of coders and each selected coder codes $n' (= n/t)$ randomly assigned forms. Denoting by $\hat{Y}_{h,r}$, and $\hat{Y}_{..r}$ the mean of coded forms coded by the h-th coder on r-th occasion and the overall mean of the coded forms on the r-th occasion respectively, we have,

$$\hat{Y}_{h,r} = \frac{1}{n'} \sum_{i=1}^{n'} Y_{hir}$$

and

$$\hat{Y}_{..r} = \frac{1}{n't} \sum_{h=1}^t \sum_{i=1}^{n'} Y_{hir}$$

Substituting from 2.1, we have

$$\hat{Y}_{..r} = \frac{1}{n't} \sum_{h=1}^t \sum_{i=1}^{n'} Y_i I_{hi} + \frac{1}{t} \sum_{h=1}^t \beta_h + \frac{1}{n't} \sum_{h=1}^t \sum_{i=1}^{n'} \eta_{hir} I_{hi} \quad (3.1)$$

where I_{hi} ($I_{hi} = 1$ if the i-th unit is coded by h-th coder, for a given sample of n forms; otherwise it is zero) indicates that which forms are assigned to the h-th coder, for a given sample of n forms.

$$E_n(I_{hi}) = \frac{n}{n'}; \text{Var}(I_{hi}) = \frac{n}{n'} \left(1 - \frac{n}{n'}\right)$$

and

$$\text{Cov}(I_{hi}) = \frac{n}{n'} \left(1 - \frac{n}{n'}\right) \left(\frac{1}{n-1}\right)$$

Let E_2 and V_2 denote the expectation and variance respectively over repeated coding of the i-th form by the h-th coder; E_1 and V_1 denote the expectation and variance respectively over coders (it includes both the selection and assignment of the coders); E_n and V_n denote the expectation and variance respectively over all possible partitions of a given sample of size n into random sub-samples; E_N and V_N denote the expectation and variance respectively over all possible samples which can be drawn from a population of N units; and E_p denote the overall expectation. Then,

$$E_2(\hat{Y}_{..r}) = \frac{1}{n't} \sum_{h=1}^t \sum_{i=1}^{n'} Y_i I_{hi} + \frac{1}{t} \sum_{h=1}^t \beta_h + \frac{1}{n't} \sum_{h=1}^t \sum_{i=1}^{n'} I_{hi} E_2(\eta_{hir})$$

$$= \frac{1}{n't} \sum_{h=1}^t \sum_{i=1}^{n'} Y_i I_{hi} + \frac{1}{t} \sum_{h=1}^t \beta_h \quad (3.2)$$

$$\begin{aligned} E_1 E_2 (\hat{Y}_{..r}) &= E_1 \left(\frac{1}{n't} \sum_{h=1}^t \sum_{i=1}^{n'} Y_i I_{hi} + \frac{1}{t} \sum_{h=1}^t \beta_h \right) \\ &= \frac{1}{n} \sum_{k=1}^t \sum_{i=1}^{n'} Y_i I_{hi} \end{aligned} \quad (3.3)$$

$$\begin{aligned} E_n E_1 E_2 (\hat{Y}_{..r}) &= E_n \left(\frac{1}{n't} \sum_{h=1}^t \sum_{i=1}^{n'} Y_i I_{hi} \right) \\ &= \frac{1}{n} \sum_{i=1}^n Y_i \end{aligned} \quad (3.4)$$

$$E_N E_n E_1 E_2 (\hat{Y}_{..r}) = E_N \left(\frac{1}{n} \sum_{i=1}^n Y_i \right) \frac{1}{N} \sum_{i=1}^N Y_i = \bar{Y}$$

Thus $E_p(\hat{Y}_{..r}) = \bar{Y}$

Let $\text{Var}(\hat{Y}_{..r})$ denote the variance of the sample mean, then

$$\text{Var}(\hat{Y}_{..r}) = V_N E_n E_1 E_2 (\hat{Y}_{..r}) + E_N V_n E_1 E_2 (\hat{Y}_{..r}) + E_N E_n V_1 E_2 (\hat{Y}_{..r}) + E_N E_n E_1 V_2 (\hat{Y}_{..r})$$

Using 3.1 and 3.2-4 we have

$$\begin{aligned} \text{Var}(\hat{Y}_{..r}) &= V_N \left(\frac{1}{n't} \sum_{h=1}^t \sum_{i=1}^{n'} Y_{hir} \right) + E_N V_n \left(\frac{1}{n} \sum_{h=1}^t \sum_{i=1}^{n'} Y_i I_{hi} \right) \\ &+ E_N E_n V_1 \left(\frac{1}{n't} \sum_{h=1}^t \sum_{i=1}^{n'} Y_i I_{hi} + \frac{1}{t} \sum_{h=1}^t \beta_h \right) \\ &+ E_N E_n E_1 V_2 \left(\frac{1}{n't} \sum_{h=1}^t \sum_{i=1}^{n'} Y_i I_{hi} + \frac{1}{t} \sum_{h=1}^t \beta_h + \frac{1}{n't} \sum_{h=1}^t \sum_{i=1}^{n'} \eta_{hir} I_{hi} \right) \\ &= \frac{1}{t} \left(\frac{N-n}{N.n} \right) S_Y^2 + \frac{1}{t} \left(\frac{n-n'}{n.n'} \right) E_N (S_Y^2) + E_N E_n \left(\frac{\sigma_\beta^2}{t} \right) + E_N E_n E_1 \left(\frac{\sigma_\eta^2}{n} \right) \\ &= \frac{1}{t} \left(\frac{N-n'}{N.n'} \right) S_Y^2 + \frac{\sigma_\beta^2}{t} + \frac{\sigma_\eta^2}{n} \end{aligned}$$

Assuming that N is large relative to n , we have

$$\text{Var}(\hat{Y}_{..r}) = \frac{1}{n} S_Y^2 + \frac{\sigma_\beta^2}{t} + \frac{\sigma_\eta^2}{n}$$

This is the total variance of the sample mean, based on the model for the coder's effects, when a simple random sample of size n is drawn from a large population of size N .

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BÉZIER LIKE QUARTIC CURVES WITH APPLICATIONS TO SWEEP AND SWUNG SURFACES

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Key Words: blending, rational, swept, swung surfaces.

ABSTRACT

In this paper, we will discuss the Bézier like quartic curves, which are an extension of Bézier like cubic curves. These curves will be used to generate swept and swung surfaces. A desirable feature of the Bézier like quartic curves is the extra shape parameter that will be useful in designing these type of surfaces.

1. INTRODUCTION

In curve design it is desirable to have curves where the mathematical and geometrical properties can be easily understood. Normally, for a simple CAD system curves are in polynomial forms and perhaps the cubic is the most popular. However for more shape design cubic curve is not enough. To add more shape parameters we extend the representation by blending two cubic curves. The blending of two cubic curves in order to add shape parameters is discussed in Jamaludin *et al* (1995). There are also others means of curve representation for example in the rational form to create more shape parameters for designing purposes as discussed by Sarfraz (1993, 1994, 1995, 2003). Our paper will only deal with non-rational curves and we feel that it is sufficient to generate the required surfaces. That is we will apply these curves to create swept and swung

surfaces. The method of swept surfaces have been discussed quite extensively by a number of researchers to state a few such as Choi (1991), Bloomenthal M *et. al* (1991), Coquillart S (1987), Wang (1999) and Wang (2001).

2. BEZIER LIKE QUARTIC CURVES

Given four control points $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2$ and \mathbf{P}_3 a segment of Bézier cubic curve is given by

$$r(t) = (1-t)^3\mathbf{P}_0 + 3(1-t)^2t\mathbf{P}_1 + 3(1-t)t^2\mathbf{P}_2 + t^3\mathbf{P}_3; \quad 0 \leq t \leq 1 \quad (1)$$

With this definition we can modify the shape of the curve by adjusting the control points. If the end points are to be interpolated then only the inner control points \mathbf{P}_1 and \mathbf{P}_2 can be adjusted to obtain a required shape. This may not be easy if the end directions are fixed. If the end points and end tangents are given then a cubic curve in the Hermite form is given by

$$r(t) = (1-t)^2(1+2t)\mathbf{P}_0 + t(1-t)^2\mathbf{m}_0 + t^2(t-1)\mathbf{m}_1 + t^2(3-2t)\mathbf{P}_3; \quad 0 \leq t \leq 1 \quad (2)$$

where \mathbf{m}_0 and \mathbf{m}_1 are the end tangents, \mathbf{P}_0 and \mathbf{P}_3 are the end points. We found out that a cubic curve in the Hermite form is not suitable for our purpose of designing swept and swung surfaces. Thus a modification in defining a cubic curve is made to cater our needs. If we let the end tangents be written in the form of

$$\mathbf{m}_0 = \beta(\mathbf{P}_1 - \mathbf{P}_0)$$

$$\mathbf{m}_1 = \beta(\mathbf{P}_3 - \mathbf{P}_2)$$

and substitute back into the Hermit form and rearranged as a combination of $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2$ and \mathbf{P}_3 we obtain

$$r(t) = F_0(t)\mathbf{P}_0 + F_1(t)\mathbf{P}_1 + F_2(t)\mathbf{P}_2 + F_3(t)\mathbf{P}_3; \quad 0 \leq t \leq 1 \quad (3)$$

where

$$F_0(t) = (1-t)^2(1+t(2-\alpha))$$

$$F_1(t) = \alpha(1-t)^2t$$

$$F_2(t) = \beta(1-t)t^2$$

$$F_3(t) = t^2(1+(1-t)(2-\beta))$$

with α and β are real numbers.

With this representation, we can modify the shape of the curve by adjusting the values of α and β . We can show that if $\alpha = 3$ and $\beta = 3$ the cubic curve is in the Bézier form. Suppose we have constructed a cubic Bézier curve and need to modify its shape by keeping its end directions then it is much easier to adjust the values of α and β instead of the control points. The following figures illustrate the cubic curves using different values of α and β without changing the control points.

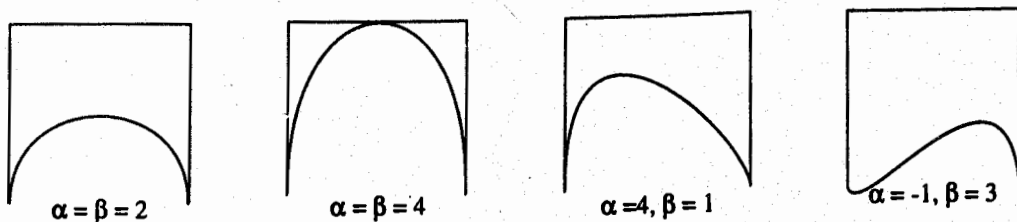


Figure 1

However with the cubic representation the availability of shape is not wide enough. For example, a single segment of a cubic curve cannot generate a flat curve such as the cross section of a car roof. With this limitation, we consider of blending two cubic curves to get extra shape parameters. The blending is discussed in Jamaludin *et al* (1995) and Lawrence (1997). Jamaludin linear functions and Lawrence use trigonometric functions to blend the cubic curves. In this paper we use the method proposed by Jamaludin, because it is simpler.

3. BLENDING TWO CUBIC CURVES

Suppose $r_1(t)$ and $r_2(t)$ are two cubic curves given by

$$r_1(t) = F_{10}(t)\mathbf{P}_0 + F_{11}(t)\mathbf{P}_1 + F_{12}(t)\mathbf{P}_2 + F_{13}(t)\mathbf{P}_3; \quad 0 \leq t \leq 1 \quad (4)$$

where

$$\begin{aligned} F_{10}(t) &= (1-t)(1+t(2-\alpha_1)) \\ F_{11}(t) &= \alpha_1(1-t)^2t \\ F_{12}(t) &= \beta_1(1-t)t^2 \\ F_{13}(t) &= t^2(1+(1-t)(2-\beta_1)) \end{aligned}$$

and

$$\mathbf{r}_2(t) = F_{20}(t)\mathbf{P}_0 + F_{21}(t)\mathbf{Q}_1 + F_{22}(t)\mathbf{Q}_2 + F_{23}(t)\mathbf{P}_3; \quad 0 \leq t \leq 1 \quad (5)$$

where

$$\begin{aligned} F_{20}(t) &= (1-t)^2(1+t(2-\alpha_2)) \\ F_{21}(t) &= \alpha_2(1-t)^2t \\ F_{22}(t) &= \beta_2(1-t)t^2 \\ F_{23}(t) &= t^2(1+(1-t)(2-\beta_2)). \end{aligned}$$

Then we define a blending curve $r(t)$ as

$$\mathbf{r}(t) = (1-t)\mathbf{r}_1(t) + t\mathbf{r}_2(t); \quad 0 \leq t \leq 1 \quad (6)$$

The curve defined by equation (6) is a quartic curve. We will call this curve a Bézier like quartic curve. To make it easier for designing purposes, we let $\alpha_1 = \alpha$, $\beta_2 = \beta$ and $\beta_1 = \alpha_2 = \gamma$. With this assumption, we call α and β the tension parameters while γ is called the shape parameter of the curve. In Figure 2, the dashed curves are the cubic curves with their respective control polygons and the solid curve is a Bézier like quartic curve. We can see that the initial tangent direction of the quartic curve depends on the first cubic curve and its end tangent direction depends on the second cubic curve. In Figure 3, we illustrate the effect of the parameters on the shape of the curve. In this illustration we fixed the values of α and β equal to 4, but with different values of γ . The top dashed curve has a value of $\gamma = 2$, and the bottom one $\gamma = -1$. Solid curves from bottom to top have the values of γ from zero to one.

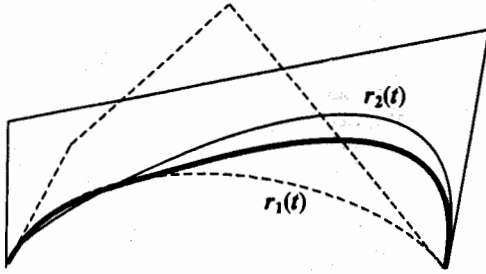


Figure 2: Two cubic curves and their blend

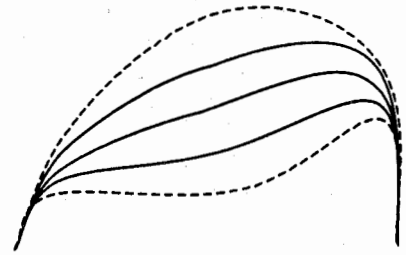


Figure 3: Different Bézier quartic like curves

However the number of control points given are redundant. We require only five control points, which can be illustrated in the following proposition.

Proposition

Suppose $r(t)$ is being defined by P_0, P_1, P_2, P_3 and P_0, Q_1, Q_2, P_3 and $r(t)$ by P_0, P_1, P^*, P_3 and P_0, P^*, Q_2, P_3 with the same tension and shape parameters, $r^*(t)$ and $r(t)$ are the same curve if and only if P^* is the midpoint of P_2 and Q_1 .

Proof

Considering $r(t)$ and $r^*(t)$, and we take the difference then we have

$$\gamma(1-t)^2t^2(P_2 + Q_1) = \gamma(1-t)^2t^2(2P^*). \text{ Hence the two curves are the same if and only if } (P_2 + Q_1) = 2P^* \text{ or } P^* \text{ is the midpoint of } P_2 \text{ and } Q_1.$$

From the proposition we only require five control points P_0, P_1, P^*, Q_2, P_3 to generate a Bézier like quartic curve. We can also note that if P^* is the midpoint of P_1 and Q_2 , the curve reduces to the one proposed by Jamaludin. It can be shown that the curve reduces to a Bézier quartic curve if $\alpha = \beta = 4$ and $\gamma = 3$. Figure 3 illustrates different Bézier like quartic curves with five control points.

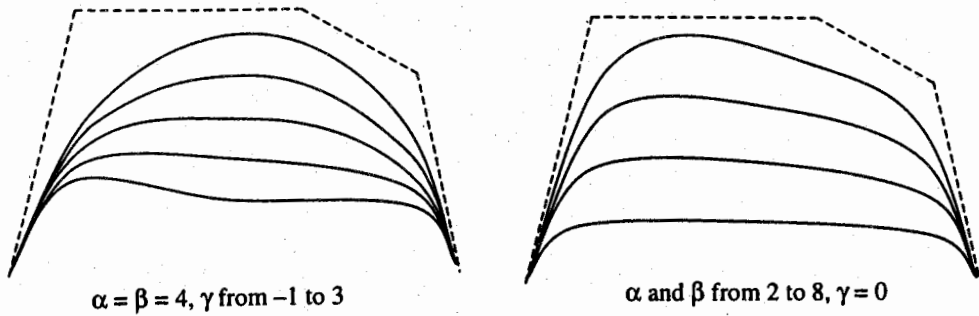


Figure 3: Different shapes of curves with different parameters

4. SWEEP AND SWUNG SURFACES

From the above examples of curve design, we propose a simple surface design similar to swept and swung techniques where the cross sectional and the spline curve are the Bézier like quartic curves given by equation (6). Surface design by swept and swung techniques are quite common in the die and mould industry.

A swept surface is given by

$$\mathbf{S}(u, v) = \mathbf{r}(v) + C_1(u, v)\mathbf{B} + C_2(u, v)\mathbf{N} \quad (7)$$

where \mathbf{r} is the spine or the trajectory of the surface, C_1 and C_2 are the planar contours. Vectors \mathbf{N} and \mathbf{B} are unit vectors that form the moving frame perpendicular to the spine. We are not using equation (7) for the swept surface. Our approach is easier. What we do is to place the control points of the Bézier like quartic curves on curves that we refer as control curves. Let us assume that the control curves are in parameter u . That is the control points vary in the u -direction. With a particular u , we construct Bézier like quartic curves with parameter v .

If the control curves are assigned by $q_0(u)$, $q_1(u)$, $q_2(u)$, $q_3(u)$ and $q_4(u)$, then the surface $\mathbf{S}(u, v)$ is given by

$$\mathbf{S}(u, v) = (1 - v)\mathbf{S}_1(u, v) + v\mathbf{S}_2(u, v), \quad 0 \leq u, v \leq 1 \quad (8)$$

where

$$S_1(u, v) = F_{10}(v)q_0(u) + F_{11}(v)q_1(u) + F_{12}(v)q_2(u) + F_{13}(v)q_4(u)$$

and

$$S_2(u, v) = F_{20}(v)q_0(u) + F_{21}(v)q_1(u) + F_{22}(v)q_3(u) + F_{23}(v)q_4(u)$$

The blending functions $F_{ji}(v)$, $j = 1, 2$ and $i = 0, 1, 2, 3$ are given by equation (4) and (5). The shape parameters are α, β and γ which can vary with respect to u .

Figure 4 illustrates the idea. In this illustration $q_0(u), q_1(u), q_2(u)$ are the same curves.

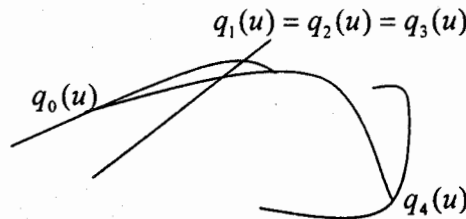
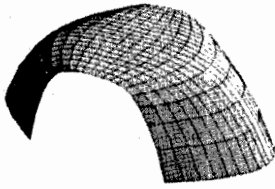
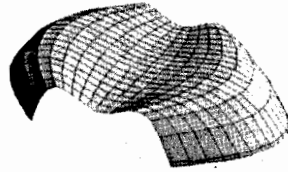
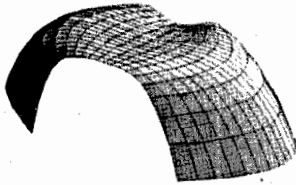
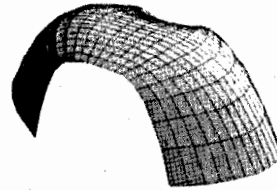


Figure 4: The control curves and cross sectional curve.

We give some examples of the proposed surface. Let fix α and β to be 4, we will illustrate how a surfaces can be designed with variable γ . Figure 5 illustrates the examples of swept surfaces. In Figure 5a, the value of γ is a constant zero. In Figure 5b, the value of γ is -2. In Figure 5c γ varies linearly with respect to u from 0 to -2, that is $\gamma(u) = -2u$. In Figure 5d, we have $\gamma(u) = -2u^2 + 2u(1 - u)$. In fact we can use γ as any function with respect to u .

Figure 5a. γ is zeroFigure 5b. γ is -2Figure 5c. g varies linearlyFigure 5c. γ varies quadraticallyFigure 5. swept surfaces with different value of γ .

In the case of the swung surface we rotate the control points and then construct the Bézier like quartic curves. For the swung surface we kept the end points fixed but the middle point revolved with respect to the axis made by the end points, for simplicity we let the axis be the z -axis, then the swung surface $S(u, v)$ is given by equation (8) with $q_1(u), q_2(u), q_3(u)$ given by

$$q_1(u) = q_2(u) = q_3(u) = (a \cos(2\pi u), a \sin(2\pi u), b)$$

a and b are positive real numbers.

Figure 6 illustrates examples of swung surfaces obtained with different values of γ .

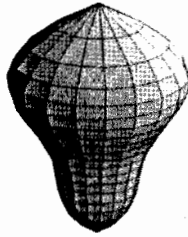


Figure 6 a: γ is -1

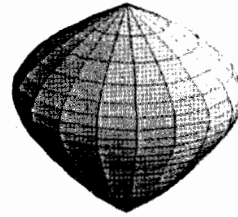


Figure 6b: γ is 1

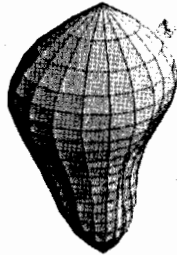


Figure 6c: γ varies but have same end values

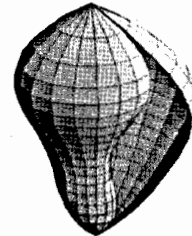


Figure 6d: end γ of different values

Figure 6: swung surfaces with different values of γ

5. CONCLUSION

Using blending technique, we have derived an alternative quartic curve definition to the Bézier. The new quartic curve representation has additional shape parameters that may be desirable in CAD environment. The additional shape parameters will be useful in curve and surface design because the Bézier like quartic curve is easy to control as compared to the Bézier quartic curve. By

assigning the parameter γ as a function we can generate interesting swept and swung surfaces which may be useful for surface design needed in the die and mould making. Eventhough the proposed method of curve and surface design easy to comprehend, other techniques of curve design such as rational cubics, b-splines or even NURBS are worth to be investigated for their suitability for swept and swung surfaces. These curves are mathematically more sophisticated but offer a lot of flexibility that may be desirable to curve and surface design.

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DISTORTION THEOREM AND KEOBE DOMAIN FOR STARLIKE FUNCTIONS OF COMPLEX ORDER

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Abstract

Let $S(1 - b)$ ($b \neq 0$, complex), denotes the class of functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots \quad \text{in } D = \{z \mid |z| < 1\}$$

which satisfies for $z = r e^{i\theta} \in D$, and

$$\operatorname{Re} \left[1 + \frac{1}{b} \left(z \frac{f'(z)}{f(z)} - 1 \right) \right] > 0, \quad z \in D$$

Then $f(z)$ is said to be starlike functions of complex order b . The author gives the new distortion theorem for the class of starlike functions of complex order. This result is used to obtain the Koebe domain for the same class under the conditions.

$$|b| \leq \frac{\operatorname{Re} \left(\frac{1}{b} \right) - |z| \operatorname{Re} \left(\frac{1}{b} \right) - 2|z|^2}{2|z| \left[\operatorname{Re} \frac{1}{b} + (2 - \operatorname{Re} \frac{1}{b}) |z|^2 \right]} \cdot \operatorname{Re} \left(\frac{1}{b} - 1 \right) > 0, \quad \forall |z| < 1$$

1. INTRODUCTION

Let A denote the class of functions normalized by

$$f(z) = z + a_2 z^2 + \dots$$

which are analytic in D . We will let $S(1 - b)$ represent the class of functions contained in A which are starlike function of complex order i.e.

$f(z) \in S(1-b)$ if $f(z) = z + a_2z^2 + a_3z^3 + \dots$ is analytic and satisfying

$$\operatorname{Re} \left(1 + \frac{1}{b} \left(z \frac{f'(z)}{f(z)} - 1 \right) \right) > 0, \quad (z \in D)$$

Also, let P denote the class of analytic functions normalized by

$$p(z) = 1 + p_1z + p_2z^2 + \dots$$

such

$$\operatorname{Re} p(z) > 0, \quad p(0) = 1$$

Function in P are often called caratheodory functions.

A function $f(z) \in A$ is said to be convex function of complex order $b(b \neq 0$ is complex), that is, $f(z) \in C(b)$ if and only if

$$\operatorname{Re} \left(1 + \frac{1}{b} z \frac{f''(z)}{f'(z)} \right) > 0, \quad (z \in D)$$

It follows from the definition of starlike functions of complex order that $f(z) \in S(1-b)$ if and only if there exists a function $p(z) \in P$ such that

$$z \frac{f'(z)}{f(z)} = b(p(z) - 1) + 1, \quad z \in D \quad (1.1)$$

DEFINITION 1

Let A be a set of functions $f(z)$ each regular in D . The Koebe domain for a set A is denoted by $K(A)$ and is the collection of points w such that w is in $f(D)$ for every function $f(z)$ in A . In symbols.

$$K(A) = \bigcap_{f \in A} f(D)$$

Supposing that the set A is invariant under the rotation, so $e^{-1\alpha} f(e^{-1\alpha} z)$ is in A whenever $f(z)$ in A . Then the Koebe domain will be either the single point $w = 0$ or an open disk $|w| < R$. In the second case R is often easy to find. Indeed supposing that we have a sharp lower bound $M(r)$ for $f(re^{i\theta})$ for all functions in A , and A contains only univalent functions then

$$R = \lim_{r \rightarrow 1^-} M(r)$$

gives the disc $|w| < R$ as the Koebe domain for the set A .

II. KOEBE DOMAIN FOR THE CLASS $S(1-b)$

In this section we shall give the new ditortion theorem and Koebe domain for the class of starlike functions of complex order.

It should be noticed that the Koebe domain is obtained under the conditions

$$|b| \leq \frac{Re\left(\frac{1}{b}\right) - |z|\left(\frac{1}{b}\right) - 2|z|^2}{2|z|\left[Re\frac{1}{b} + (2 - Re\frac{1}{b})|z|^2\right]}, \quad Re\left(\frac{1}{b} - 1\right) > 0, \quad |z| < 1$$

LEMMA 2.1

A sufficient condition for the univalence of $f(z)$ in $S(1 - b)$ is

$$|b| \leq \frac{Re\left(\frac{1}{b}\right) - |z|\left(\frac{1}{b}\right) - 2|z|^2}{2|z|\left[Re\frac{1}{b} + (2 - Re\frac{1}{b})|z|^2\right]}, \quad Re\left(\frac{1}{b} - 1\right) > 0, \quad \text{for all } |z| < 1 \quad (2.1)$$

PROOF

Let $f(z) \in S(1 - b)$. If we take the logarithmic derivative from (1.1) we obtain

$$\frac{f''(z)}{f'(z)} = b(p(z) - 1) + \frac{zp'(z)}{p(z) + \frac{1}{b} - 1}, \quad z \in D \quad (2.2)$$

where $p(z) \in P$. On the other hand Duren, Shapiro and Shields proved the univalence criterion (see [2]page 171). Let $g(z)$ be analytic in D and $g'(0) \neq 0$. If

$$(1 - |z|)^2 \left| z \frac{g''(z)}{g'(z)} \right| < 1, \quad z \in D \quad (2.3)$$

then $g(z)$ is univalent in D , and the following inequalities are proved by Bernardi and Robertson, respectively [5], [4].

$$\left| \frac{zp'(z)}{p(z) + \frac{1}{b} - 1} \right| \leq \frac{2|z|}{(1 - |z|)(1 + \beta + (1 - \beta)|z|)} \quad (2.4)$$

$$|p(z) - 1| \leq \frac{2|z|}{(1 - |z|)} \quad (2.5)$$

where

$$\beta = \operatorname{Re} \left(\frac{1}{b} - 1 \right) > 0, \quad p(z) \in P$$

If we apply inequalities (2,4) and (2,5) to the inequality (2,3) we obtain (2,1).

LEMMA 2.2

Let $f(z)$ be regular in unit circle and normalized so that

$$f(0) = f'(0) - 1 = 0$$

A necessary and sufficient condition for $f(z) \in C(b)$, is that for each member $s(z) \in S(1 - b)$ the equation

$$s(z) = z \left(\frac{f(z) - f(\eta)}{z - \eta} \right)^2, \quad z, \eta \in D, \quad z \neq \eta \quad (2.6)$$

must be satisfied.

PROOF

Let $f(z)$ convex function of complex order in D , then the function $s(z)$ which is defined by the relation (2.6) is analytic, regular and continuous in the unit disc. Therefore by using continuity the equation (2.6) can be written in the form

$$s(z) = z(f'(z))^2 \quad (2.7)$$

If we take the logarithmic derivative from (2.7) and simple calculations shows that

$$\operatorname{Re} \left[\frac{1}{2b} \left(z \frac{s'(z)}{s(z)} - 1 \right) + 1 \right] = \operatorname{Re} \left[1 + \frac{1}{b} z \frac{f''(z)}{f'(z)} \right] \quad (2.8)$$

Considering the relation (2.8) and the definition of convex functions complex order, the definition of starlike function of complex order together we conclude the function $s(z)$ is starlike functions of complex order.

CONVERSELY

Let $s(z)$ is starlike functions of complex order in D , then simple calculations from (2.6) we obtain that

$$\frac{1}{b} \left(z \frac{s'(z)}{s(z)} - 1 \right) + 1 = \frac{1}{b} \left[\frac{2zf'(z)}{f(z) - f(\eta)} - \frac{z + \eta}{z - \eta} \right] + \frac{b - 1}{b} \quad (2.9)$$

If we write

$$F(z, \eta) = \frac{1}{b} \left[\frac{2zf'(z)}{f(z) - f(\eta)} - \frac{z + \eta}{z - \eta} \right] + \frac{b - 1}{b}$$

the relation (2.9) can be written in the form

$$F(z, \eta) = \frac{1}{b} \left(z \frac{s'(z)}{s(z)} - 1 \right) + 1 \tag{2.10}$$

Considering the relation (2.10) and the definition of starlike function of complex order together we obtain

$$ReF(z, \eta) > 0 \tag{2.11}$$

$$F(z, \eta) = 1 + \frac{1}{b} \left(\frac{1}{f(\eta)} - \frac{2}{\eta} \right) z + \dots \tag{2.12}$$

$$\lim_{\eta \rightarrow z} F(z, \eta) = 1 + \frac{1}{b} z \frac{f''(z)}{f'(z)} \tag{2.13}$$

Therefore by using continuity the claim is proved. Hence it follows that $f(z)$ is convex function of complex order.

THEOREM 2.1

Let $f(z) \in S(1 - b)$. Then

$$\frac{2r}{(1 + |b|)(1 + r)^2} < |f(z)| < \frac{2r}{(1 + |b|)(1 - r)^2} \tag{2.14}$$

The limits are attained by the function

$$f_*(z) = \frac{z}{(1 + |b|)(1 + z)^2}$$

PROOF

Let $h(z) \in C(b)$, then from lemma (2.2), the function

$$F(z, \eta) = \frac{1}{b} \left\{ \frac{2zh'(z)}{h(z) - h(\eta)} - \frac{z + \eta}{z - \eta} \right\} + \frac{b - 1}{b} = 1 + \frac{1}{b} \left(\frac{2}{h(\eta)} - \frac{2}{\eta} \right) z + \dots$$

belongs the class P . Therefore from the charatheodory inequality we can write

$$\left| \frac{1}{b} \left(\frac{2}{h(\eta)} - \frac{2}{\eta} \right) \right| < 2$$

The last inequality can be written in the form

$$\operatorname{Re} \left(\frac{1+|b|}{2} \cdot \frac{h(z)}{z} \right) > \frac{1}{2} \quad (2.15)$$

Therefore the function

$$\left(\frac{1+|b|}{2} \cdot \frac{h(z)}{z} \right)$$

is subordinate to the function

$$\left(\frac{1}{1-z} \right),$$

using the subordination principle we can write

$$\frac{1+|b|}{2} h(z) = \frac{z}{1-\phi(z)}, \quad (2.16)$$

where $\phi(z)$ is analitic in D and satisfies the conditions $\phi(0) = 0$, $|\phi(z)| < 1$. If we take differentiating from (2.8) we obtain

$$\frac{1+|b|}{2} h'(z) = \frac{1+z\phi'(z) - \phi(z)}{(1-\phi(z))^2} \quad (2.17)$$

If we use Jack's Lemma [3] in (2.17) we find that

$$\frac{1+|b|}{2} zh'(z) = \frac{z}{(1-\phi(z))^2} \quad (2.18)$$

It is clear that the relation between the class $S(1-b)$ and $C(b)$ is

$$h(z) \in C(b) \Leftrightarrow zh'(z) = f(z) \in S(1-b) \quad (2.19)$$

Therefore the equality (2.18) can be written in the form

$$\frac{1+|b|}{2} f(z) = \frac{z}{(1-\phi(z))^2}, \quad (2.20)$$

where $f(z) \in S(1-b)$ (2.20) shows that the function,

$$\left(\frac{1+|b|}{2} \cdot \frac{f(z)}{z} \right),$$

is subordinate to the Keobe function

$$\left(\frac{1}{(1-z)^2} \right).$$

Finally using the subordination principle we obtain (2.14). This is a new distortion theorem for the class $S(1-b)$.

COROLLARY 2.1

The following special cases are obtained by giving special values to b .

$$(i) \quad b = 1, \quad \frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}$$

This result is well known, which is distortion theorem of starlike functions [1].

$$(ii) \quad b = 1 - \alpha, \quad \frac{2r}{(2-\alpha)(1+r)^2} \leq |f(z)| \leq \frac{2r}{(2-\alpha)(1-r)^2}, \quad (0 \leq \alpha < 1)$$

This is a new distortion for the class of starlike functions of order α , $0 \leq \alpha < 1$

$$(iii) \quad b = (1-\alpha)e^{-i\lambda} \cos \lambda, \quad \frac{2r}{[1+(1-\alpha)\cos \lambda](1+r)^2} \leq |f(z)| \\ \leq \frac{2r}{[1+(1-\alpha)\cos \lambda](1-r)^2}$$

This result is a new distortion for the class of spirallike functions of order α

$$0 \leq \alpha < 1, \quad |\lambda| < \frac{\pi}{2}$$

(iv) if $b = e^{-i\lambda} \cos \lambda$ then (2.14) reduces to

$$\frac{2r}{(1+\cos \lambda)(1+r^2)} \leq |f(z)| \leq \frac{2r}{(1-\cos \lambda)(1-r^2)}$$

This result is a new distortion for the class of spirallike functions.

COROLLARY 2.2

If we take the limit from $r \rightarrow 1$ and using the definition (1.3) we obtain the Keobe domain for the class $S(1 - b)$, which is

$$R = \frac{1}{2(1 + |b|)}$$

If we give special values to b we obtain the following results.

(i) $b = 1$, $R = 1/4$ is a well known result. This is the Keobe domain for the class of Starlike functions.

(ii) $b = 1 - \alpha$

$$R e = \frac{1}{2(2 - \alpha)}$$

This result is Keobe domain for the class of starlike functions of order α ($0 \leq \alpha < 1$)

(iii) $b = (1 - \alpha)e^{-i\lambda} \cos \lambda$ ($0 \leq \alpha < 1$, $|\lambda| < \pi/2$)

$$R = \frac{1}{2[1 + (1 - \alpha) \cos \lambda]}$$

This is the Keobe domain for the class of spirallike functions of order α .

(iv) $b = e^{-i\lambda} \cos \lambda$ ($|\lambda| < \pi/2$)

$$R = \frac{1}{2(1 + \cos \lambda)}$$

This is the Keobe domain for starlike functions of spirallike functions

REMARK

Robertson ([1]) proved the Keobe domain of starlike functions of order α to be

$$R_1 = \frac{1}{4^{1-\alpha}}$$

and we find that the Keobe domain for the same class is

$$R_2 = \frac{1}{2(2 - \alpha)}$$

If we compare the result of R_1 and R_2 we can clearly see the numerical difference between them.

$$R_1 = \frac{1}{4^{1-\alpha}}$$

$\alpha = 1/2$	$R_1 = 0.500000000$	$\alpha = 1/14$	$R_1 = 0.276022378$
$\alpha = 1/3$	$R_1 = 0.369850263$	$\alpha = 1/15$	$R_1 = 0.274206244$
$\alpha = 1/4$	$R_1 = 0.353553390$	$\alpha = 1\alpha = 1/13$	$R_1 = 0.278813286$
$\alpha = 1/5$	$R_1 = 0.329876977$	$\alpha = 1/17$	$R_1 = 0.271240978$
$\alpha = 1/6$	$R_1 = 0.314980262$	$\alpha = 1/18$	$R_1 = 0.270014934$
$\alpha = 1/7$	$R_1 = 0.304753413$	$\alpha = 1/19$	$R_1 = 0.268922646$
$\alpha = 1/8$	$R_1 = 0.297301778$	$\alpha = 1/20$	$R_1 = 0.267933365$
$\alpha = 1/9$	$R_1 = 0.291632259$	$\alpha = 1/21$	$R_1 = 0.267060422$
$\alpha = 1/10$	$R_1 = 0.287174588$	$\alpha = 1/22$	$R_1 = 0.266260272$
$\alpha = 1/11$	$R_1 = 0.28357813$	$\alpha = 1/23$	$R_1 = 0.265531749$
$\alpha = 1/12$	$R_1 = 0.280615512$	$\alpha = 1/24$	$R_1 = 0.264865773$

$$R_2 = \frac{1}{2(2 - \alpha)}$$

$\alpha = 1/2$	$R_2 = 0.333333333$	$\alpha = 1/14$	$R_2 = 0.276022378$
$\alpha = 1/3$	$R_2 = 0.300000000$	$\alpha = 1/15$	$R_2 = 0.258620689$
$\alpha = 1/4$	$R_2 = 0.285714285$	$\alpha = 1/16$	$R_2 = 0.258064516$
$\alpha = 1/5$	$R_2 = 0.277777777$	$\alpha = 1/17$	$R_2 = 0.257575757$
$\alpha = 1/6$	$R_2 = 0.272727272$	$\alpha = 1/18$	$R_2 = 0.257142857$
$\alpha = 1/7$	$R_2 = 0.269230769$	$\alpha = 1/19$	$R_2 = 0.256756756$
$\alpha = 1/8$	$R_2 = 0.266666666$	$\alpha = 1/20$	$R_2 = 0.256410256$
$\alpha = 1/9$	$R_2 = 0.264705882$	$\alpha = 1/21$	$R_2 = 0.256097561$
$\alpha = 1/10$	$R_2 = 0.263157894$	$\alpha = 1/22$	$R_2 = 0.255813953$
$\alpha = 1/11$	$R_2 = 0.261904761$	$\alpha = 1/23$	$R_2 = 0.255555555$
$\alpha = 1/12$	$R_2 = 0.260869565$	$\alpha = 1/24$	$R_2 = 0.255319148$
$\alpha = 1/13$	$R_2 = 0.226000000$	$\alpha = 1/25$	$R_2 = 0.255100204$

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**ON CHARACTERIZATION OF THE CHEVALLEY GROUP
 $F_4(2)$ BY THE SMALLER CENTRALISER OF A CENTRAL
INVOLUTION III**

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Key Words: Chevalley group $F_4(2)$, involutinos, conjugacy classes, central series.

AMS Subject Classification: 20 D

Abstract:

In this paper we obtain a characterization of the Chevalley group $F_4(2)$ in terms of the smaller centraliser of a central involution in $F_4(2)$.

1. INTRODUCTION:

Let $F_4(2)$ denote the Chevalley group of type F_4 over the field $T = \{0, 1\}$. The centre of a Sylow subgroup S of F_4 is a four group. The elements of order two in this subgroup of S lie in three distinct conjugacy classes in $F_4(2)$. Let t_1, t_2 and $t_3 = t_1 t_2$ be these involutions in the centre of S . Now in $F_4(2)$

$$C(t) \cong C(t_2)$$

and

$$C(t_1) \cap C(t_2) \cong C(t_3)$$

For necessary details about the group $F_4(2)$, we refer the reader to [8].

We shall refer tables 1, 2, & 3 of [8].

Table 5

y	$F_2(y)$
$x_{21}(\alpha)$	$\{x_{21}(\alpha)\}$
$x_{24}(1)$	$\{x_i(\alpha_i) i = 18, 23, 24\}$
$x_{21}(\alpha)x_{24}(1)$	$\{u_{21}(\alpha) u \in F_1(x_{24}(1))\}$
$x_{17}(1)$	$\{x_i(\alpha_i) I = 13, 17\}$
$x_{17}(1)x_{24}(1)$	$\{\prod_{i \in I} x_i(\alpha_i) I = \{17, 24\}, \{17, 23\}, \{13, 18\}, \{13, 23\}\}$
$x_{16}(1)x_{22}(1)$	$\{\prod_{i \in I} x_i(\alpha_i) I = \{16, 22\}, \{13, 24\}, \{13, 20\}, \{17, 18\}, \{13, 18, 24\}, \{13, 23, 24\}, \{13, 18, 20\}\}$
$x_9(1)$	$\{x_i(\alpha_i) i = 4, 9\}$
$x_9(1)x_{21}(\alpha)$	$\{ux_{21}(\alpha) u \in F_i(x_9(1))\}$
$x_9(1)x_{17}(\alpha)$	$\{\prod_{i \in I} x_i(\alpha_i) I = \{9, 17\}, \{4, 18\}, \{4, 24\}\}$
$x_9(1)x_{15}(1)$	$\{\prod_{i \in I} x_i(\alpha_i) I = \{9, 15\}, \{9, 13\}, \{4, 12\}, \{4, 16\}, \{4, 16, 18\}\}$
$x_7(1)$	$\{x_i(\alpha_i) i = 3, 7\}$
$x_7(1)x_{24}$	$\{x_3(\alpha_3)x_{21}(\alpha)x_7(\alpha_7)x_{24}(\alpha)\}$
$x_7(1)x_{23}(1)$	$\{\prod_{i \in I} x_i(\alpha_i) I = \{7, 23\}, \{3, 18\}, \{3, 22\}, \{3, 24\}, \{3, 21, 24\}\}$
$x_7(1)x_9(1)$	$\{\prod_{i \in I} x_i(\alpha_i)x_{21}(\alpha)\}$
	$\{x_7(\alpha_7)x_9(\alpha_9)x_{21}(\alpha), x_3(\alpha_3)x_4(\alpha_4)x_{21}(\alpha), x_3(\alpha)x_7(\alpha_7)x_{24}(\alpha)\}$
$x_7(1)x_9(1)x_{15}(1)$	$\{\prod_{i \in I} x_i(\alpha_i) I = \{7, 9, 15\}, \{3, 4, 18\}, \{3, 17, 22\}, \{3, 4, 24\}, \{3, 4, 21, 24\}\}$
$x_4(1)x_7(1)$	$\{\prod_{i \in I} x_i(\alpha_i) I = \{4, 7\}, \{3, 8\}, \{3, 4, 7\}\}$

Let $D_{10} = C(x_{17})$ and $D_5 = C_5(x_{23})$. We write M for D_{10} . Then M and D_5 are the only subgroups of S of order 2^{23} with centres of order 2^3 . For our convenience we write D for D_5 . We identify C with C_3 .

It is easily observed that $Z(S) = S_{21}S_{24}$. The Chevalley group $F_4(2^n)$ of type F_4 over the field of 2^n elements have been characterized by Guterma [3] in terms of the centralisers of 2-central involutions and this characterization is given by the following theorem.

Theorem I

Let G be a finite group. Suppose the centre of a Sylow₂ subgroup of G contains elements y_1, y_2 and $y_3 = y_1y_2$ of order two such that

$$C_G(y_i) \cong C(t_i); \quad i = 1, 2, 3,$$

Then

$$G \cong F_4(2^n)$$

In [10] Thomas has given an improved characterization of the Chevalley group $F_4(2^n)$ in terms of only the centraliser $C(t_1)$ for all $n > 2$.

Later Husnine [4] treated the case for $n = 1$.

We began our improved characterization of $F_4(2)$ in [8] and in [9]. The following result have been proved.

Theorem B

There is an involution u in $N_g(D)$ so that

$$x_{23}^u = x_{24}, \quad x_{14}^u = x_{12}, \quad x_{15}^u = x_{13}$$

$$x_{18}^u = x_{19}x_{21}(\epsilon), \quad x_8^u = x_4$$

and u centralizes $S_9S_{20}S_{16}S_{17}S_{21}S_{22}$.

In Section 2, we complete the proof of the following theorem.

Theorem C

Let G be a finite simple group with an involution y_3 lying in the centre of a Sylow₂ - subgroup. Suppose $C = C_G(y_3)$ is isomorphic to $C(t_3)$, the centraliser of t_3 in $F_4(2)$. Then G is isomorphic to $F_4(2)$.

2. ACTION OF $N_G(D)$ ON $Z_7(D)$

In this section we prove the following results.

Lemma 2.1

There is an involution u in $N_G(D)$ so that u takes x_6 to $x_2x_{16}(\epsilon)x_8$ to x_4, x_{19} to $x_8x_{21}(\epsilon)$ and permutes all other $S_i (i \neq 5)$ as w_5 does in $F_4(2)$

Proof

Let g_1 be the involution satisfying in [2] since

$$Z_6(D) = S_3, S_7, S_{11}, Z_5(D)$$

and

$$Z_7(D) = S_1S_2S_6S_{10}Z_{16}(D)$$

Since $x_7^{g_1} = x_3z_5(D)$

Let x_4 appear in $x_7^{g_1}$

$$\begin{aligned} [x_7^{g_1}, x_{20}] &= [x_4, x_{20}] = x_{24} \\ \Rightarrow [x_7^{g_1}, x_{20}] &= x_{23} \\ \Rightarrow [x_7, x_{20}^{g_1}] &= x_{24} \end{aligned}$$

This shows that x_7 is conjugate to x_7, x_{24} . But under graph automorphism ϕ .

Put

$$\begin{aligned} \phi(x_7) &= x_{18} \\ \phi(x_{24}) &= x_{21} \end{aligned}$$

we found $x_{18} \sim x_{18}x_{21}$. This is contradiction from structure of S .

Let x_8 appear in $x_7^{g_1}$.

$$\begin{aligned} [x_7^{g_1}, x_{20}] &= [x_8, x_{20}] = x_{24} \\ [x_7^{g_1}, x_{20}]^{g_1} &= x_{23} \\ \Rightarrow x_{20}^{g_1}, x_7x_{20}^{g_1} &= x_7x_{23} \end{aligned}$$

This shows that x_7 and x_7x_{23} are conjugate in S but this is a contradiction from table 3 of [1].

Let x_{12} appear in $x_7^{g_1}$

$$\begin{aligned}
 & [x_7^{g_1}, x_{15}] = [x_{12}, x_{15}] = x_{21} \\
 \Rightarrow & [x_7, x_{15}^{g_1}] = x_{21} \\
 & x_7 \sim x_7 x_{21} \quad \text{but under graph automorphism} \\
 \Rightarrow & x_{18} \sim x_{18} x_{24} \sim x_{18} x_{21} \\
 \Rightarrow & x_{18} \sim x_{18} x_{21} \quad \text{but this contradicts the table 3 of [1]}
 \end{aligned}$$

Let x_{14} appear in $x_7^{g_1}$ thus

$$\begin{aligned}
 & [x_7^{g_1}, x_{13}] = [x_{14}, x_{13}] = x_{21} \\
 & [x_7, x_{13}^{g_1}] = x_{21} \\
 & x_7 \sim x_7 x_{21} \quad \text{but under graph automorphism} \\
 & x_{18} \sim x_{18} x_{24} \sim x_{18} x_{21} \\
 \text{therefore} & \quad x_{18} \sim x_{18} x_{21} \quad \text{this is contradiction}
 \end{aligned}$$

By the use of similar arguments, it can be shown that $x_9, x_{15}, x_{17}, x_{18}, x_{20}$ are not involved in Z .

Let x_{22} appear in $x_7^{g_1}$

$$\begin{aligned}
 & [x_7^{g_1}, x_6] = [x_{22}, x_6] = x_{24} \\
 & [x_7^{g_1}, x_6^{g_1}] = x_{24}^{g_1} = x_{23} \\
 \Rightarrow & \quad x_7 \sim x_7 x_{23} \\
 \Rightarrow & \quad x_{18} \sim x_{18} x_{21} \quad \text{which is a contradiction}
 \end{aligned}$$

Let

$$x_7^{g_1} = x_3 x_{24}$$

then

$$x_3^{g_1} = x_7 x_{23}$$

$$\Rightarrow (x_3 x_7)^{g_1} = x_3 x_7 x_{23} x_{24}$$

Therefore $g_2 = g_1 x_5 g_1$ belong to C_1 as g_2 centralizes x_{21} .

By considering

$$\begin{aligned}
 (x_7x_{23})^{g_2} &= (x_7x_{23})^{g_1x_5g_1} \\
 &= (x_3)^{x_5g_1} \\
 &= (x_3x_7x_9)^{g_1} \\
 &= x_3x_7x_9x_{23}x_{24}
 \end{aligned}$$

So

$$x_7x_{23} \sim x_3x_7x_9x_{23}x_{24} \sim x_3x_{23}$$

but

$$\begin{aligned}
 (x_3x_{23})^{w_5} &= x_7x_{24} \\
 \Rightarrow x_3x_{23} &\sim x_7x_{24}
 \end{aligned}$$

this is a contradiction to table no. 5. next,

Let

$$x_7^{g_1} = x_3x_{21}$$

then

$$(x_7x_{21})^{g_1} = x_3$$

Therefore

$$g_2x_5g_1 \in C_3$$

taking

$$\begin{aligned}
 (x_7x_{21})^{g_2} &= (x_7x_{21})^{g_1x_5g_1} \\
 &= (x_3)^{x_5g_1} \\
 &= (x_3x_7x_9)^{g_1} \\
 &= x_3x_7x_9
 \end{aligned}$$

Therefore

$$x_7x_{21} \sim x_3x_7x_9 \sim x_3$$

but in C_1

$$(x_7x_{21})^{w_5} = x_3x_{21}$$

This shows that $x_3x_{21} \sim x_3$

But from table no. 5, we see x_3 is conjugate to x_7 and x_3x_{21} is conjugate to x_7x_{24} . Which provides contradiction. Hence we found that

$$x_7^{g_1} = x_3$$

next, since

$$x_{11}^{g_1} = x_{11}z, \quad z \in z_6$$

Let x_4 appear in $x_{11}^{g_1}$, then from commutator relations in S ,

$$\begin{aligned} [x_{11}^{g_1}, x_{20}] &= [x_4, x_{20}] = x_{23} \\ \Rightarrow [x_{11}^{g_1}, x_{20}] &= x_{23} \\ [x_{11}, x_{20}^{g_1}] &= x_{24} \\ \Rightarrow x_{11} &\sim x_{11}x_{24} \end{aligned}$$

but under graph automorphism ϕ .

put

$$\begin{aligned} \phi(x_{11}) &= x_6 \\ \phi(x_{24}) &= x_{21} \end{aligned}$$

So, we got, $x_6 \sim x_{21}$ in S but this is contradiction from table no. 3.

Let x_8 appear in $x_{11}^{g_1}$. Thus structure of S implies.

$$[x_{11}^{g_1}, x_{20}] = [x_8, x_{20}] = x_{24}$$

$\Rightarrow x_{11}$ is conjugate to $x_{11}x_{23}$ under the action of $x_{20}^{g_1}$. Under graph automorphism ϕ , we have $x_6 \sim x_6x_{17}$ in S . but this contradicts table 3.

On same pattern, one finds that $x_9, x_{12}, x_{14}, x_{15}, x_{16}, x_{18}$ cannot appear in Z .

Let x_{20} appear in $x_{11}^{g_1}$ then

$$\begin{aligned} [x_{11}^{g_1}, x_8] &= [x_{20}, x_8] = x_{24} \\ \Rightarrow [x_{11}^{g_1}, x_8] &= x_{24} \\ \Rightarrow [x_{11}, x_8^{g_1}] &= x_{23} \end{aligned}$$

Hence $x_8^{g_1^{-1}}$ takes x_{11} to $x_{11}x_{23}$ which is not true according to commutator relations in S and graph automorphism next, Let x_{22} occur in $x_{11}^{g_1}$. Then from table 2.

$$[x_{11}^{g_1}, x_6] = [x_{22}, x_6] = x_{24}$$

$$[x_{11}^{g_1}, x_6] = x_{24}$$

$\Rightarrow x_{11}$ is conjugate to $x_{11}x_{23}$ by $x_6^{g_1}$ but under graph automorphism, x_6 is conjugate to x_6x_{17} which is contradiction from table 2. Hence x_{22} cannot appear in $x_{11}^{g_1}$.

Let

$$x_{11}^{g_1} = x_{11}x_{21}$$

If x_{21} occur in $x_{11}^{g_1}$. We write $g = g_1g_{16}$ that is

$$\begin{aligned} x_{11}^g &= x_{11}^{g_1x_{16}} \\ &= (x_{11}x_{21})^{x_{16}} \\ &= x_{11} \end{aligned}$$

g takes x_{24} to x_{23}, x_{15} , to x_{13}, x_{14} to x_{12}, x_8 to x_4, x_7 to x_3 and centralizes $x_{22}, x_{21}, x_{20}, x_{17}, x_{16}, x_{11}, x_{10}, x_9$.

now, we are left

$$x_{11}^{g_1} = x_{11}x_{17}x_{24}$$

In order to go further and determine the involvement from Z_6 in the conjugate of x_{10} i.e.

$$x_{10}^{g_1} = x_{10}z, \quad z \in Z_6$$

One finds that $x_3, x_4, x_7, x_8, x_{11}, x_{13}, x_{15}, x_{19}, x_{20}$ cannot appear in Z .

So we have $x_{10}^{g_1} = x_{10}S_{17}S_{21}S_{24}$

If x_{21} appear in $x_{10}^{g_1}$, we write $g = g_1g_{17}$ and g acts on elements of S in the same manner as g in above lemma.

Let

$$x_{10}^{g_1} = x_{10}x_{17}$$

Since, $g_1w_1g_1 \in C_3$, so

$$\begin{aligned} x_{10}^{g_1w_1g_1} &= (x_{10}x_{17})^{w_1g_1} \\ &= (x_{11}x_{17})^{g_1} \\ &= x_{11}x_{24} \end{aligned}$$

Thus x_{10} is conjugate to $x_{11}x_{24}$ in C_3 but under graph automorphism ϕ ,

$$\begin{aligned} \phi(x_{10}) &= x_5 \\ \phi(x_{11}) &= x_6 \\ \phi(x_{24}) &= x_{21} \end{aligned}$$

$\Rightarrow x_5$ is conjugate to x_6x_{21} in C_3 , but $(x_6, x_{21})^{w_1} = x_5x_{21}$

Hence in C_3 , x_5 is conjugate to x_5x_{21} which is not true as x_5 and x_5x_{21} belong to different conjugacy classes in C_3 , this is again contradiction.

next, Let

$$x_{10}^{g_1} = x_{10}x_{24}$$

since $g_1w_1g_1 \in C_3$ as $g_1w_1g_1$ centralizes x_{21} and x_{24} .

$$\begin{aligned} x_{10}^{g_1w_1g_1} &= (x_{10}x_{24})^{w_1g_1} \\ &= (x_{11}x_{24})^{g_1} \\ &= x_{11}x_{17}x_{24}x_{23} \end{aligned}$$

So

$$\begin{aligned} x_{10} &\sim x_1x_{17}x_{23}x_{24} \sim x_{10}x_{17}x_{23} \\ &\Rightarrow x_{10} \sim x_{11}x_{17}x_{23} \end{aligned}$$

but under graph automorphism ϕ , it will be of form

$$x_5 \sim x_6x_{17}x_{23}$$

Since

$$x_5 \sim x_6x_{17}x_{23} \sim x_6x_{16}x_{17}$$

but in C_3 , we see that x_5 and $x_6x_{16}x_{17}$ belong to different conjugacy classes, a contradiction. Finally, we found that $x_{10}^{g_1} = x_{10}$.

next,

$$x_1^{g_1} = x_1z, \quad z \in Z_6$$

and we found that $x_3, x_9, x_{11}, x_{12}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}, x_{23}$ cannot appear in Z .

Now

Next, Let

$$x_6^{g_1} = x_2x_{21}$$

Therefore

$$g_1w_1g_1 \in C_3$$

as $g_1w_1g_1$ centralizes x_{21} and x_{24} .

$$x_6^{g_1w_1g_1} = (x_2x_{21})^{w_1g_1} = (x_4x_{21})^{g_1} = x_8x_{21}$$

$\Rightarrow x_6$ is conjugate to x_8x_{21} in C_3 but we found x_8x_{21} is conjugate to x_9x_{21} and x_6 is conjugate to x_9 in C_3 . This is contradiction. Hence x_{21} cannot occur in $x_6^{g_1}$.

Next, suppose that

$$x_6^{g_1} = x_2x_{24}$$

Therefore

$$g_1w_1g_1 \in C_3$$

$$\begin{aligned} x_6^{g_1w_1g_1} &= (x_2x_{24})^{w_1g_1} \\ &= (x_4x_{24})^{g_1} \\ &= (x_8x_{23}) \text{ but } x_8x_{23} \sim x_8x_{17} \end{aligned}$$

Hence x_6 is conjugate to x_8x_{17} in C_3 but Table No. 4 shows that x_8x_{17} is conjugate to x_4x_{17} and x_6 is conjugate to x_9 , again a contradiction.

Hence x_{24} cannot occur in $x_6^{g_1}$.

Hence from structure of S , we found

$$x_6^{g_1} = x_2$$

As we have seen that $x_{11}^{g_1} = x_{11}x_{17}x_{24}$

Let

$$x_{11}^{g_1} = x_{11}x_{17}$$

Therefore

$$g_1w_1g_1 \in C_3$$

and

$$x_{21}^{g_1w_1g_1} = x_{21} \& x_{24}^{g_1w_1g_1} = x_{24}$$

$$x_{11}^{g_1w_1g_1} = (x_{11}x_{17})^{w_1g_1} = (x_{10}x_{16})^{g_1} = x_{10}x_{16}$$

$\Rightarrow x_{11}$ is conjugate to $x_{10}x_{16}$ in C_3 . Under graph automorphism ϕ ,

Put

$$\begin{aligned} \phi(x_{11}) &= x_6 \\ \phi(x_{10}) &= x_5 \\ \phi(x_{16}) &= x_{22} \end{aligned}$$

$\Rightarrow x_6$ is conjugate to x_5x_{22} in C_3 . We found, x_5, x_{22} is conjugate to x_5x_{15} by x_{13} . Hence x_6 is conjugate to x_5x_{15} but in table 4 we see that x_6 is conjugate to x_9 and x_5x_{15} is conjugate to x_9x_{17} . Hence there is contradiction.

Next, Suppose that

$$x_{11}^{g_1} = x_{11}x_{24}$$

Therefore

$$g_1w_1g_1 \in C_3$$

$$x_{11}^{g_1w_1g_1} = (x_{11}x_{24})^{w_1g_1} = (x_{10}x_{24})^{g_1} = x_{10}x_{23}$$

$\Rightarrow x_{11}$ is conjugate to $x_{10}x_{23}$ in C_3 and graph automorphism implies that x_6 is conjugate to x_5x_{17} but in C_3 according to table 4, we see that x_6 is conjugate to x_9 and x_5x_{17} is conjugate to x_9x_{15} . Hence we got contradiction, so x_{24} cannot appear in $x_{11}^{g_1}$.

now, we are left

$$x_{11}^{g_1} = x_{11}$$

This completes the proof of Lemma 2.1.

Lemma 2.2

There is an involution u in $N_G(D)$ which permutes the $S_i (i \neq 5)$ exactly in the way w_5 does in $F_4(2)$ and $(ux_5)^3 = 1$

Proof

According to [5 (4.6)], C_3 admits an automorphism θ such that $\theta(x_2) = x_2x_{16}(\epsilon)$, $\theta(x_4) = x_4x_{17}(\epsilon)$, $\theta(x_{18}) = x_{18}x_{21}(\epsilon)$, $\theta = 1$ on all other S_i , $\theta(w_1) = w_1$, $\theta(w) = w_2w_{24}$ for all $\epsilon \in T$. Thus by relabeling the elements of C_3 if necessary we can assume that an u_1 satisfying (2.1) acts upon D exactly as w_5 does in $F_4(2)$.

now, $(u_1x_5)^3$ centralise all $S_i (i \neq 2, 5, 6)$

Thus $(u_1x_5)^3 \in Z(D)$ and $|(u_1x_5)^3| \leq 2$. Since u_1 and x_5 centralise $(u_1x_5)^3$, we have $(u_1x_5)^3 = x_{21}(\delta)$. Write $u = x_1x_{21}(\delta)$. Then u acts upon D exactly as w_5 in $F_4(2)$ and $(ux_5)^3 = 1$.

IDENTIFICATION OF G

From now we write \bar{x} for $\theta_1(x)$ for all $x \in C_3$, θ_1 being the isomorphism mentioned in theorem C.

3.1 Lemma

Let u be an involution satisfying (2.2), then $(u\bar{w}_2)^2 = (u\bar{w}_2)^3 = 1$. Hence there exists a homomorphism σ from W_1 onto $(\bar{w}_1, \bar{w}_2, u)$ which takes w_5 to u and θ_1 on W_3 .

Proof

$$(u\bar{w}_1)^2 \text{ centralises } \Pi\bar{S}_i (i \neq 2, 5)$$

i.e.

$$(u\bar{w}_1)^2 = \bar{x}_{23}(\alpha)\bar{x}_{21}(\alpha)\bar{x}_{24}(\beta)$$

i.e.

$$\bar{x}_{23}(\alpha)\bar{x}_{21}(\alpha)\bar{x}_{24}(\beta)u = \bar{x}_1\bar{w}_1u\bar{w}_1x_1 = \bar{w}_1\bar{x}_1\bar{w}_1\bar{x}_1u\bar{x}_1\bar{w}_1\bar{x}_1$$

Since $(\bar{x}_1\bar{w}_1)^3 = 1$.

This implies

$$1 = (\bar{x}_1(\bar{w}_1u))^2(\bar{x}_1 \text{ or } (\bar{w}_1u)^2 = 1$$

centralizes

$$\Pi\bar{S}_i (i \neq 2, 5, 6)$$

Thus the structure of C_3 implies

$$(u\bar{w})^2 \in \bar{S}_{22}, \bar{S}_{23}, \bar{S}_{21}, \bar{S}_{24}$$

and since u and \bar{w}_2 centralise $(u\bar{w}_2)^3$, we have

$$(u\bar{w}_2)^3 = \bar{x}_{22}(\epsilon)\bar{x}_{23}(\epsilon)\bar{x}_{24}(\epsilon)(\delta)$$

i.e.

$$(u\bar{w}_2u)\bar{w}_2(u\bar{w}u) = \bar{u}_{22}(\epsilon)\bar{x}_{23}(\epsilon)\bar{x}_{21}(\delta)u \dots$$

Now

$$(\bar{x}_2\bar{w}_2)^3 = 1$$

Thus

$$[(\bar{x}_2\bar{w}_2)^{u\bar{w}_2u}]^3 = 1$$

i.e.

$$\begin{aligned} 1 &= (\bar{x}_2^{u\bar{w}_2u}\bar{x}_{22}(\epsilon)\bar{x}_{23}(\epsilon)\bar{x}_{21}(\delta)\bar{x}_{24}(\delta)u)^3 \\ &= (u\bar{x}_5\bar{x}_{22}(\epsilon)\bar{x}_{23}(\epsilon)\bar{x}_{21}(\epsilon)\bar{x}_{24}(\delta))^3 \\ &= (u\bar{x}_5)^3\bar{x}_{22}(\epsilon)\bar{x}_{21}(\delta) = \bar{x}_{16}(\epsilon)\bar{x}_{21}(\delta) \end{aligned}$$

This implies $\epsilon = \delta = 0$. Thus $(u\bar{w}_2)^3 = 1$

Proof of Theorem C

According to 4.2 of Husinine [4] $K = \langle \bar{w}_1, \bar{w}_2, u, \bar{S} \rangle \subseteq C_G(\bar{x}_{21})$ and $\cong C_1$. Thus G satisfies the hypothesis of Theorem B of [1.1] and is thus isomorphic to $F_4(2)$.

This completes the proof of Theorem C.

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LINEAR BOUNDARY PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS WITH DEVIATED ARGUMENTS

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1. INTRODUCTION

Let $C(J, R^p)$ denote the set of all continuous functions $x : J \rightarrow R^p$ with $J = [0, b]$, $b > 0$. Similarly, by $C^1(J, R^p)$, we mean the family of all functions $x : J \rightarrow R^p$ having continuous first derivative. Let $p \times p$ matrix functions A_1 and A_2 be continuous on J , $g \in C(J, R^p)$ and $\alpha \in C(J, J)$.

Under the above assumptions, we consider the linear system of ordinary differential equations with deviated arguments of the form

$$y'(t) = A_1(t)y(t) + A_2(t)y(\alpha(t)) + g(t), \quad t \in J, \quad (1a)$$

subject to the linear boundary condition

$$B_1y(0) + B_2y(b) = D \in R^p. \quad (1b)$$

Here B_1 and B_2 are constant square matrices of order p . By a solution of (1) we mean a function $y^* \in C^1(J, R^p)$ which satisfies (1). One of our tasks is to establish conditions by which problem (1) has a unique solution.

To prove the existence of a solution of problem (1) it is convenient to transform problem (1) into an integral equation. For initial value problems, which are special cases of (1), an integral equation of Volterra type can be obtained from (1) by using the substitution $y'(t) = z(t)$. If anyone assumes only that α is

continuous, then this integral equation has a unique solution if a corresponding condition holds. It is worth noticing that this condition is superfluous if $\alpha(t) \leq t$ (see Theorem 1).

As we see in part 3, the previous substitution $y'(t) = z(t)$ is not so useful for boundary value problems of type (1). In this case, much better is to represent any solution of problem (1) in a similar form as for linear boundary value problems without the retardations, i.e. by using the notion of a fundamental solution. By a such approach, we transform now problem (1) into an integral equation of Fredholm type. Solving such equations and proving the existence of solutions is usually a much more complicated problem than in the case of Volterra type equations.

A knowledge of how perturbations of g and D affect the solution y of (1) is obviously important when solving problem (1) numerically. It is characterised by a stability constant (see, for example [1]). The aim of part 4 is to examine the role played by this stability constant. In part 5, we present the extension of the above problems to multistep boundary value problems.

There are some papers available which deal with the problems of integral representation of linear boundary value problems without retardations and the role of stability constants for them. The book by Ascher et al. [1] contains such considerations and the list of selected references which provide an orientation to the field. For functional differential problems see, for example, [3], [6] and the references therein.

My paper is an extension to linear problems with deviated arguments of some results obtained for linear boundary problems without retardations.

2. INITIAL VALUE PROBLEMS

In this paragraph, we consider problem (1) when B_2 equals the zero matrix. It means that problem (1) is now a linear initial value problem. In addition, we assume that B_1 is nonsingular. To establish existence conditions it is convenient to transform problem (1) into the integral equation of Volterra type by the substitution $y'(t) = z(t)$, $t \in J$. Hence, we have the integral equation for z of the form

$$z(t) = (Lz)(t) + g_1(t), \quad t \in J, \quad (2)$$

where the operator L is defined by

$$(Lz)(t) = A_1(t) \int_0^t z(\tau) d\tau + A_2(t) \int_0^{\alpha(t)} z(\tau) d\tau, \quad t \in J,$$

and

$$g_1(t) = g(t) + [A_1(t) + A_2(t)] (B_1)^{-1} D, \quad t \in J.$$

Notice that, if z is a continuous solution of (2), then the corresponding solution $y \in C^1(J, R^p)$ of problem (1) is given by the following formula

$$y(t) = (B_1)^{-1} D + \int_0^t z(\tau) d\tau, \quad t \in J.$$

It is known that problem (2) has a unique continuous solution if the condition

$$\max_{t \in J} [\|A_1(t)\|t + \|A_2(t)\|\alpha(t)] < 1$$

holds. This condition can be weakened if $\alpha(t) \leq t, t \in J$. In this case, we apply Bielecki's norm, i.e.

$$\|u\|_* = \max_{t \in J} \|u(t)\| \exp(-\rho t), \quad u \in C(J, R^p).$$

We can formulate

THEOREM 1

Let $p \times p$ matrix functions A_1, A_2 be continuous on J and $g_1 \in C(J, R^p)$. Let $\alpha \in C(J, J)$ and $\alpha(t) \leq t$. Then problem (2) has the unique continuous solution on J .

Proof

Let $\rho \geq \max_{t \in J} [\|A_1(t)\| + \|A_2(t)\|]$. It is simple to prove that the operator T defined by the right-hand side of equation (2) is a contraction

$$\|Tu - Tw\|_* \leq [1 - \exp(-\rho b)] \|u - w\|_*, \quad u, w \in C(J, R^p).$$

By the well-known Banach fixed point theorem, equation (2) has the unique solution.

3. BOUNDARY VALUE PROBLEMS

Let us use the same substitution $y'(t) = z(t)$, $t \in J$ to boundary value problem (1). Then, (1a) leads to

$$z(t) = (Lz)(t) + [A_1(t) + A_2(t)]y(0) + g(t), \quad t \in J. \quad (3)$$

To define $y(0)$ we apply condition (1b). Then, under the assumption that $B_1 + B_2$ is nonsingular, we have

$$y(0) = (B_1 + B_2)^{-1} \left[D - B_2 \int_0^b z(\tau) d\tau \right].$$

Substituting it into (3), we obtain the following Volterra-Fredholm type equation

$$z(t) = (Lz)(t) + [A_1(t) + A_2(t)](B_1 + B_2)^{-1} \left[D - B_2 \int_0^b z(\tau) d\tau \right] + g(t) \quad (4)$$

for $t \in J$. Problem (1) is now replaced by (4). The matrix $B_1 + B_2$ must be nonsingular. This assumption is very restrictive. For example, already in the case $p = 2$, it is not satisfied for such simple boundary conditions as $y_1(0) = c_1$, $y_1(b) = c_2$, i.e. when

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

It results from the above, that the substitution $y'(t) = z(t)$, $t \in J$, which is very popular and useful for initial value problems, it is not so useful to boundary value problems. Due to this fact, to characterize a solution of (1) we apply the notion of a fundamental solution. Let Y be any fundamental solution connected with A_1 , so

$$Y'(t) = A_1(t)Y(t). \quad (5)$$

It will be convenient to call a fundamental solution any solution Y satisfying equation (5) and having linearly independent columns, but not necessarily this solution for which $Y(0) = I_{p \times p}$. Sometimes, it is useful to require the condition

$$B_1Y(0) + B_2Y(b) = I \quad (6)$$

to be satisfied. We seek a solution of (1) in the form

$$y(t) = Y(t) \left[k + \int_0^t Y^{-1}(\tau)z(\tau)d\tau \right], \quad \text{where } z \in C(J, R^p). \quad (7)$$

Hence,

$$y'(t) = Y'(t) \left[k + \int_0^t Y^{-1}(\tau)z(\tau)d\tau \right] + z(t), \quad t \in J, \quad (8)$$

and substituting it into condition (1a), we obtain

$$k = Q^{-1} \left[D - B_2Y(b) \int_0^b Y^{-1}(\tau)z(\tau)d\tau \right]$$

provided that the matrix $Q = B_1Y(0) + B_2Y(b)$ is nonsingular. Now, for $t \in J$, problem (1) takes the form

$$y(t) = Y(t)Q^{-1} \left[D + B_1Y(0) \int_0^t Y^{-1}(\tau)z(\tau)d\tau - B_2Y(b) \int_t^b Y^{-1}(\tau)z(\tau)d\tau \right], \quad t \in J.$$

Combining the integrals, it has a simple form, namely

$$y(t) = \Phi(t)D + \int_0^b G_1(t, s)z(s)ds, \quad t \in J, \quad (9)$$

where G_1 is the $p \times p$ Green's function, defined by

$$G_1(t, s) = \begin{cases} \Phi(t)B_1\Phi(0)\Phi^{-1}(s) & \text{if } s \leq t, \\ -\Phi(t)B_2\Phi(b)\Phi^{-1}(s) & \text{if } s > t, \end{cases}$$

and $\Phi(t) = Y(t)Q^{-1}$, $t \in J$. Now, we need to define the corresponding equation for z . Substituting (7) and (8) into equation (1a), we get

$$z(t) = A_2(t)Y(\alpha(t))Q^{-1} \left[D + B_1Y(0) \int_0^{\alpha(t)} Y^{-1}(\tau)z(\tau)d\tau - B_2Y(b) \int_{\alpha(t)}^b Y^{-1}(\tau)z(\tau)d\tau \right] + g(t).$$

Hence, we have

$$z(t) = \int_0^b G_2(t, s)z(s)ds + Q_1(t), \quad t \in J, \quad (10)$$

where G_2 is the $p \times p$ Green's function, defined by

$$G_2(t, s) = \begin{cases} A_2(t)\Phi(\alpha(t))B_1\Phi(0)\Phi^{-1}(s) & \text{if } s \leq \alpha(t), \\ -A_2(t)\Phi(\alpha(t))B_2\Phi(b)\Phi^{-1}(s) & \text{if } s > \alpha(t), \end{cases}$$

and

$$Q_1(t) = A_2(t)\Phi(\alpha(t))D + g(t), \quad t \in J.$$

Summing up the discussion conducted above, we conclude:

THEOREM 2

Let $p \times p$ matrix functions A_1, A_2 be continuous on J , $g \in C(J, R^p)$ and $\alpha \in C(J, J)$. Let $\det Q \neq 0$. If problem (10) has a unique solution $z \in C(J, R^p)$, then problem (1) has the unique solution $y \in C^1(J, R^p)$ given by formula (9).

4. EXISTENCE, UNIQUENESS RESULTS

In this section, we will give results concerning existence and uniqueness of solutions of problem (10). Notice that the Green's function G_2 is continuous with respect to t for fixed s and it has a first kind discontinuity along the line $s = \alpha(t)$. It is known that problem (10) has the unique continuous solution if the condition

$$K_1 = \max_{t \in J} \int_0^b \|G_2(t, s)\| ds < 1 \quad (11)$$

holds. This result is obtained by the Banach fixed point theorem (use the norm $\|u\|_\infty = \max_{t \in J} \|u(t)\|$).

Remark 1

Assume that the matrix $A_2(t) = O_{p \times p}$, $t \in J$, and Q is nonsingular. Then $G_2(t, s) = O_{p \times p}$, $t, s \in J$. In this case, $K_1 = 0$, so $z(t) = g(t)$, $t \in J$ is the unique solution of equation (10). Problem (1) has the unique solution y given by formula (9) with z as above, i.e.

$$y(t) = \Phi(t)D + \int_0^b G_1(t, s)g(s)ds, \quad t \in J.$$

This linear case, without the retardations, was considered by many authors, see for example [1], p.94.

Remark 2

Sometimes condition (11) may be weakened too. It will be possible when there

exists a constant $\rho > 0$ for which the following condition

$$\rho^{-1} (\exp(\rho b) - 1) \max_{t \in J} \left(\exp(-\rho t) \sup_{s \in J} \|G_2(t, s)\| \right) < 1$$

holds. The last condition will appear when one uses the Banach fixed point theorem with Bielecki's norm. It is worth noticing that problem (10) has a unique continuous solution also in the case when the condition

$$\sup_{s \in J} \|G_2(t, s)\| \leq K (\exp(\rho t) - 1) [\exp(\rho b) - 1]^{-1}$$

is satisfied. Then, apply Bielecki's norm with $\rho \geq K$. Now we need to estimate y appearing in (9). Indeed, we see that

$$\|y\|_\infty = \max_{t \in J} \|y(t)\| \leq \|\Phi\|_\infty \|D\| + \max_{t \in J} \int_0^b \|G_1(t, s)z(s)\| ds. \tag{12}$$

Notice that the expression

$$\max_{t \in J} \int_0^b \|G_1(t, s)z(s)\| ds \tag{13}$$

may be estimated on different ways. One of them is the following:

$$\max_{t \in J} \int_0^b \|G_1(t, s)z(s)\| ds \leq \|z\|_\infty \max_{t \in J} \int_0^b \|G_1(t, s)\| ds.$$

To complete the above estimate we need to add the corresponding estimate for z from (10). It is simple to note that (10) yields

$$\begin{aligned} \|z\|_\infty &\leq \|Q_1\|_\infty + \max_{t \in J} \int_0^b \|G_2(t, s)\| \|z(s)\| ds \\ &\leq \|Q_1\|_\infty + K_1 \|z\|_\infty. \end{aligned}$$

Hence

$$\|z\|_\infty \leq \frac{1}{1 - K_1} \|Q_1\|_\infty,$$

and

$$\max_{t \in J} \int_0^b \|G_1(t, s)z(s)\| ds \leq \frac{K_2}{1 - K_1} \|Q_1\|_\infty,$$

where the constant K_2 is defined by

$$K_2 = \max_{t \in J} \int_0^b \|G_1(t, s)\| ds.$$

Adding this to (12) we obtain

$$\|y\|_\infty \leq \|\Phi\|_\infty \|D\| + \frac{K_2}{1 - K_1} \|Q_1\|_\infty \leq \rho_1 (\|D\| + \|Q_1\|_\infty), \quad (14)$$

where

$$\rho_1 = \max \left(\|\Phi\|_\infty, \frac{K_2}{1 - K_1} \right).$$

The expression (13) can also be estimated in another way, namely

$$\max_{t \in J} \int_0^b \|G_1(t, s)z(s)\| ds \leq \max_{t, s \in J} \|G_1(t, s)\| \int_0^b \|z(s)\| ds = \|G_1\|_\infty \|z\|_1,$$

where

$$\|G_1\|_\infty = \max_{t, s \in J} \|G_1(t, s)\|, \quad \|z\|_1 = \int_0^b \|z(s)\| ds.$$

Moreover, using the norm $\|\cdot\|_1$ for equation (10), we obtain

$$\begin{aligned} \|z\|_1 &\leq \int_0^b \int_0^b \|G_2(t, s)\| \|z(s)\| ds dt + \|Q_1\|_1 \\ &\leq \max_{s \in J} \int_0^b \|G_2(t, s)\| dt \|z\|_1 + \|Q_1\|_1. \end{aligned}$$

If condition (11) is satisfied, then

$$\|z\|_1 \leq \frac{1}{1 - K_1} \|Q_1\|_1,$$

so

$$\begin{aligned} \|y\|_\infty &\leq \|\Phi\|_\infty \|D\| + \frac{\|G_1\|_\infty}{1 - K_1} \|Q_1\|_1 \\ &\leq \rho_2 (\|D\| + \|Q_1\|_1), \end{aligned} \quad (15)$$

where

$$\rho_2 = \max \left(\|\Phi\|_\infty, \frac{\|G_1\|_\infty}{1 - K_1} \right).$$

The constant ρ_1 (or ρ_2) is called in the literature as a stability constant or the conditioning constant. If the conditioning constant of problem (1) is a

constant of moderate size, then problem (1) is well-conditioned. Problem (1) is well-conditioned if a small change in the data should produce only a small change in the solution y of problem (1). Notice that knowing ρ_1 (or ρ_2), we may say what is the resulting perturbation of y when the data D and g are perturbed (see (14) or (15)). It is important in problems of stability in numerical methods (see, for example [2], [4], [5]).

Remark 3

Consider the case from Remark 1, i.e. when $z(t) = g(t), t \in J$ is the unique solution of equation (10). Then, by using Hölder's inequality, we have

$$\max_{t \in J} \int_0^b \|G_1(t, s)g(s)\| ds \leq c_q \|g\|_{\bar{q}}, \quad \frac{1}{\bar{q}} + \frac{1}{q} = 1,$$

where

$$c_q = \max_{t \in J} \left(\int_0^b \|G_1(t, s)\|^q ds \right)^{\frac{1}{q}}, \quad \|g\|_{\bar{q}} = \left(\int_0^b \|g(s)\|^{\bar{q}} ds \right)^{\frac{1}{\bar{q}}}.$$

It yields the following result

$$\|y\|_{\infty} \leq \rho_3 (\|D\| + \|g\|_{\bar{q}})$$

with $\rho_3 = \max(\|\Phi\|_{\infty}, c_q)$ (see, for example [1]).

Consider the perturbed problem of the form

$$\begin{aligned} w'(t) &= A_1(t)w(t) + A_2(t)w(\alpha(t)) + g^*(t), \quad t \in J, \\ B_1w(0) + B_2w(b) &= D^* \in R^p, \end{aligned} \tag{16}$$

where $g^* \in C(J, R^p)$. Define the error by $e(t) = w(t) - y(t), t \in J$, where y denotes the solution of problem (1). Indeed, e satisfies the following problem:

$$\begin{aligned} e'(t) &= A_1(t)e(t) + A_2(t)e(\alpha(t)) + g^*(t) - g(t), \quad t \in J, \\ B_1e(0) + B_2e(b) &= D^* - D. \end{aligned} \tag{17}$$

By (9)-(12) and (14)-(15), we obtain

$$\|e\|_{\infty} \leq \rho_1 (\|D^* - D\| + \|Q^*\|_{\infty}),$$

or

$$\|e\|_{\infty} \leq \rho_2 (\|D^* - D\| + \|Q^*\|_1)$$

with

$$Q^*(t) = A_2(t)\Phi(\alpha(t))(D^* - D) + g^*(t) - g(t), \quad t \in J.$$

5. MULTIPOINT BOUNDARY PROBLEMS

Let us extend the previous discussion to multipoint boundary problems of the form

$$\begin{aligned} y'(t) &= A_0(t)y(t) + \sum_{i=1}^r A_i(t)y(\alpha_i(t)) + g(t), \quad t \in J, \\ \sum_{i=1}^m B_i y(\xi_i) &= D \in R^p, \quad m \geq 2, \end{aligned} \tag{18}$$

where B_1, \dots, B_m are constant square matrices of order p ; A_0, \dots, A_r are continuous matrices of order $p \times p$ and $g \in C(J, R^p)$, $\alpha_i \in C(J, J)$, $i = 1, 2, \dots, r$. Here, $\xi_i \in J$, $i = 1, 2, \dots, m$ and $a = \xi_1 < \xi_2 < \dots < \xi_m = b$. The points $\xi_1, \xi_2, \dots, \xi_m$ are called switching points. Let now Y be any fundamental solution connected with the matrix A_0 , i.e.

$$Y'(t) = A_0(t)Y(t), \quad t \in J.$$

Repeating the discussion of the previous sections we can express any solution y of problem (18) in the form

$$y(t) = \Phi(t)D + \int_0^b G_1^*(t, s)z(s)ds, \quad t \in J \tag{19}$$

if the $p \times p$ matrix

$$Q = \sum_{i=1}^m B_i Y(\xi_i),$$

appearing in Φ , is nonsingular. The Green's function, G_1^* is, now defined for $t \in [\xi_1, \xi_2]$ by

$$G_1^*(t, s) = \begin{cases} \Phi(t)B_1\Phi(0)\Phi^{-1}(s) & \text{if } s \leq t, \\ - \Phi(t) \sum_{i=2}^m B_i\Phi(\xi_i)\Phi^{-1}(s) & \text{if } t < s \leq \xi_2 \\ - \Phi(t) \sum_{i=k+1}^m B_i\Phi(\xi_i)\Phi^{-1}(s) & \text{if } \xi_k < s \leq \xi_{k+1}, k = 2, \dots, m-1 \end{cases}$$

and if $t \in (\xi_j, \xi_{j+1}]$ for $j = 2, \dots, m-1$, then

$$G_1^*(t, s) = \begin{cases} \Phi(t) \sum_{i=1}^k B_i\Phi(\xi_i)\Phi^{-1}(s) & \text{if } \xi_k < s \leq \xi_{k+1}, k = 1, \dots, j-1, \\ \Phi(t) \sum_{i=1}^j B_i\Phi(\xi_i)\Phi^{-1}(s) & \text{if } \xi_j < s \leq t, \\ - \Phi(t) \sum_{i=j+1}^m B_i\Phi(\xi_i)\Phi^{-1}(s) & \text{if } t < s \leq \xi_{j+1}, \\ - \Phi(t) \sum_{i=n+1}^m B_i\Phi(\xi_i)\Phi^{-1}(s) & \text{if } \xi_n < s \leq \xi_{n+1}, n = j+1, \dots, m-1 \end{cases}$$

The function z appearing in (19) is considered as a solution of the equation

$$z(t) = \int_0^b G_2^*(t, s)z(s)ds + Q_1^*(t), \quad t \in J, \tag{20}$$

where

$$Q_1^*(t) = \sum_{i=1}^r A_i(t)\Phi(\alpha_i(t))D + g(t), \quad t \in J,$$

$$G_2^*(t, s) = \sum_{j=1}^r H_j(t, s), \quad t, s \in J.$$

The Green's functions H_j , for $j = 1, 2, \dots, r$, are defined by:

$$H_j(t, s) = \begin{cases} A_j(t)\Phi(\alpha_j(t))B_1\Phi(0)\Phi^{-1}(s) & \text{if } s \leq \alpha_j(t), \\ - A_j(t)\Phi(\alpha_j(t)) \sum_{i=2}^m B_i\Phi(\xi_i)\Phi^{-1}(s) & \text{if } \alpha_j(t) < s \leq \xi_2, \\ - A_j(t)\Phi(\alpha_j(t)) \sum_{i=k+1}^m B_i\Phi(\xi_i)\Phi^{-1}(s) & \text{if } \xi_k < s \leq \xi_{k+1}, \\ & k = 2, \dots, m-1 \end{cases}$$

provided that $\alpha_j(t) \leq \xi_2$, and by

$$H_j(t, s) = \begin{cases} A_j(t)\Phi(\alpha_j(t))B_1\Phi(0)\Phi^{-1}(s) & \text{if } s \leq \xi_2, \\ A_j(t)\Phi(\alpha_j(t))\sum_{i=1}^k B_i\Phi(\xi_i)\Phi^{-1}(s) & \text{if } \xi_k < s \leq \xi_{k+1}, \\ & k = 2, \dots, n-1, \\ A_j(t)\Phi(\alpha_j(t))\sum_{i=1}^n B_i\Phi(\xi_i)\Phi^{-1}(s) & \text{if } \xi_n < s \leq \alpha_j(t), \\ - A_j(t)\Phi(\alpha_j(t))\sum_{i=n+1}^m B_i\Phi(\xi_i)\Phi^{-1}(s) & \text{if } \alpha_j(t) < s \leq \xi_{n+1}, \\ - A_j(t)\Phi(\alpha_j(t))\sum_{i=k+1}^m B_i\Phi(\xi_i)\Phi^{-1}(s) & \text{if } \xi_k < s \leq \xi_{k+1}, \\ & k = n+1, \dots, m-1 \end{cases}$$

if $\xi_n < \alpha_j(t) \leq \xi_{n+1}$, $n = 2, 3, \dots, m-1$. Notice that equations (19) and (20) have the same construction as equations (9) and (10), respectively. Hence, indeed, we can obtain immediately results which are similar to those from sections 3 and 4.

At the end of this paper, we are going to mention about some benefits when differential equations of type (18) are replaced by their integral representation of type (19)-(20). If one wants to solve problem (18) by numerical techniques operating on the differential equation then may meet numerical instabilities. Such problems may be avoided when we replace the differential form by the equivalent integral one. Notice that finite difference methods or shooting and multiple shooting methods applied to differential problems require the solution of large systems of linear or nonlinear equations. Such aspects are considered in paper [7] for two-point boundary value problems of differential equations of second order.

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ACTION OF THE GROUP $M = \langle x, y : x^2 = y^6 = 1 \rangle$ ON CERTAIN REAL QUADRATIC FIELDS

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Abstract:

Studying groups through their actions on different sets and algebraic structures has become a useful technique to know about the structure of the groups. In this paper we examine the action of the infinite group $M = \langle x, y : x^2 = y^6 = 1 \rangle$ where $x : z \rightarrow \frac{-1}{3z}$ and $y : z \rightarrow \frac{-1}{3(z+1)}$, on real quadratic field $Q(\sqrt{n})$ and find invariant subsets of $Q(\sqrt{n})$ under the action of the group M . Moreover we discuss the properties of real quadratic irrational numbers under the action of the group M .

1. INTRODUCTION

Throughout this paper, for any two integers a and b , (a, b) denotes the greatest common divisor of a and b and n denotes a non square positive integer. We take:

$$Q^*(\sqrt{n}) = \left\{ \frac{a+\sqrt{n}}{c} : \frac{a^2-n}{c} \text{ is an integer and } (a, \frac{a^2-n}{c}, c) = 1 \right\}.$$

$$Q'''(\sqrt{n}) = \left\{ \frac{a}{t} : a \in Q^*(\sqrt{n}), t = 1, 3 \right\}$$

$$Q^{***}(\sqrt{n}) = \left\{ \frac{a+\sqrt{n}}{c} : \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n}) \text{ and } 3|c \right\}$$

$$Q^{**}(\sqrt{n}) = \left\{ \frac{a+\sqrt{n}}{c} : \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n}) \text{ and } 2|c \right\} \text{ and}$$

$$Q'(\sqrt{n}) = \left\{ \frac{a+\sqrt{n}}{c} : \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n}), a \text{ odd and } b, c \text{ both even integers} \right\}$$

An element $a + b\sqrt{n}$, where $b \neq 0$, of $Q(\sqrt{n}) = \{a + b\sqrt{n} : a, b \in Q\}$ is called a real quadratic irrational number.

Let $\alpha = \frac{a+\sqrt{n}}{c}$. If $a, \frac{a^2-n}{c}$ and c are relatively prime integers, we say that α is in canonical form. Also if $\alpha = \frac{a+\sqrt{n}}{c}$ then we write $\bar{\alpha} = \frac{a-\sqrt{n}}{c}$ so that $-\bar{\alpha} = \frac{-a+\sqrt{n}}{c}$. For $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$; α and its conjugate $\bar{\alpha}$ may or may not have the same sign. If α and $\bar{\alpha}$, as real numbers, have different signs then α is called an ambiguous number [3]. If $\alpha = \frac{a+\sqrt{n}}{c}$, then $N(\alpha) = \alpha\bar{\alpha} = \frac{a^2-n}{c^2}$ is called the norm of α .

Let $R' = R \cup \{\pm\infty\}$ be the extended real line. The action of the modular group $PSL(2, Z)$ on the real quadratic fields, subsets of R' , has been studied in [2]. In this paper we study the action of M on such fields. We have used coset diagrams to study this action.

In our case a coset diagram is just a graphical representation of a permutation action of the group M : the 6-cycles of the transformation y are denoted by six vertices of a hexagon permuted anti-clockwise by y and the two vertices which are interchanged by x are joined by an edge. Fixed points of x and y , if they exist, are denoted by heavy dots.

Coset diagrams for the orbit of the group M acting on real quadratic fields give some interesting information. By using these coset diagrams, Mushtaq and Aslam [4] have shown that, in the orbit α^M , the non square positive integer n does not change its value and the ambiguous numbers in canonical form are finite in number and that part of the coset diagram containing such numbers forms a single closed path. It is important to bear in mind that closed path in the orbit of α is unique except for the hexagons.

Now it becomes interesting to investigate the invariant subsets of $Q(\sqrt{n})$ under the action of M on one hand and to discuss the properties of real quadratic irrational numbers under the action of the group M on the other.

We start our discussion with the following lemma which gives a necessary and sufficient for both α and $\frac{\alpha}{3}$ to be in $Q^*(\sqrt{n})$.

LEMMA 1.1

Let $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ and $b = \frac{a^2-n}{c}$. Then:

1. $\frac{\alpha}{3}$ belongs to $Q^*(\sqrt{n})$ if and only if $3|b$.
2. $\frac{\alpha}{3}$ belongs to $Q^*(\sqrt{9n})$ if and only if $3 \nmid b$.

PROOF

1. Let $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ be such that $3|b$. Then $\frac{b}{3}$ is an integer, and $(a, b, c) = 1 \Leftrightarrow (a, \frac{b}{3}, 3c) = 1$. Hence $\frac{\alpha}{3} = \frac{a+\sqrt{n}}{3c}$ belongs to $Q^*(\sqrt{n})$.

Conversely suppose that $\frac{\alpha}{3} = \frac{a+\sqrt{n}}{3c}$ belongs to $Q^*(\sqrt{n})$. Then clearly $\frac{a^2-n}{3c} = \frac{b}{3}$ is an integer and hence $3|b$.

2. Let $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ and $3 \nmid b$. Obviously $(a, \frac{a^2-n}{c}, c) = 1 \Leftrightarrow (3a, \frac{a^2-n}{c}, 9c) = (3a, \frac{(3a)^2-9n}{9c}, 9c) = 1$. So $\frac{\alpha}{3} = \frac{a+\sqrt{n}}{3c} = \frac{3a+\sqrt{9n}}{9c}$ belongs to $Q^*(\sqrt{9n})$.

Conversely let $\frac{\alpha}{3} = \frac{a+\sqrt{n}}{3c} \in Q^*(\sqrt{9n})$. Then $\frac{a+\sqrt{n}}{3c} = \frac{3a+\sqrt{9n}}{9c}$ and $\frac{(3a)^2-9n}{9c} = \frac{9a^2-9n}{9c} = \frac{a^2-n}{c}$ is an integer.

Moreover $(3a, \frac{a^2-n}{c}, 9c) = 1$ which is possible only if $3 \nmid b$.

REMARKS 1.2

1. Since $A = \left\{ \frac{a+\sqrt{n}}{3c} : \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n}), \frac{a^2-n}{c} \text{ is not divisible by } 3 \right\}$ is a subset of $Q^*(\sqrt{9n})$, by lemma 1.1, we have $Q^*(\sqrt{n})$ and A are disjoint and $Q'''(\sqrt{n}) = Q^*(\sqrt{n}) \cup A$.
Hence $\{Q^*(\sqrt{n}), A\}$ is a partition of $Q'''(\sqrt{n})$.
It is important to note that $Q'''(\sqrt{n}) \subset Q^*(\sqrt{n}) \cup Q^*(\sqrt{9n})$.
2. If n and m are two distinct non-square positive integers then $Q^*(\sqrt{n})$ and $Q^*(\sqrt{m})$ are disjoint sets, whereas $Q'''(\sqrt{n})$ and $Q'''(\sqrt{m})$ are not necessarily disjoint.
In particular, $Q^*(\sqrt{9n})$ and $Q^*(\sqrt{n})$ are disjoint whereas $Q'''(\sqrt{n})$ and $Q'''(\sqrt{9n})$ are overlapping (i.e. not disjoint).
3. In the light of lemma 1.1 we may write $Q'''(\sqrt{n}) = \left\{ \frac{\alpha}{t} : \alpha \in Q^*(\sqrt{n}), t = 1, 3 \right\}$

2. PROPERTIES OF REAL QUADRATIC IRRATIONAL NUMBERS UNDER THE ACTION OF THE GROUP M

Let $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ with $b = \frac{a^2-n}{c}$. We list the actions on α of $x, y, xy, xy^2, xy^3, xy^4, xy^5$ and various other combinations of x, y in the following table 1. These will be used in the sequel.

The following lemma establishes a relationship between $x(\alpha)$ and $y(\alpha), \alpha \in Q'''(\sqrt{n})$.

LEMMA 2.1

Let α be a fixed element of $Q'''(\sqrt{n})$. Then we have

$$y^5(\alpha) = -1 + x(\alpha), xy^2(\alpha) = 1 + y(\alpha) \text{ and } y^4x(\alpha) = -1 - y(-\alpha)$$

PROOF

Proof is straightforward and follows from table 1.

Note that these are fundamental relations between $x(\alpha)$ and $y(\alpha)$ and we can derive more relations from these fundamental relations.

TABLE 1

α	a	b	c
$x(\alpha) = \frac{-1}{3\alpha}$	$-a$	$\frac{c}{3}$	$3b$
$y(\alpha) = \frac{-1}{3(\alpha+1)}$	$-a - c$	$\frac{c}{3}$	$3(2a + b + c)$
$y^2(\alpha) = \frac{-(\alpha+1)}{3\alpha+2}$	$-5a - 3b - 2c$	$2a + b + c$	$12a + 9b + 4c$
$y^3(\alpha) = \frac{-(3\alpha+2)}{3(2\alpha+1)}$	$-7a - 6b - 2c$	$\frac{12a+9b+4c}{3}$	$3(4a + 4b + c)$
$y^4(\alpha) = \frac{-(2\alpha+1)}{3\alpha+1}$	$-5a - 6b - c$	$4a + 4b + c$	$6a + 9b + c$
$y^5(\alpha) = \frac{-(3\alpha+1)}{3\alpha}$	$-a - 3b$	$\frac{(6a+9b+c)}{3}$	$3b$
$xy(\alpha) = \alpha + 1$	$a + c$	$2a + b + c$	c
$xy^2(\alpha) = \frac{3\alpha+2}{3(\alpha+1)}$	$5a + 3b + 2c$	$\frac{(12a+9b+4c)}{3}$	$3(2a + b + c)$
$xy^3(\alpha) = \frac{2\alpha+1}{3\alpha+2}$	$7a + 6b + 2c$	$4a + 4b + c$	$12a + 9b + 4c$
$xy^4(\alpha) = \frac{3\alpha+1}{3(2\alpha+1)}$	$5a + 6b + c$	$\frac{(6a+9b+c)}{3}$	$3(4a + 4b + c)$
$xy^5(\alpha) = \frac{\alpha}{3\alpha+1}$	$a + 3b$	b	$6a + 9b + c$
$yx(\alpha) = \frac{\alpha}{1-3\alpha}$	$a - 3b$	b	$-6a + 9b + c$
$y^2x(\alpha) = \frac{1-3\alpha}{3(-1+2\alpha)}$	$5a - 6b - c$	$\frac{(-6a+9b+c)}{3}$	$3(-4a + 4b + c)$
$y^3x(\alpha) = \frac{1-2\alpha}{-2+3\alpha}$	$7a - 6b - 2c$	$-4a + 4b + c$	$-12a + 9b + 4c$
$y^4x(\alpha) = \frac{2-3\alpha}{3(-1+\alpha)}$	$5a - 3b - 2c$	$\frac{(-12a+9b+4c)}{3}$	$3(-2a + b + c)$
$y^5x(\alpha) = \alpha - 1$	$a - c$	$-2a + b + c$	c

For $x, y \in M$, we have $x^{-1} = x, y^{-1} = y^5, (y^2)^{-1} = y^4, (y^3)^{-1} = y^3, (xy)^{-1} = y^5x, (xy^2)^{-1} = y^4x, (xy^3)^{-1} = y^3x, (xy^4)^{-1} = y^2x$ and $(xy^5)^{-1} = yx$.

As each $g \in M$ is a word in x, y, y^2, y^3, y^4, y^5 so $x, y, y^2, y^3, y^4, y^5, xy, xy^2, xy^3, xy^4$ and xy^5 are important elements of M . Hence we discuss the elements $x(\alpha), y(\alpha), y^2(\alpha), y^3(\alpha), xy(\alpha), xy^2(\alpha), xy^3(\alpha), xy^4(\alpha)$ and $xy^5(\alpha), \alpha \in Q'''(\sqrt{n})$. Properties of elements of $Q'''(\sqrt{n})$ under the action of M , covered in this section, are expressed in terms of following results. Most of these properties

can be viewed as a bridge by which ambiguous numbers in a particular orbit α^M , where $\alpha \in Q'''(\sqrt{n})$, are connected to form a unique closed path in the coset diagram.

Before a discussion of these elements of $Q'''(\sqrt{n})$ we prove the following lemma which shows that the orbit α^M of $\alpha = \frac{a+\sqrt{n}}{c}$ contains all $\alpha' = \frac{(a+kc)+\sqrt{n}}{c}$, $k \in Z$.

LEMMA 2.2

Let $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ and c fixed. Then elements of the form $\frac{a'+\sqrt{n}}{c}$ of $Q^*(\sqrt{n})$, $a' = a + kc$, $k \in Z$, belong to α^M .

PROOF

Proof is straightforward.

Before we come to the properties of elements of $Q'''(\sqrt{n})$ under the action of M , we need to establish the following theorem, which gives us a superset of $Q^*(\sqrt{n})$ invariant under the action of M .

THEOREM 2.3

Let $M = \langle x, y : x^2 = y^6 = 1 \rangle$ and $Q'''(\sqrt{n}) = \{ \frac{\alpha}{t} : \alpha \in Q^*(\sqrt{n}), t = 1, 3 \}$

Then $Q'''(\sqrt{n})$ is invariant under the action of M .

PROOF

In [3] it has been proved that $Q^*(\sqrt{n})$ is invariant under the action of the modular group $PSL(2, Z) = \langle x', y' : x'^2 = y'^3 = 1 \rangle$ where $x'(\alpha) = \frac{-1}{\alpha}$, $y'(\alpha) = \frac{\alpha-1}{\alpha}$.

If $y'' = x'y'x'$, then $y''(\alpha) = \frac{-1}{\alpha+1}$

and $PSL(2, Z) = \langle x', y'' : (x')^2 = (y'')^3 = 1 \rangle$.

Now it is clear that $(x')^2 = 1$, $(y'')^3 = 1$, $(y'')^2(\alpha) = -(\frac{\alpha+1}{\alpha})$, $(y'')^3(\alpha) = \alpha$.

So $x'(\alpha)$, $y''(\alpha)$, $(y'')^2(\alpha)$ belong to $Q^*(\sqrt{n}) \forall \alpha \in Q^*(\sqrt{n})$.

Now $x(\alpha) = \frac{1}{3}x'(\alpha)$, $y(\alpha) = \frac{1}{3}y''(\alpha)$ both belong to $Q'''(\sqrt{n})$, $\forall \alpha \in Q^*(\sqrt{n})$.

As $x(\frac{\alpha}{3}) = \frac{-1}{\alpha}$, $y(\frac{\alpha}{3}) = \frac{-1}{3[\frac{\alpha}{3}+1]} = \frac{-1}{\alpha+3} = \frac{-1}{\alpha'}$, $\alpha' = \alpha + 3$ belong to $Q^*(\sqrt{n})$,

so $x(\frac{\alpha}{3}), y(\frac{\alpha}{3})$ both belong to $Q^*(\sqrt{n})$, $\forall \alpha \in Q^*(\sqrt{n})$.

Since every element of the group M is a word in the generators x, y of the group M and every element of $Q'''(\sqrt{n})$ is mapped onto an element of $Q'''(\sqrt{n})$ by x and y , so $Q'''(\sqrt{n})$ is invariant under the action of M .

In what follows, some properties of the elements of $Q'''(\sqrt{n})$ are discussed under the action of M . Most of these properties can be viewed as a bridge by which ambiguous numbers in a particular orbit α^M , where $\alpha \in Q'''(\sqrt{n})$, are connected to form a unique closed path in the coset diagram.

The next lemma shows that image of a conjugate of element of $Q'''(\sqrt{n})$ under an element of M is the conjugate of the image.

LEMMA 2.4

Let $\alpha \in Q'''(\sqrt{n})$, then $g(\bar{\alpha}) = \overline{g(\alpha)}$, $\forall g \in M$

PROOF

For any $\alpha \in Q'''(\sqrt{n})$, the lemma follows from the equations

$$\overline{x(\alpha)} = \overline{\left[\frac{-1}{3\alpha}\right]} = \frac{-1}{3\bar{\alpha}} = x(\bar{\alpha}),$$

$$y(\bar{\alpha}) = \frac{-1}{3(1+\bar{\alpha})} = \overline{\left[\frac{-1}{3(1+\alpha)}\right]} = \overline{y(\alpha)},$$

$$\overline{y^2(\alpha)} = \overline{y(y(\alpha))} = y(\overline{y(\alpha)}) = y(y(\bar{\alpha})) = y^2(\bar{\alpha})$$

and

$$\overline{y^3(\alpha)} = \overline{y(y^2(\alpha))} = y(\overline{y^2(\alpha)}) = y(y^2(\bar{\alpha})) = y^3(\bar{\alpha})$$

because each $g \in M$ is a word in x, y, y^2, y^3, y^4 and y^5 .

DEFINITION 2.5

Let $\alpha \in Q'''(\sqrt{n})$. Then the number of ambiguous numbers in the orbit α^M is called the ambiguous length of α with respect to M .

LEMMA 2.6

For $\alpha \in Q'''(\sqrt{n})$ and any β in α^M

1. $x(-\beta) = -x(\beta)$

2. $yx(-\beta) = -[xy^5(\beta)]$
3. $y^2x(-\beta) = -[xy^4(\beta)],$
4. $xy^3(-\beta) = -[y^3x(\beta)]$
5. $y^4x(-\beta) = -[xy^2(\beta)],$
6. $y^5x(-\beta) = -[xy(\beta)]$

PROOF

1. Here, for $\alpha \in Q'''(\sqrt{n})$ and β in α^M , $x(\beta) = \frac{-1}{3\beta} = \frac{1}{-3\beta} = -x(-\beta)$

Proofs of (2),(3),(4),(5) and (6) follow from table 1. The following corollary is an immediate consequence of lemmas 2.4 and 2.6.

COROLLARY 2.7

For a β in α^M , $\alpha \in Q'''(\sqrt{n})$.

1. $x(-\bar{\beta}) = \overline{x(-\beta)} = -\overline{x(\beta)} = -x(\bar{\beta})$
2. $xy^2(-\bar{\beta}) = \overline{xy^2(-\beta)} = -\overline{[y^4x(\beta)]} = -[y^4x(\bar{\beta})]$
3. $yx(-\bar{\beta}) = \overline{yx(-\beta)} = -\overline{[xy^5(\beta)]} = -[xy^5(\bar{\beta})]$
4. $y^3x(-\bar{\beta}) = \overline{y^3x(-\beta)} = -\overline{[xy^3(\beta)]} = -[xy^3(\bar{\beta})]$
5. $y^2x(-\bar{\beta}) = -\overline{[xy^4(\beta)]} = \overline{[y^2x(-\beta)]} = -[xy^4(\bar{\beta})]$ and
6. $y^5x(-\bar{\beta}) = \overline{xy(-\beta)} = -\overline{[y^5x(\beta)]} = -[xy(\bar{\beta})]$

Following remark is a combination of results from lemmas 2.4,2.6, and corollary 2.7 and gives a connection between elements of $Q'''(\sqrt{n})$ in the coset diagram under the action of M .

REMARK 2.8

1. Using lemma 2.4, it is easy to see that: For $\alpha \in Q'''(\sqrt{n})$, if $\bar{\alpha} \in \alpha^M$ then, for all $\beta \in \alpha^M, \bar{\beta} \in \alpha^M$
2. Using lemma 2.6, we have: For $\alpha \in Q'''(\sqrt{n})$, if $-\alpha \in \alpha^M$ then, for all $\beta \in \alpha^M, -\beta \in \alpha^M$
3. Corollary 2.7 gives: For $\alpha \in Q'''(\sqrt{n})$, if $-\bar{\alpha} \in \alpha^M$ then, for all $\beta \in \alpha^M, -\bar{\beta} \in \alpha^M$

REMARK 2.9

For any $\alpha \in Q'''(\sqrt{n})$, since $g(\bar{\alpha}) = \overline{g(\alpha)}$ for all $g \in M$, $[\bar{\alpha}]^M$ consists of just conjugates of elements of α^M and vice versa. So, for each $\alpha \in Q'''(\sqrt{n})$, the ambiguous lengths of α and $\bar{\alpha}$ are the same.

A necessary condition for the orbits α^M and $[\bar{\alpha}]^M$ to be identical is given in the lemma that follows.

LEMMA 2.10

For $\alpha \in Q'''(\sqrt{n})$ let $N(\alpha) = \alpha\bar{\alpha} = \frac{-1}{3}$, then $\alpha^M = (\bar{\alpha})^M$.

PROOF

Here $N(\alpha) = \alpha\bar{\alpha} = \frac{-1}{3} \Leftrightarrow \bar{\alpha} = \frac{-1}{3\alpha} = x(\alpha)$ and $x(\bar{\alpha}) = \alpha^M$. So $\alpha \in (\bar{\alpha})^M, \alpha^M$ and $\bar{\alpha} \in \alpha^M$. As $\alpha \in \alpha^M, \alpha^M$ and $(\bar{\alpha})^M$ are not disjoint so $\alpha^M = (\bar{\alpha})^M$.

The converse of lemma 2.10 is false. That is if $\alpha^M = (\bar{\alpha})^M$, then $N(\alpha)$ may or may not be $\frac{-1}{3}$. For example $(\frac{3+\sqrt{3}}{3})^M = (\frac{3-\sqrt{3}}{3})^M$, but $N(\frac{3-\sqrt{3}}{3}) = \frac{2}{3} \neq \frac{-1}{3}$.

LEMMA 2.11

For each ambiguous number $\alpha \in Q'''(\sqrt{n})$, we have:

$x(\alpha) \neq \alpha, y(\alpha) \neq \alpha, y^2(\alpha) \neq \alpha, y^3(\alpha) \neq \alpha, xy(\alpha) \neq \pm\alpha, xy^2(\alpha) \neq \pm\alpha, xy^4(\alpha) \neq -\alpha, xy^5(\alpha) \neq \pm\alpha, xy^3(\alpha) \neq -\alpha, x(\alpha) \neq -\bar{\alpha}, y(\alpha) \neq \pm\bar{\alpha}, y^2(\alpha) \neq \pm\bar{\alpha}, y^3(\alpha) \neq -\bar{\alpha}, xy(\alpha) \neq \bar{\alpha}, xy^2(\alpha) \neq \bar{\alpha}, xy^3(\alpha) \neq \bar{\alpha}, xy^4(\alpha) \neq \bar{\alpha}, xy^5(\alpha) \neq \bar{\alpha}$ and $x(\alpha) = \bar{\alpha} \Leftrightarrow N(\alpha) = \frac{\alpha^2 - n}{c^2} = \frac{-1}{3}$. In such a case $3n = 3a^2 + c^2$.

PROOF

The proof is straightforward and follows from table 1. Moreover proofs are analogous to the proofs of following lemmas 2.12 and 2.14.

LEMMA 2.12

Let $\alpha \in Q'''(\sqrt{n})$. Then :

1. $x(\alpha) = -\alpha \Leftrightarrow \alpha = \frac{\pm\sqrt{3}}{3}$.
2. $y(\alpha) = -\alpha \Leftrightarrow \alpha = \frac{-3 \pm \sqrt{21}}{6}$.
3. $y^2(\alpha) = -\alpha \Leftrightarrow \alpha = \frac{-1 \pm \sqrt{13}}{6}$.
4. $y^3(\alpha) = -\alpha \Leftrightarrow \alpha = \frac{\pm\sqrt{3}}{3}$.
5. $xy^3(\alpha) = \alpha \Leftrightarrow \alpha = \frac{\pm\sqrt{3}}{3}$.
6. $xy^4(\alpha) = \alpha \Leftrightarrow \alpha = \frac{\pm\sqrt{6}}{6}$.

PROOF

Let $\alpha \in Q'''(\sqrt{n})$. Then:

1. $x(\alpha) = -\alpha \Leftrightarrow \frac{-1}{3\alpha} = -\alpha \Leftrightarrow \alpha^2 = \frac{1}{3} \Leftrightarrow \alpha = \frac{\pm\sqrt{3}}{3}$.
2. $y(\alpha) = -\alpha \Leftrightarrow \frac{-1}{3(\alpha+1)} = -\alpha \Leftrightarrow 3\alpha^2 + 3\alpha - 1 = 0 \Leftrightarrow \alpha = \frac{-3 \pm \sqrt{21}}{6}$.
3. $y^2(\alpha) = -\alpha \Leftrightarrow \frac{-(\alpha+1)}{3\alpha+2} = -\alpha \Leftrightarrow 3\alpha^2 + \alpha - 1 = 0 \Leftrightarrow \alpha = \frac{-1 \pm \sqrt{13}}{6}$.
4. $y^3(\alpha) = -\alpha \Leftrightarrow \frac{-(3\alpha+2)}{6\alpha+3} = -\alpha \Leftrightarrow \alpha^2 = \frac{1}{3} \Leftrightarrow \alpha = \frac{\pm\sqrt{3}}{3}$.
5. $xy^3(\alpha) = \alpha \Leftrightarrow \frac{2\alpha+1}{3\alpha+2} = \alpha \Leftrightarrow 2\alpha + 1 = 3\alpha^2 + 2\alpha \Leftrightarrow \alpha = \frac{\pm\sqrt{3}}{3}$.
6. $xy^4(\alpha) = \alpha \Leftrightarrow \frac{3\alpha+1}{6\alpha+3} = \alpha \Leftrightarrow \alpha^2 = \frac{1}{6} \Leftrightarrow \alpha = \frac{\pm\sqrt{6}}{6}$.

The following are the cosequences of lemma 2.12.

REMARKS 2.13

1. $x(\alpha) \neq -\alpha, \forall \alpha \in Q'''(\sqrt{n}), n \neq 3.$
2. $y(\alpha) \neq -\alpha, \forall \alpha \in Q'''(\sqrt{n}), n \neq 21.$
3. $xy^3(\alpha) \neq \alpha, \forall \alpha \in Q'''(\sqrt{n}), n \neq 3.$
4. $xy^4(\alpha) \neq \alpha, \forall \alpha \in Q'''(\sqrt{n}), n \neq 6.$
5. $y^2(\alpha) \neq -\alpha, \forall \alpha \in Q'''(\sqrt{n}), n \neq 13.$
6. $y^3(\alpha) \neq -\alpha, \forall \alpha \in Q'''(\sqrt{n}), n \neq 3.$

In particular for $n = p, p$ a prime number, all the results of remarks 2.13 hold.

LEMMA 2.14

Let p be an odd prime and $\alpha = \frac{a+\sqrt{p}}{c}$ be an ambiguous number of $Q^*(\sqrt{p})$. Then $xy(\alpha) = \bar{\alpha} \Leftrightarrow \alpha = \frac{-1+\sqrt{p}}{2}$ or $\frac{1+\sqrt{p}}{-2}$. Moreover if $\alpha = \frac{a+\sqrt{p}}{c}$ with $p > 10$ is an ambiguous number of $Q'''(\sqrt{p})$, then $xy(\alpha) = -\bar{\alpha} \Leftrightarrow \alpha = \frac{-1+\sqrt{p}}{2}$ or $\frac{1+\sqrt{p}}{-2}$ or $\frac{-3+\sqrt{p}}{6}$ or $\frac{3+\sqrt{p}}{-6}$.

PROOF

The proof of this lemma is straightforward.

LEMMA 2.15

Let $\alpha = \frac{a+\sqrt{n}}{c} \in Q'''(\sqrt{n})$. Then :

1. $x(\alpha) = \bar{\alpha} \Leftrightarrow 3(n - a^2) = c^2$
2. $y^3(\alpha) = \bar{\alpha} \Leftrightarrow 3[n + a(a + c)] = c^2$
3. $xy^2(\alpha) = -\bar{\alpha} \Leftrightarrow 3[(c + a)^2 - n] = c^2$
4. $xy^3(\alpha) = -\bar{\alpha} \Leftrightarrow (c + 2a)^2 = 3n + a^2$

$$5. xy^4(\alpha) = -\bar{\alpha} \Leftrightarrow (c + 2a)^2 = 2[3n - a(c + a)]$$

$$6. xy^5(\alpha) = -\bar{\alpha} \Leftrightarrow (c + a)^2 = c^2 + 3n - 2a^2$$

PROOF

Let $\alpha = \frac{a+\sqrt{n}}{c} \in Q'''(\sqrt{n})$. Then :

$$1. x(\alpha) = \bar{\alpha} \Leftrightarrow \frac{-1}{3\alpha} = \bar{\alpha} \Leftrightarrow \alpha\bar{\alpha} = \frac{-1}{3} \Leftrightarrow \frac{a^2-n}{c^2} = \frac{-1}{3} \Leftrightarrow 3(n - a^2) = c^2$$

$$2. y^3\left(\frac{a+\sqrt{n}}{c}\right) = \frac{-a+\sqrt{n}}{-c} \Leftrightarrow \frac{-7a-6b-2c+\sqrt{n}}{12a+12b+3c} = \frac{-a+\sqrt{n}}{-c}$$

$$\Leftrightarrow -7a - 6b - 2c = -a, 12a + 12b + 3c = -c \Leftrightarrow 3a + 3b + c = 0$$

$$\Leftrightarrow c = -3(a+b) = -3\left[a + \left(\frac{a^2-n}{c}\right)\right] \Leftrightarrow c^2 = -3ac + 3a^2 + 3n = 3[n + a^2 + ac]$$

$$\text{Hence } y^3(\alpha) = \bar{\alpha} \Leftrightarrow 3[n + a(a + c)] = c^2$$

$$3. xy^2\left(\frac{a+\sqrt{n}}{c}\right) = \frac{-a+\sqrt{n}}{c} \Leftrightarrow \frac{(5a+3b+2c)+\sqrt{n}}{6a+3b+3c} = \frac{-a+\sqrt{n}}{c}$$

$$\Leftrightarrow 5a + 3b + 2c = -a, 6a + 3b + 3c = c \Leftrightarrow 6a + 3b + 2c = 0$$

$$\Leftrightarrow c = \frac{-1}{2}(6a + 3b) = \frac{-1}{2}\left[6a + 3\left(\frac{a^2-n}{c}\right)\right], \Leftrightarrow -6ac - 3a^2 + 3n = 2c^2$$

$$\Leftrightarrow c^2 = 3c^2 + 3a^2 + 6ac - 3n \Leftrightarrow c^2 = 3[(a + c)^2 - n].$$

$$\text{Hence } xy^2(\alpha) = -\bar{\alpha} \Leftrightarrow 3[(c + a)^2 - n] = c^2$$

$$4. xy^3\left(\frac{a+\sqrt{n}}{c}\right) = \frac{-a+\sqrt{n}}{c} \Leftrightarrow \frac{7a+6b+2c+\sqrt{n}}{12a+9b+4c} = \frac{-a+\sqrt{n}}{c}$$

$$\Leftrightarrow 7a + 6b + 2c = -a, 12a + 9b + 4c = c \Leftrightarrow 4a + 3b + c = 0 \Leftrightarrow c = -(4a + 3b) = -4a - 3\left(\frac{a^2-n}{c}\right) \Leftrightarrow c^2 = -4ac - 3a^2 + 3n \Leftrightarrow c^2 + 4ac + 4a^2 = 3n + a^2 \Leftrightarrow (c + 2a)^2 = 3n + a^2.$$

$$5. xy^4\left(\frac{a+\sqrt{n}}{c}\right) = \frac{-a+\sqrt{n}}{c} \Leftrightarrow \frac{5a+6b+c+\sqrt{n}}{12a+12b+3c} = \frac{-a+\sqrt{n}}{c}$$

$$\Leftrightarrow 6a + 6b + c = 0, 12a + 12b + 3c = c \Leftrightarrow 6a + 6b + c = 0$$

$$\Leftrightarrow c = -6(a + b) = -6\left[a + \left(\frac{a^2-n}{c}\right)\right], \Leftrightarrow c^2 = -6(ac + a^2 - n)$$

$$\Leftrightarrow c^2 + 6ac + 6a^2 = 6n \Leftrightarrow c^2 + 4ac + 4a^2 = 6n - 2ac - 2a^2$$

$$\Leftrightarrow (c + 2a)^2 = 2[3n - a(c + a)].$$

$$\text{Hence } xy^4(\alpha) = -\bar{\alpha} \Leftrightarrow (c + 2a)^2 = 2[3n - a(c + a)]$$

The proof of 6 is straightforward and is similar to 5.

Lemma 2.15 yields the following important results.

COROLLARIES 2.16

1. If $n \equiv 2 \pmod{4}$, $a \in Z$, then $(a^2 + 3n)$ is not a perfect square. Because square of an integer is either $\equiv 0$ or $1 \pmod{4}$ and $3n \equiv 2 \pmod{4}$, so $(a^2 + 3n)$ is either congruent to 2 or 3 $\pmod{4}$. Thus, $xy^3(\alpha) \neq -\bar{\alpha}, \forall \alpha \in Q'''(\sqrt{n}), n \equiv 2 \pmod{4}$.
2. If $n \equiv 3 \pmod{4}$, $a \in Z$, then $3[(c+a)^2 - n]$ is not a perfect square. Because square of an integer a is either $\equiv 0$ or $1 \pmod{4}$, so $3[(c+a)^2 - n]$ is either congruent to 2 or 3 $\pmod{4}$. Thus, $xy^2(\alpha) \neq -\bar{\alpha}, \forall \alpha \in Q'''(\sqrt{n}), n \equiv 3 \pmod{4}$.
3. If $n \equiv 2 \pmod{4}$, then $3(n - a^2)$ is not a perfect square. Because square of an integer is either $\equiv 0$ or $1 \pmod{4}$, so $3(n - a^2)$ is either congruent to 2 or 3 $\pmod{4}$. Thus, $x(\alpha) \neq \bar{\alpha}, \forall \alpha \in Q'''(\sqrt{n}), n \equiv 2 \pmod{4}$.

REMARKS 2.17

The following are the consequences of Lemmas 2.14 and 2.15.

1. $xy(\alpha) \neq -\bar{\alpha}, \forall \alpha \in Q'''(\sqrt{p}) \setminus Q^{**}(\sqrt{p})$.
2. $xy^2(\alpha) \neq -\bar{\alpha}, \forall \alpha \in Q'''(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$.
3. $xy^4(\alpha) \neq -\bar{\alpha}, \forall \alpha \in Q'''(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$.
4. $y^3(\alpha) \neq -\bar{\alpha}, \forall \alpha \in Q'''(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$.

In particular, for $n = p, p$ a prime, all the results of remarks 2.17 hold.

3. INVARIANT SUBSETS OF $Q(\sqrt{n})$ UNDER THE ACTION OF THE GROUP M

Subsets of an M -set $Q(\sqrt{n})$ may or may not be M -subsets. Some important subsets of M -sets which qualify to be the M -subsets of M -sets are of particular

interest in this section. Recollect that a subset Ω' (say) of an M -set $Q(\sqrt{n})$ is an M -subset if and only if Ω' is itself invariant under the action of the group M . We have proved in the previous section that $Q'''(\sqrt{n})$ is invariant under the action of M . Hence $Q'''(\sqrt{n})$ is an M -set within an M -set $Q(\sqrt{n})$. Such an M -set is called an M -subset of an M -set.

It may be mentioned that a subset Ω' of an M -set $Q'''(\sqrt{n})$ may itself be invariant under the action of a group different from that of a group M . In this case Ω' need not be an M -subset of an M -set $Q'''(\sqrt{n})$. For instance the set $Q^*(\sqrt{n})$ is a subset of $Q'''(\sqrt{n})$ whereas $Q^*(\sqrt{n})$ is not an M -subset of an M -set $Q'''(\sqrt{n})$, because $Q^*(\sqrt{n})$ is not invariant under the action of M . However $Q^*(\sqrt{n})$ is invariant under the action of the modular group G .

Now we investigate some non trivial (or proper) M -subsets of an M -set $Q'''(\sqrt{n})$.

LEMMA 3.1

Let $n \equiv 2 \pmod{3}$ be a non square positive integer.

Then $(a^2 - n) \not\equiv 0 \pmod{3}, \forall a \in \mathbb{Z}$.

PROOF

Since square of an integer is congruent to either 0 or 1 (mod 3), so two cases arise

Case I If $a^2 \equiv 0 \pmod{3}$, then $(a^2 - n) \equiv (0 - 2) \pmod{3} \equiv 1 \pmod{3}$.

Case II If $a^2 \equiv 1 \pmod{3}$, then $(a^2 - n) \equiv (1 - 2) \pmod{3} \equiv 2 \pmod{3}$.

So in no case $(a^2 - n) \equiv 0 \pmod{3}$.

COROLLARY 3.2

Let $n \equiv 2 \pmod{3}$ be a non square positive integer. Then

$$Q^{***}(\sqrt{n}) = \left\{ \frac{a+\sqrt{n}}{c} : \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n}) \text{ and } 3|c \right\} = \phi.$$

PROOF

Here $bc = (a^2 - n) \equiv 1 \pmod{3} \Leftrightarrow$ either both $b, c \equiv 1 \pmod{3}$ or both $b, c \equiv 2 \pmod{3}$, and $bc = (a^2 - n) \equiv 2 \pmod{3} \Leftrightarrow$ exactly one of $b, c \equiv 1 \pmod{3}$ and the other $\equiv 2 \pmod{3}$.

Hence, by the above lemma 3.1, if $c \equiv 0 \pmod{3}$, then $\frac{a^2-n}{c}$ is not an integer. Thus $Q^{***}(\sqrt{n}) = \phi$.

In the following theorem we determine a proper subset of $Q^*(\sqrt{n})$ which is an M -subset of $Q'''(\sqrt{n})$.

THEOREM 3.3

If a non square positive integer $n \equiv 0$ or $1 \pmod{3}$, then

$$Q^{***}(\sqrt{n}) = \left\{ \frac{a+\sqrt{n}}{c} : \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n}) \text{ and } 3|c \right\} \text{ is an } M\text{-subset of } Q'''(\sqrt{n}).$$

PROOF:

$$\text{Let } \alpha \in Q^{***}(\sqrt{n}) = \left\{ \frac{a+\sqrt{n}}{c} : \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n}) \text{ and } 3|c \right\}$$

Since every element of the group M is a word in the generators x, y of the group M , to prove that $Q^{***}(\sqrt{n})$ is invariant under the action of M , it is enough to show that every element of $Q^{***}(\sqrt{n})$ is mapped onto an element of $Q^{***}(\sqrt{n})$ by x and y .

Now $x(\alpha) = \frac{-a+\sqrt{n}}{3b} = \frac{a_1+\sqrt{n}}{c_1}$ with $a_1 = -a, b_1 = \frac{c}{3}, c_1 = 3b$ and $y(\alpha) = \frac{-a-c+\sqrt{n}}{3(2a+b+c)} = \frac{a_2+\sqrt{n}}{c_2}$ with $a_2 = -a-c, b_2 = \frac{c}{3}, c_2 = 3(2a+b+c)$. Since b and $\frac{c}{3}$ are integers and $(a, b, c) = 1$, so $b_1, b_2, \frac{c_1}{3}$ and $\frac{c_2}{3}$ are integers. Also we know that $(a, b, c) = 1 \Leftrightarrow (a_1, b_1, c_1) = 1 \Leftrightarrow (a_2, b_2, c_2) = 1$.

Hence $x(\alpha)$ and $y(\alpha)$ are elements of $Q^{***}(\sqrt{n})$, for all $\alpha \in Q^{***}(\sqrt{n})$.

In $y(\alpha), 3|c_2$. So $y^2(\alpha)$ and similarly $y^3(\alpha)$ are in $Q^{***}(\sqrt{n})$. This shows that $Q^{***}(\sqrt{n})$ is invariant under the action of M and $Q^{***}(\sqrt{n})$ is a proper M -subset of $Q'''(\sqrt{n})$.

Now we are in a position to find proper M -subsets of $Q^{***}(\sqrt{n})$. If $n \equiv 1 \pmod{4}$, then $Q'(\sqrt{n}) = \left\{ \frac{a+\sqrt{n}}{c} : \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n}), a \text{ odd and } b, c \text{ both even integers} \right\}$ is a proper subset of $Q^*(\sqrt{n})$. It is important to note that $Q'(\sqrt{n}), n \equiv 1 \pmod{4}$ is invariant under the action of modular group G . But $Q'(\sqrt{n})$ is not invariant under the action of M . For if $\frac{a+\sqrt{n}}{c} \in Q'(\sqrt{n})$ then $x\left(\frac{a+\sqrt{n}}{c}\right) = \frac{-a+\sqrt{n}}{3b} = \frac{a_1+\sqrt{n}}{c_1}$ where $a_1 = -a$ is odd, $c_1 = 3b$ is even, whereas $b_1 = \frac{c}{3}$ may not necessarily be even integer. However if $\frac{c}{3}$ is an even integer, then we prove that the set of all such elements of $Q'(\sqrt{n})$ is invariant under the action of M .

We know that $Q'(\sqrt{n})$ is non empty if and only if $n \equiv 1 \pmod{4}$ and $Q^{***}(\sqrt{n})$

is non empty if and only if $n \equiv 0$ or $1 \pmod{3}$.

We now determine conditions under which a subset of $Q^{***}(\sqrt{n})$ is invariant under the action of M .

THEOREM 3.4

For each non square positive integer $n \equiv 1$ or $9 \pmod{12}$,

$$Q^{***'}(\sqrt{n}) = \left\{ \frac{a+\sqrt{n}}{c} : \frac{a+\sqrt{n}}{c} \in Q'(\sqrt{n}) \text{ and } 6|c \right\} \text{ is an } M\text{-subset of } Q^{***}(\sqrt{n}).$$

PROOF:

The set $Q^{***'}(\sqrt{n}) = \left\{ \frac{a+\sqrt{n}}{c} : \frac{a+\sqrt{n}}{c} \in Q'(\sqrt{n}) \text{ and } 6|c \right\}$ is non empty if and only if $n \equiv 1$ or $9 \pmod{12}$ and is a subset of $Q^{***}(\sqrt{n})$. By theorem 3.1, $Q^{***}(\sqrt{n})$ is invariant under the action of M . We prove that $Q^{***'}(\sqrt{n})$ is invariant under the action of M .

$$\text{Let } \alpha = \frac{a+\sqrt{n}}{c} \in Q^{***'}(\sqrt{n})$$

Since every element of the group M is a word in its generators x, y to prove that $Q^{***'}(\sqrt{n})$ is invariant under the action of M , it is enough to show that every element of $Q^{***'}(\sqrt{n})$ is mapped onto an element of $Q^{***'}(\sqrt{n})$ by x and y .

$$\text{Now } x(\alpha) = \frac{-a+\sqrt{n}}{3b} = \frac{a_1+\sqrt{n}}{c_1} \text{ (say), gives } a_1 = -a, b_1 = \frac{c}{3}, c_1 = 3b \text{ and}$$

$$y(\alpha) = \frac{-a-c+\sqrt{n}}{3(2a+b+c)} = \frac{a_2+\sqrt{n}}{c_2} \text{ (say), yields } a_2 = -a-c, b_2 = \frac{c}{3}, c_2 = 3(2a+b+c).$$

Since a is odd while $b, \frac{c}{3}$ are both even integers and $(a, b, c) = 1$, so a_1 and a_2 are both odd integers, $b_1, b_2, \frac{c_1}{3}$ and $\frac{c_2}{3}$ are all even. Also

$$(a, b, c) = 1 \Leftrightarrow (a_1, b_1, c_1) = 1 \Leftrightarrow (a_2, b_2, c_2) = 1.$$

So $x(\alpha)$ and $y(\alpha)$ are elements of $Q^{***'}(\sqrt{n})$.

Hence $Q^{***'}(\sqrt{n})$ is invariant under the action of M and is a proper M -subset of $Q^{***}(\sqrt{n})$. This completes the proof.

THEOREM 3.5

Let $n \equiv 1 \pmod{4}$ be non square positive integer. Then the set

$$Q''''(\sqrt{n}) = \left\{ \frac{\alpha}{t} : \alpha \in Q'(\sqrt{n}) \right\}$$

and $t = 1, 3\}$ is invariant under the action of M .

PROOF

This theorem can be proved by the arguments similar to those used in theorem 2.3.

Now we put together theorems 2.3, 3.3, 3.4 and 3.5 to obtain the following remarks.

REMARKS 3.6

1. Let $\alpha \in Q^{***}(\sqrt{n})$, then the orbit $\alpha^M \subseteq Q^{***}(\sqrt{n})$.
2. Let $\alpha \in Q^{***'}(\sqrt{n})$, then the orbit $\alpha^M \subseteq Q^{***'}(\sqrt{n})$.
3. $Q'''(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$ is invariant under the action of M and hence is a proper M -subset of $Q'''(\sqrt{n})$.
4. $Q^{***}(\sqrt{n}) \setminus Q^{***'}(\sqrt{n})$ is invariant under the action of M and hence is a proper G -subset of $Q^{***}(\sqrt{n})$. Similarly $Q'''(\sqrt{n}) \setminus Q^{***'}(\sqrt{n})$ is invariant under the action of M and hence is a proper M -subset of $Q'''(\sqrt{n})$.

5. If $n \equiv 1$ or $9 \pmod{12}$ then $Q'''(\sqrt{n}), Q^{***'}(\sqrt{n}),$
 $Q^{***}(\sqrt{n}), Q^{***'}(\sqrt{n}) \setminus Q^{***}(\sqrt{n}), Q'''(\sqrt{n}) \setminus Q^{***'}(\sqrt{n}),$
 $Q'''(\sqrt{n}) \setminus Q^{***'}(\sqrt{n}), Q''''(\sqrt{n}), Q'''(\sqrt{n}) \setminus Q''''(\sqrt{n}),$
 $Q^{***}(\sqrt{n}) \cup Q''''(\sqrt{n}), Q'''(\sqrt{n}) \setminus (Q^{***}(\sqrt{n}) \cup Q''''(\sqrt{n})) = Q^*(\sqrt{n})$

and $Q''''(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$ are at least eleven proper M -subsets of an M -set $Q(\sqrt{n})$. If $n \equiv 0, 3, 4, 6, 7$ or $10 \pmod{12}$, then $Q'''(\sqrt{n}), Q^{***}(\sqrt{n})$ and $Q'''(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$ are at least three proper M -subsets of an M -set $Q(\sqrt{n})$. If $n \equiv 5 \pmod{12}$, then $Q'''(\sqrt{n}), Q''''(\sqrt{n}),$

$Q'''(\sqrt{n}) \setminus Q''''(\sqrt{n})$ are at least three proper M -subsets of an M -set $Q(\sqrt{n})$ whereas if $n \equiv 2, 8$ or $11 \pmod{12}$, then $Q'''(\sqrt{n})$ is at least one proper M -subset of an M -set $Q(\sqrt{n})$.

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ON CERTAIN CLASS OF UNIVALENT FUNCTIONS WITH VARYING ARGUMENTS

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Abstract. In this paper we consider the class $V_n(A, B)$ consisting of analytic and univalent functions with varying arguments. The object of the present paper is to show coefficient estimates and some distortion theorems for the function $f(z)$ in the class $V_n(A, B)$.

1. INTRODUCTION

Let S denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1.1)$$

which are analytic and univalent in the unit disc $U = \{z : |z| < 1\}$. Given two functions $f(z), g(z) \in S$, where $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad (1.2)$$

The Hadamard product or convolution $f * g(z)$, is defined by

$$f * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \quad z \in U \quad (1.3)$$

By using the Hadamard product, Ruscheweyh [7] defined

$$D^\gamma f(z) = \frac{z}{(1-z)^{\gamma+1}} * f(z) \quad (\gamma \geq -1). \quad (1.4)$$

Ruscheweyh [7] observed that

$$D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!} \quad (1.5)$$

when $n = \gamma \in N_0 = N \cup \{0\}$, where $N = \{1, 2, \dots\}$. This symbol $D^n f(z)$ ($n \in N_0$) was called the n -th order Ruscheweyh derivative of $f(z)$ by Al-Amiri [1]. We note that $D^0 f(z) = f(z)$ and $D^1 f(z) = z f'(z)$. It is easy to see that

$$D^n f(z) = z + \sum_{k=2}^{\infty} \delta(n, k) a_k z^k, \quad (1.6)$$

where

$$\delta(n, k) = \binom{n+k-1}{n}. \quad (1.7)$$

Let $R_n(A, B)$ denote the class of functions $f(z) \in S$ such that

$$\left| \frac{(n+1) \left(\frac{D^{n+1} f(z)}{D^n f(z)} - 1 \right)}{B(n+1) \frac{D^{n+1} f(z)}{D^n f(z)} - (Bn + A)} \right| < 1 \quad (1.8)$$

for $z \in U$, where $-1 \leq A < B \leq 1$, $0 < B \leq 1$ and $n \in N_0$.

We note that

- (i) $R_n(-1, 1) = R_n$ (Singh and Singh [10]).
- (ii) $R_0(2\alpha - 1, -1) = S(\alpha)$ ($0 \leq \alpha < 1$) (Robertson [6]).
- (iii) $R_0((2\alpha - 1)\beta, \beta) = S(\alpha, \beta)$ ($0 \leq \alpha < 1, 0 < \beta \leq 1$) (Juneja and Mogra [3]).
- (iv) Owa and Aouf [5], Owa [4], Silverman [8] and Gupta and Jain [2] studied the subclasses of $R_n(A, B)$, R_n , $S(\alpha)$ and $S(\alpha, \beta)$ consisting of functions with negative coefficients.

Definition 1

(Silverman [9]). A function $f(z)$ defined by (1.1) is said to be in the class $V(\theta_n)$ if $f(z) \in S$ and $\arg(a_n) = \theta_n$ for all $n \geq 2$. If, furthermore, there exists a real number β such that

$$\theta_n + (n - 1)\beta \cong \pi \text{ mod}(2\pi), \tag{1.9}$$

then $f(z)$ is said to be in the class $V(\theta_n, \beta)$. The union of $V(\theta_n, \beta)$ taken over all possible sequences $\{\theta_n\}$ and all possible real numbers β is denoted by V .

Let $V_n(A, B)$ denote the subclass of V consisting of functions $f(z)$ in $R_n(A, B)$

2. COEFFICIENT ESTIMATES

Theorem 1

Let the function $f(z)$ defined by (1.1) be in V . Then $f(z) \in V_n(A, B)$ if and only if

$$\sum_{k=2}^{\infty} [(1 + B)k - (A + 1)]\delta(n, k) |a_k| \leq (B - A). \tag{2.1}$$

The result is sharp.

Proof

Suppose $f(z) \in V_n(A, B)$. Then

$$\left| \frac{(n + 1) \left(\frac{D^{n+1}f(z)}{D^n f(z)} - 1 \right)}{B(n + 1) \frac{D^{n+1}f(z)}{D^n f(z)} - (Bn + A)} \right| = \left| \frac{- \sum_{k=2}^{\infty} (k - 1)\delta(n, k)a_k z^{k-1}}{(B - A) + \sum_{k=2}^{\infty} (Bk - A)\delta(n, k)a_k z^{k-1}} \right| \leq 1, \quad z \in U. \tag{2.2}$$

Since $Re\{w(z)\} < |w(z)| < 1$, we obtain on simplification

$$Re \left\{ \frac{- \sum_{k=2}^{\infty} (k - 1)\delta(n, k)a_k z^{k-1}}{(B - A) + \sum_{k=2}^{\infty} (Bk - A)\delta(n, k)a_k z^{k-1}} \right\} \leq 1.. \tag{2.3}$$

Since $f(z) \in V$, $f(z)$ lies in $V(\theta_n, \beta)$. for some sequence $\{\theta_k\}$ and a real number β such that

$$\theta_k + (k-1)\beta \cong \pi \pmod{2\pi}$$

Set $z = re^{i\beta}$ in (2.3), we get

$$\operatorname{Re} \left\{ \frac{\sum_{k=2}^{\infty} (k-1)\delta(n, k)|a_k|r^{k-1}}{(B-A) - \sum_{k=2}^{\infty} (Bk-A)\delta(n, k)|a_k|r^{k-1}} \right\} \leq 1.. \quad (2.4)$$

Hence,

$$\sum_{k=2}^{\infty} [(1+B)k - (A+1)]\delta(n, k)|a_k|r^{k-1} \leq (B-A). \quad (2.5)$$

Letting $r \rightarrow 1$ in (2.5), we get (2.1).

Conversely, suppose that $f(z) \in V$ and satisfies (2.1). In view of (2.5)

$$\begin{aligned} \left| -\sum_{k=2}^{\infty} (k-1)\delta(n, k)a_k z^{k-1} \right| &\leq \sum_{k=2}^{\infty} (k-1)\delta(n, k)|a_k|r^{k-1} \\ &< (B-A) - \sum_{k=2}^{\infty} (Bk-A)\delta(n, k)|a_k|r^{k-1} \\ &< \left| (B-A) - \sum_{k=2}^{\infty} (Bk-A)\delta(n, k)|a_k|z^{k-1} \right|, \end{aligned}$$

which gives (2.2) and hence follows that $f(z) \in V_n(A, B)$. The equality in the result (2.1) holds for the function $f(z)$ defined by

$$f(z) = z + \frac{(B-A)}{[(1+B)k - (A+1)]\delta(n, k)} e^{i\theta_k} z^k \quad (k \geq 2). \quad (2.6)$$

Corollary 1

Let the function $f(z)$ defined by (1.1) be in the class $V_n(A, B)$. Then

$$|a_k| \leq \frac{(B-A)}{[(1+B)k - (A+1)]\delta(n, k)} \quad (k \geq 2). \quad (2.7)$$

The result is sharp for the function $f(z)$ define by (2.6).

Theorem 2

Let the function $f(z)$ defined by (1.1) be in the class $V_n(A, B)$, with $\arg a_k = \theta_k$, where $[\theta_k + (k - 1)\beta] \cong \pi \text{ mod } (2\pi)$. Define

$$f_1(z) = z$$

and

$$f_k(z) = z + \frac{(B - A)}{[(1 + B)k - (A + 1)]\delta(n, k)} e^{i\theta_k} z^k \quad (k \geq 2), z \in U.$$

Then $f(z) \in V_n(A, B)$ if and only if $f(z)$ can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z), \tag{2.8}$$

where $\mu_k \geq 0 (k \geq 1)$ and $\sum_{k=1}^{\infty} \mu_k = 1$.

Proof

If $f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z)$ with $\sum_{k=1}^{\infty} \mu_k = 1$ and $\mu_k \geq 0$, then

$$\begin{aligned} & \sum_{k=2}^{\infty} [(1 + B)k - (A + 1)]\delta(n, k) \cdot \frac{(B - A)\mu_k}{[(1 + B)k - (A + 1)]\delta(n, k)} \\ &= \sum_{k=2}^{\infty} (B - A)\mu_k = (B - A)(1 - \mu_1) \leq (B - A). \end{aligned}$$

Hence $f(z) \in V_n(A, B)$.

Conversely, let the function $f(z)$ defined by (1.1) be in the class $V_n(A, B)$, define

$$\mu_k = \frac{[(1 + B)k - (A + 1)]\delta(n, k)}{(B - A)} |a_k|, (k \geq 2)$$

and

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k.$$

From Theorem 1, $\sum_{k=2}^{\infty} \mu_k \leq 1$ and so $\mu_1 \geq 0$. Since $\mu_k f_k(z) = \mu_k z + a_k z^k$,

$$\sum_{k=1}^{\infty} \mu_k f_k(z) = z + \sum_{k=2}^{\infty} a_k z^k = f(z).$$

This completes the proof of the theorem.

3. DISTORTION THEOREMS

Theorem 3

Let the function $f(z)$ defined by (1.1) be in the class $V_n(A, B)$. Then

$$|z| - \frac{(B-A)}{[2B-A+1](n+1)} |z|^2 \leq |f(z)| \leq |z| + \frac{(B-A)}{[2B-A+1](n+1)} |z|^2. \quad (3.1)$$

The result is sharp.

Proof

We employ the same technique as used by Silverman [9]. In view of Theorem 1, since $[(1+B)k - (A+1)]\delta(n, k)$ is an increasing function of k ($k \geq 2$), we have

$$[2B-A+1](n+1) \sum_{k=2}^{\infty} |a_k| \leq \sum_{k=2}^{\infty} [(1+B)k - (A+1)]\delta(n, k) |a_k| \leq (B-A).$$

that is,

$$\sum_{k=2}^{\infty} |a_k| \leq \frac{(B-A)}{[2B-A+1](n+1)} \quad (3.2)$$

Thus

$$|f(z)| = \left| z + \sum_{k=2}^{\infty} a_k z^k \right| \leq |z| + |z|^2 \sum_{k=2}^{\infty} |a_k| \leq |z| + \frac{(B-A)}{[2B-A+1](n+1)} |z|^2. \quad (3.3)$$

Similarly, we get

$$|f(z)| \geq |z| - |z|^2 \sum_{k=2}^{\infty} |a_k| \geq |z| - \frac{(B-A)}{[2B-A+1](n+1)} |z|^2. \quad (3.4)$$

This completes the proof of Theorem 3. Finally the result is sharp for the function

$$f(z) = z + \frac{(B - A)}{[2B - A + 1](n + 1)} e^{i\theta_2} z^2. \tag{3.5}$$

at $z = \pm |z| e^{i\theta_2}$.

Corollary 1

Under the hypotheses of Theorem 3, $f(z)$ is included in a disc with its center at the origin and radius r_1 given by

$$r_1 = 1 + \frac{(B - A)}{[2B - A + 1](n + 1)}. \tag{3.6}$$

Theorem 4

Let the function $f(z)$ defined by (1.1) be in the class $V_n(A, B)$. Then

$$1 - \frac{2(B - A)}{[2B - A + 1](n + 1)} |z| \leq |f'(z)| \leq 1 + \frac{2(B - A)}{[2B - A + 1](n + 1)} |z|. \tag{3.7}$$

The result is sharp.

Proof

Since $k[(1 + B)k - (A + 1)]\delta(n, k)$ is an increasing function of $k (k \geq 2)$, in view of Theorem 1, we have

$$\frac{1}{2} [2B - A + 1](n + 1) \sum_{k=2}^{\infty} k |a_k| \leq \sum_{k=2}^{\infty} [(1 + B)k - (A + 1)]\delta(n, k) |a_k| \leq (B - A),$$

that is,

$$\sum_{k=2}^{\infty} k |a_k| \leq \frac{2(B - A)}{[2B - A + 1](n + 1)}. \tag{3.8}$$

Thus

$$|f'(z)| = \left| 1 + \sum_{k=2}^{\infty} a_k z^{k-1} \right| \leq 1 + |z| \sum_{k=2}^{\infty} k |a_k| \leq 1 + \frac{2(B - A)}{[2B - A + 1](n + 1)} |z|. \tag{3.9}$$

Similarly, we get

$$|f'(z)| \geq 1 - |z| \sum_{k=2}^{\infty} k |a_k| \geq 1 - \frac{2(B-A)}{[2B-A+1](n+1)} |z|. \quad (3.10)$$

This completes the proof of Theorem 4. Finally the result is sharp for the function (3.5)

Corollary 2

Let the functions $f(z)$ defined by (1.1) be in the class $V_n(A, B)$. Then $f'(z)$ is include in a disc with its center at the origin and radius r_2 given by

$$r_2 = 1 + \frac{2(B-A)}{[2B-A+1](n+1)}.$$

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FUZZY IDEALS IN B -ALGEBRAS

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Abstract: In this paper the notion of fuzzy k -ideals in B -algebras is introduced, and related results are investigated.

1. INTRODUCTION

Imai and Iséki introduced two classes of logical algebras: BCK -algebras and BCI -algebras ([7, 9]). It is known that the class of BCK -algebras is a proper subclass of the class of BCI -algebras. In ([5, 6]), Hu and Li introduced a wide class of logical algebras: BCH -algebras. They have shown that the class of BCI -algebras is a proper subclass of the class of BCH -algebras. Neggers and Kim ([14]) introduced the notion of d -algebras which is another generalization of BCK -algebras, and they also introduced the notion of B -algebras ([13]), which is equivalent in some sense to the groups.

Fuzzy set was introduced by Zadeh ([18]). Since then there have been wide-ranging applications of the fuzzy set theory. Many research workers have

fuzzified the various mathematical structures : topological spaces, functional analysis, loop, group, ring, near ring, vector spaces, automation. In 1991, Xi ([17]) applied the concept of fuzzy set to BCK-algebras and studied their properties. In 1993, Ahmad and Jun ([1, 12]) applied it to BCI-algebras and discussed their properties. In ([10]), Jun, Roh and Kim studied the fuzzification of (normal) B -subalgebras. In ([2]), Akram and Kim have shown that the notions of a (fuzzy) subalgebra, a (fuzzy) normal subalgebra and a (fuzzy) k -ideal are equivalent in 0-commutative B -algebras. In this paper we discuss fuzzy k -ideals and obtain some results in B -algebras.

2. PRELIMINARIES

A groupoid $(X; *, 0)$ with a special element 0 is called a B -algebra ([13]) if, for all $x, y, z \in X$, the following axioms hold:

$$(B_1) \quad x * x = 0,$$

$$(B_2) \quad x * 0 = x,$$

$$(B_3) \quad (x * y) * z = x * (z * (0 * y)).$$

Proposition 2.1.

In B -algebras $(X; *, 0)$, the following identities hold:

$$(B_4) \quad 0 * (0 * x) = x \quad ([13])$$

$$(B_5) \quad 0 * (x * y) = y * x \quad ([15])$$

$$(B_6) \quad x * (y * z) = (x * (0 * z)) * y \quad ([16])$$

$$(B_7) \quad (x * y) * (0 * y) = x, \quad ([13]), \text{ for all } x, y, z \in X.$$

A non-empty subset S of a B -algebra $(X; *, 0)$ is called a *subalgebra* ([10]) of X if $x * y \in S$ for any $x, y \in S$.

A subset I of a B -algebra $(X; *, 0)$ is called a *k -ideal* ([2]) of X if

$$(K1) \quad 0 \in I,$$

(K2) if $x \in I$, then $0 * x \in I$,

(K3) if $x * y, y * z \in I$, then $x * z \in I$, for any $x, y, z \in I$.

Example 2.2.([2])

Let $X := \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:

*	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then $(X; *, 0)$ is a B -algebra ([13]). It is easy to show that $I := \{0, 1, 2\}$ is a k -ideal of X , but $J := \{0, 1\}$ is not a k -ideal of X , since $0 * 2 = 1, 2 * 1 = 1 \in I$, but $0 * 1 = 2 \notin J$.

Let X be a non-empty set. A *fuzzy (sub)set* μ of X is a mapping

$$\mu : X \longrightarrow [0, 1].$$

Let μ be the fuzzy set of a set X . For a fixed $s \in [0, 1]$, the set $\mu_s = \{x \in X : \mu(x) \geq s\}$ is called an *upper level* of μ . A mapping $f : X \longrightarrow Y$ of B -algebras is called a *homomorphism* if $f(x*y) = f(x)*f(y), \forall x, y \in X$. Note that if f is a homomorphism, then $f(0)=0$.

3. FUZZY k -IDEALS

Definition 3.1.

Let X be a B -algebra and μ be a fuzzy subset of X . Then μ is called a *fuzzy k -ideal* of X if it satisfies the following inequalities:

$$(FK_1) \mu(0) \geq \mu(x), \text{ for all } x \in X,$$

$$(FK_2) \mu(0 * x) \geq \mu(x), \text{ for all } x \in X,$$

(FK₃) $\mu(x * z) \geq \min\{\mu(x * y), \mu(y * z)\}$, for all $x, y, z \in X$.

Definition 3.2.

Let μ be a fuzzy set of a B-algebra X . Then μ is called a *fuzzy subalgebra* ([10]) of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in X$.

Example 3.3.

Let $X := \{0, 1, 2, 3\}$ be a B-algebra given by the following Cayley table:

*	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

We define a fuzzy set μ in X by

$\mu(0) = 0.651$ and $\mu(x) = 0.016$, for all $x \neq 0$ in X . Then it is easy to show that μ is a fuzzy k -ideal of X .

Lemma 3.4.

Every fuzzy k -ideal of a B-algebra X is a fuzzy subalgebra of X .

Proof.

Routine. \square

Theorem 3.5.

If each non-empty level subset $U(\mu; s)$ of μ is a k -ideal of X , then μ is a fuzzy k -ideal of X , where $s \in [0, 1]$.

Proof.

This is a consequence of ([2]). \square

Definition 3.6. ([3])

Let λ and μ be the fuzzy sets in a set X . The cartesian product $\lambda \times \mu : X \times X \rightarrow [0, 1]$ is defined by $(\lambda \times \mu)(x, y) = \min\{\lambda(x), \mu(y)\}$, for all $x, y \in X$.

Proposition 3.7.

If λ and μ are fuzzy k -ideals of a B-algebra X , then $\lambda \times \mu$ is a fuzzy k -ideal of $X \times X$.

Proof.

For any $(x, y) \in X \times X$, we have

$$(\lambda \times \mu)(0, 0) = \min\{\lambda(0), \mu(0)\} \geq \min\{\lambda(x), \mu(y)\} = (\lambda \times \mu)(x, y)$$

$$\begin{aligned} \text{and } (\lambda \times \mu)((0, 0) * (x, y)) &= (\lambda \times \mu)((0 * x, 0 * y)) \\ &= \min\{\lambda(0 * x), \mu(0 * y)\} \end{aligned}$$

$$\geq \min\{\lambda(x), \mu(y)\} = (\lambda \times \mu)(x, y).$$

Let $(x_1, x_2), (y_1, y_2)$ and $(z_1, z_2) \in X \times X$. Then

$$\begin{aligned} (\lambda \times \mu)((x_1, x_2) * (z_1, z_2)) &= (\lambda \times \mu)(x_1 * z_1, x_2 * z_2) \\ &= \min\{\lambda(x_1 * z_1), \mu(x_2 * z_2)\} \end{aligned}$$

$$\geq \min\{\min\{\lambda(x_1 * y_1), \lambda(y_1 * z_1)\}, \min\{\mu(x_2 * y_2), \mu(y_2 * z_2)\}\}$$

$$= \min\{\min\{\lambda(x_1 * y_1), \mu(x_2 * y_2)\}, \min\{\lambda(y_1 * z_1), \mu(y_2 * z_2)\}\}$$

$$= \min\{(\lambda \times \mu)((x_1 * y_1, x_2 * y_2)), (\lambda \times \mu)((y_1 * z_1, y_2 * z_2))\}$$

$$= \min\{(\lambda \times \mu)((x_1, x_2) * (y_1, y_2)), (\lambda \times \mu)((y_1, y_2) * (z_1, z_2))\}.$$

Hence $\lambda \times \mu$ is a fuzzy k -ideal of $X \times X$. \square

Proposition 3.8.

Let λ and μ be fuzzy sets in a B-algebra X such that $\lambda \times \mu$ is a fuzzy k -ideal of $X \times X$. Then

- (i) Either $\lambda(0) \geq \lambda(x)$ or $\mu(0) \geq \mu(x)$, $\forall x \in X$.
- (ii) If $\lambda(0) \geq \lambda(x)$, $\forall x \in X$, then either $\mu(0) \geq \lambda(x)$ or $\mu(0) \geq \mu(x)$.
- (iii) If $\mu(0) \geq \mu(x)$, $\forall x \in X$, then either $\lambda(0) \geq \lambda(x)$ or $\lambda(0) \geq \mu(x)$.
- (iv) λ or μ is a fuzzy k -ideal of X .

Proof.

Straightforward. \square

Definition 3.9.([3])

Let A be a fuzzy set in a set S , the *strongest fuzzy relation* on S that is fuzzy relation on A is μ_A given by $\mu_A(x, y) = \min\{A(x), A(y)\}$, for all $x, y \in S$.

Proposition 3.10.

Let A be a fuzzy set in a B -algebra X and μ_A be the strongest fuzzy relation on X . Then A is a fuzzy k -ideal of X if and only if μ_A is a fuzzy k -ideal of $X \times X$.

Proof.

Suppose that A is a fuzzy k -ideal of X . Then

$$\mu_A(0, 0) = \min\{A(0), A(0)\} \geq \min\{A(x), A(y)\} = \mu_A(x, y), \text{ and } \mu_A((0, 0) * (x, y)) = \mu_A(0 * x, 0 * y) = \min\{A(0 * x), A(0 * y)\} \geq \min\{A(x), A(y)\} = \mu_A(x, y), \forall (x, y) \in X \times X.$$

Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2) \in X \times X$. Then

$$\begin{aligned} \mu_A(x * z) &= \mu_A((x_1, x_2) * (z_1, z_2)) \\ &= \mu_A((x_1 * z_1, x_2 * z_2)) \\ &= \min\{A(x_1 * z_1), A(x_2 * z_2)\} \\ &\geq \min\{\min\{A(x_1 * y_1), A(y_1 * z_1)\}, \min\{A(x_2 * y_2), A(y_2 * z_2)\}\} \\ &= \min\{\min\{A(x_1 * y_1), A(x_2 * y_2)\}, \min\{A(y_1 * z_1), A(y_2 * z_2)\}\} \\ &= \min\{\mu_A(x_1 * y_1, x_2 * y_2), \mu_A(y_1 * z_1, y_2 * z_2)\} \end{aligned}$$

$$\begin{aligned}
&= \min \{ \mu_A((x_1, x_2) * (y_1, y_2)), \mu_A((y_1, y_2) * (z_1, z_2)) \} \\
&= \min \{ \mu_A(x * y), \mu_A(y * z) \}.
\end{aligned}$$

Hence μ_A is a fuzzy k -ideal of $X \times X$.

The converse is straightforward. \square

Definition 3.11.

Let $f : X \rightarrow Y$ be a mapping of B -algebras, and let μ be a fuzzy set of Y . The map μ^f is called the *pre-image* of μ under f , if $\mu^f(x) = \mu(f(x))$, $\forall x \in X$.

Proposition 3.12.

Let $f : X \rightarrow Y$ be an onto homomorphism of B -algebras. If μ is a fuzzy k -ideal of Y , then μ^f is also a fuzzy k -ideal of X .

Proof.

If we let $T := \min$ in Theorem 4.8 ([2]), then μ^f is a fuzzy k -ideal of X . \square

Proposition 3.13.

Let $f : X \rightarrow Y$ be an onto homomorphism of B -algebras. If μ^f is a fuzzy k -ideal of Y , then μ is also a fuzzy k -ideal of X .

Proof.

This is a consequence of Theorem 4.7 ([2]). \square

Definition 3.14.

A k -ideal A of B -algebra X is said to be *characteristic* if $f(A) = A$, for all $f \in \text{Aut}(X)$, where $\text{Aut}(X)$ is the set of all automorphisms of X .

Definition 3.15.

A fuzzy k -ideal μ of B -algebra X is said to be *fuzzy characteristic* if $\mu^f(x) = \mu(x)$, for all $x \in X$ and $f \in \text{Aut}(X)$.

Lemma 3.16.

Let μ be a fuzzy k -ideal of a B -algebra X and let $x \in X$. Then $\mu(x) = t$ if and only if $x \in \mu_t$ and $x \notin \mu_s$, for all $s > t$.

Theorem 3.17.

For a fuzzy k -ideal μ of a B -algebra X , the following are equivalent:

- (i) μ is fuzzy characteristic.
- (ii) each level k -ideal of μ is characteristic. par

Proof.

Suppose that μ is fuzzy characteristic and let $t \in \text{Im}(\mu)$, $f \in \text{Aut}(X)$ and $x \in \mu_t$. Then

$$\mu^f(x) = \mu(x) \geq t$$

$$\implies \mu(f(x)) \geq t$$

$$\implies f(x) \in \mu_t.$$

Thus $f(\mu_t) \subseteq \mu_t$.

Let $x \in \mu_t$ and $y \in X$ such that $f(y) = x$. Then

$$\mu(y) = \mu^f(y) = \mu(f(y)) = \mu(x) \geq t$$

$$\implies y \in \mu_t \text{ so that } x = f(y) \in \mu_t.$$

Consequently, $\mu_t \subseteq f(\mu_t)$. Hence $f(\mu_t) = \mu_t$, i.e., μ_t is characteristic.

Conversely, suppose that each level k -ideal of μ is characteristic and let $x \in X$, $f \in \text{Aut}(X)$ and $\mu(x) = t$. Then, by virtue of Lemma 3.16, $x \in \mu_t$ and $x \notin \mu_s$, for all $s > t$. It follows from the assumption that $f(x) \in f(\mu_t) = \mu_t$, so that $\mu^f(x) = \mu(f(x)) \geq t$. Let $s = \mu^f(x)$ and assume that $s > t$. Then $f(x) \in \mu_s = f(\mu_s)$, which implies from the injectivity of f that $x \in \mu_s$, a contradiction. Hence $\mu^f(x) = \mu(f(x)) = t = \mu(x)$ showing that μ is fuzzy characteristic. \square

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A SIMPLE PROOF OF A_5 AS A SUBGROUP OF ${}^2F_4(2)$

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Abstract

In this paper, the existence of one class of alternating group A_5 as a subgroup of twisted Chevalley group of Ree type ${}^2F_4(2)$ is proved.

1 INTRODUCTION

$F_4(2)$ is a well-known simple Lie group of automorphisms of the special Lie algebra $f_4(2)$. ${}^2F_4(2)$ is a twisted Chevalley group of Ree type within $F_4(2)$, as a subgroup. ${}^2F_4(2)$ is, however, non-simple of order $2^{12} \cdot 3^2 \cdot 5^2 \cdot 13$ but it contains a simple derived subgroup ${}^2F_4'(2)$ known as Tits group of index 2. Maximal subgroups of ${}^2F_4'(2)$ are studied in [6]. $F_4(2)$ has been characterized by the centralizer of an involution in the center of 2-Sylow subgroups of $F_4(2)$. The construction of Lie groups $F_4(k)$ as automorphism groups of Lie algebras of type $f_4(k)$, over the field k has been discussed in [1] and [2].

Definition.1.1

Let G be a finite group. G is defined to be a $(2, 3, 5)$ group if there exists an element x of order 2, y of order 3, such that xy is of order 5.

i- e $G = \langle x, y \rangle: x^2 = 1 = y^3 = (xy)^5$.

We prove our result in the following form.

Theorem 1.2.

Let G be a twisted Chevalley group of Ree type over Galois field of two elements. Then there exists only one class of simple groups isomorphic to A_5 within ${}^2F_4(2)$.

Proof. Let G_0 be the subgroup of ${}^2F_4(2)$, which is isomorphic to A_5 since $|G_0| = 2^2 \cdot 3 \cdot 5$, therefore by Lagranges theorem A_5 may be a candidate to exist within ${}^2F_4(2)$, as a subgroup. In order to search for the possibility of existence of A_5 within ${}^2F_4(2)$, we use information known to exist about A_5 and ${}^2F_4(2)$ both. The conjugate classes and character table of ${}^2F_4(2)$ are known in [6]. It is a well-known fact that A_5 is the smallest non-Abelian simple group and is a $(2, 3, 5)$ triangular group which is generated by the elements x and y of order 2 and 3 respectively, such that $(xy)^5 = 1$.

Thus $A_5 = \{ \langle x, y \rangle: x^2 = 1 = y^3 = (xy)^5 \}$ [3]. In order to show existence of A_5 within ${}^2F_4(2)$, we observe from [6] that there exist two conjugate classes with representatives α_1, α_2 of elements of order 2, one conjugate class of elements of order 3 and one conjugate class of elements of order 5. Either α_1 or α_2 or both classes may be involved in triangular relation $(2, 3, 5)$, if A_5 exists within ${}^2F_4(2)$. The existence of non-zero integral number of triangular relations $(2, 3, 5)$ shall ensure the existence of A_5 within ${}^2F_4(2)$, where

$$\# \langle 2, 3, 5 \rangle = \frac{|G|}{|C_G(2)||C_G(3)|} \sum \chi_i(2)\chi_i(3)\overline{\chi_i(5)}[3]$$

Where $|C_G(2)|$ stands for order of centralizer of an element of order 2. $|C_G(3)|$ stands for order of centralizer of an element of order 3. $\chi_i(2)$, $\chi_i(3)$, $\chi_i(5)$ stands for the characters of degree of the classes of order 2, 3 and complex conjugate of class of order 5 respectively.

$\chi_i(1)$ stands for degree of representation of the i^{th} character of G . From char-

acter table of ${}^2F_4(2)$, we observe that if $x \in \alpha_1$, then

$$\begin{aligned} \# \langle 2.3 = 5 \rangle &= \frac{2^{12} \cdot 3^3 \cdot 5^2 \cdot 13}{2^{12} \cdot 5 \cdot 2^3 \cdot 2^3} \left\{ 1 + 1 + \frac{(-12)(-2)(2)}{52} + \frac{(14)(-3)(3)}{78} \right. \\ &\quad \left. + \frac{(14)(-3)(3)}{78} + \frac{(-32)(6)(-2)}{1248} \right\} = 0 \end{aligned}$$

If $x \in \alpha_2$ then

$$\begin{aligned} \# \langle 2.3 = 5 \rangle &= \frac{2^{12} \cdot 3^3 \cdot 5^2 \cdot 13}{2^{10} \cdot 3 \cdot 2^3 \cdot 2^3} \left\{ 1 + 1 + \frac{(4)(-2)(2)}{52} + \frac{(-2)(-3)(3)}{78} \right. \\ &\quad \left. + \frac{(-2)(-3)(3)}{78} + \frac{(32)(6)(-2)}{1248} \right\} = 100. \end{aligned}$$

Hence the class α_1 of ${}^2F_4(2)$ is not involved in the construction of $(2, 3, 5)$ relation but α_2 does generate. Hence for each of the relation, $\# \langle 2.3 = 5 \rangle = 100$, there exists $A_5 = \{ \langle x, y \rangle : x^2 = 1 = y^3 = (xy)^5 \}$. We find that $\# \langle 2.3 = 5 \rangle = 100$. We observe that $|C_G(5)| = 100$ and all the 100 relations are conjugate by conjugating x and y both by the elements of the centralizer $C_G(5)$ of an element of order 5. Since each $(2, 3, 5)$ relation generates an A_5 therefore 100 of such A_5 are generated within ${}^2F_4(2)$ by different conjugates x and y . That is if x is of order 2 and y of order 3 in ${}^2F_4(2)$ within their specified corresponding conjugacy classes then xy is of order 5. Let $xy = t$, and $t^5 = 1$. If $c \in C_G(t)$, then,

$$c(xy)c^{-1} = ctc^{-1} = t$$

$$\Rightarrow cxc^{-1}cyc^{-1} = t$$

$$\Rightarrow (cxc^{-1})(cyc^{-1}) = t$$

The same t is produced by conjugating x and y by $c \in C_G(t)$. So the total number of pairs of x and y shall count to be equal in number to the order of the centralizer of t . Thus all the 100 relations become conjugate by conjugating x and y by the centralizer of t . This concludes that all copies of A_5 stand conjugate to each other, which shall form a single conjugate class within ${}^2F_4(2)$.

Cont.

c = char, cl = classes, R = Rank, D = Degree

c		χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}
cl	$ \chi $	675	675	1248	1404	2600	4096	54	54	600	702	702	3456
R	D							2	2	2	2	2	2
α_1	$2^{12}.5$	35	35	-32	60	40	0	-10	-10	-40	-2	-2	-128
α_2	$2^{10}.3$	3	3	32	12	-24	0	6	6	-8	-18	-18	0
4_{α^a}	2^8	3	3	0	4	-8	0	-2	-2	8	6	6	0
$4_{\alpha^b}^i$	$2^8.5$	5	-5	0	0	0	0	-2	2	0	10	-10	0
$4_{\alpha^b}^{ii}$	$2^7.3$	3	3	0	-12	0	0	6	6	-8	6	6	0
$4_{\alpha^c}^i$	2^7	5	-5	0	0	0	0	-2	2	0	-6	6	0
$4_{\alpha^d}^{ii}$	2^7	3	3	0	-4	0	0	-2	-2	8	-2	-2	0
$4_{\alpha^e}^{iv}$	$2^6.3$	-3	3	0	0	0	0	6	-6	0	-6	6	0
$8_{\alpha^a}^i$	2^6	-1	-1	0	0	0	0	-2	-2	0	-2	-2	0
$8_{\alpha^a}^{ii}$	2^5	1	-1	0	0	0	0	2	-2	0	-2	2	0
$8_{\alpha^c}^{ii}$	2^4	1	-1	0	0	0	0	2	-2	0	2	-2	0
$8_{\alpha^d}^{ii}$	2^4	-1	-1	0	0	0	0	-2	-2	0	2	2	0
16_{α^a}	2^4	1	1	0	0	0	0	0	0	0	0	0	0
16_{α^b}	2^4	-1	1	0	0	0	0	0	0	0	0	0	0
3	$2^3.3^3$	0	0	6	0	8	-8	0	0	6	0	0	0
6	$2^3.3$	0	0	2	0	0	0	0	0	-2	0	0	0
12_{α^b}	$2^2.3$	0	0	0	0	0	0	0	0	-2	0	0	0
$12_{\alpha^c}^{iv}$	$2^2.3$	0	0	0	0	0	0	0	0	0	0	0	0
5	$2^2.5^2$	0	0	-2	4	0	-4	4	4	0	2	2	6
10	$2^2.5$	0	0	-2	0	0	0	0	0	0	-2	-2	2
20_{α^b}	$2^2.5$	0	0	0	0	0	0	-2	2	0	0	0	0
13	13	-1	-1	0	0	0	1	2	2	2	0	0	-2

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