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RINGS WITH CHAIN CONDITIONS ON SMALL RIGHT IDEALS

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Abstract

In this note we study rings with ascending and descending chain conditions on small right ideals. We show that these classes are closed under finite direct sum and quotients for every non-zero small ideal, but they are not closed under extensions or taking ideals, however if I is a small ideal in a commutative ring R and if I and R/I satisfies the a. c. c. (resp. d.c.d.) on small ideals, then R satisfies the a. c. c. (resp. d. c. c.) on small ideals. Finally we show that Camillo's theorem need not be true in non-commutative rings. Then we prove that a ring is Noetherian if and only if every quotient satisfies the a. c. c. on complement and small ideals.

1. PRELIMINARIES

Let R be a ring with identity. An ideal A of R is said to be small right ideal in R (see [1, p.72]) if for all right ideals B of R the equation $A + B = R$ implies $B = R$. The right socle of R *soc*, (R) is the sum of all the minimal right ideals of R. The radical $rad(R)$ is the intersection of all maximal right ideals in R, and it is the sum of all small right ideals in R. We say that R satisfies the ascending chain condition (a. c. c.) on small right ideals if every ascending chain of small right ideals $I_1 \subseteq I_2 \subseteq \cdots$ terminates after a finite number of steps. This is equivalent to the fact that every non-empty set of small right ideals has a maximal element. The descending chain condition (d. c. c.) on small right ideals is defined similarly. We say that R is Goldie if R satisfies the a. c. c. on annihilator ideals and the a. c. c. on complement ideals.

Remark 1.1

(a) If R satisfies the a. c. c. on small ideals, then R need not to satisfy the d. c. c. on small ideals as the following example shows:

Let $R = (F[x], +, .), F$ is a field. Then R satisfies the a. c. c. on small ideals, but does not satisfy the d. c. c. on small ideals since $< x > \supset < x^2 > \supset \cdots$ is an infinite descending chain of small ideals in R.

(b) If R satisfies the d. c. c. on small ideals, then R need not to satisfy the a. c. c. on small ideals as the following example shows:

Let $R = \mathbf{Z}(p^\infty) = \left\{\frac{m}{p^n}, \in \mathbf{Q} | 0 < \frac{m}{p^n} < 1\right\}$, where p be a prime number. Then R is a ring without identity and satisfies the d. c. c. on small ideals, but does not satisfy the a. c. c. on small ideals since each small ideal is of the form

$$A_k = \left\{\frac{1}{p^k}, \frac{2}{p^k}, \cdots, \frac{p^k - 1}{p^k}\right\}$$

where k is some positive integer and, $A_1 \subset A_2 \subset \cdots$ is an infinite properly ascending chain of small ideals in R.

Lemma 1.2

Let R be a ring and $0 \neq B \lhd R$.

(a) If A is a small right ideal in R such that $A \supset B$, then A/B is a small right ideal in R/B.

(b) If B is a small right ideal in $R, \pi: R \to R/B$ be the natural homomorphism and A/B is a small right ideal in R/B, then $\pi^{-1}(A/B) = A$ is a small right ideal in R.

Proof

(a) Suppose that $A/B + L/B = R/B$, where $L/B \lhd_r R/B$, then $A + L + B = A + L = R$, but A is a small right ideal in R, hence $A \subseteq R = L$. Therefore

$A/B \subseteq L/B$ and $L/B = R/B$. Hence A/B is a small right ideal in R/B.

(b) Suppose that $A + L = R$, where $L \lhd_r R$ then $A/B + L/B = R/B$, but A/B is a small right ideal in R/B, hence $L/B = R/B$. Therefore $L + B = R$, but B is a small right ideal in R so that $L = R$. Hence A is a small right ideal in R.

Remark 1.3

If R is a ring, $A \lhd_r R$ and I be a small right ideal in R such that $A \subset I$, then A need not be small right ideal in I as the following example shows:

Let

$$R = \left(\left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in Z\right\}. + ..\right)$$

and

$$I = \left\{\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} | x \in Z\right\},$$

then I is the small ideal in R. If

$$A = \left\{\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} | a \in \mathbf{2Z}\right\},$$

then $A \lhd R$ and $A \subset I$, but A not small ideal in I.

However we have the following:

Lemma 1.4

Let R be a commutative ring. A and I are ideals in R such that $A \subset I$ and I is a small ideal in R, then A is a small ideal in I.

Proof

Since I is a small ideal in R, then $I \subseteq rad(R)$ and so $A \subseteq rad(R)$. Therefore $A = A \cap I \subseteq rad(R) \cap I = rad(I)$. Hence $A \subseteq rad(I)$ and A is a small ideal in I.

Lemma 1.5

(a) Let R be a ring, I and J are small right ideals in R, then $I + J$ is a small right ideal in R.

(b) Let $R = R_1 \oplus R_2$, where R_1 and R_2 are right ideals in R. If $I_1 \lhd_r R_i$, then $I_1 \oplus I_2$ is a small right ideal in R if and only if I, is a small right ideal in $R_i(i = 1, 2)$.

Proof

(a) Suppose that $I + J + L = R$, where $L \lhd_r R$. But I is a small right ideal in R, hence $J + L = R$, but J is a small right ideal in R. Hence $L = R$ and $I + J$ is a small right ideal in R.

(b) Let $I_1 \oplus I_2$ be a small right ideal in R and if $p_i : R \to R_i$ be projection of R on R_i along $R_j(i \neq j)$ then by Lemma 1.2 (a), $p_i(I_1+I_2) = I_i$ is a small right ideal in $R_i(i = 1, 2)$. Conversely, let I_j be a small right ideal in $R_j(j = 1, 2)$ so that if $i : R_j \to R$ be the inclusion and then by Lemma 1.2 (a), $i(I_j) = I_j$ is a small right ideal in $R_j(j = 1, 2)$. Therefore by (a) $I_1 \oplus I_2 = I_1 + I_2$ is a small right ideal in R.

2. RESULTS

Theorem 2.1

If R satisfies the a. c. c. (resp. d. c. c.) on small right ideals and I is a non-zero small ideal in R, then R/I satisfies the a.c.c. (resp. d. c. c.) on small right ideals.

Proof

Let $\bar{I}_1 \subseteq \bar{I}_2 \subseteq \cdots$ be an ascending chain of small right ideals in $\bar{R} = R/I$. Now if $\pi : R \to R/I$ is the natural homomorphism, then $I_i = \pi^{-1}(\bar{I}_i)$ is a small right ideal in R for all i, by Lemma 1.2 (b). Therefore $I_1 \subseteq I_2 \subseteq \cdots$ is an ascending chain of small right ideals in R. But R satisfies the a.c.c. on small right ideals, hence there exists $n \in N$ such that $I_n = I_m, \forall m \geq n$. Therefore $\bar{I}_n = \bar{I}_m, \forall m \geq n$ and $\bar{R}$ satisfies the a. c. c. on small right ideals.

The proof for d. c. c. is similar.

Remark 2.2

If R/I satisfies the a. c. c. on small right ideals for every non-zero ideal I in R, then R need not to satisfy the a.c.c. on small right ideals as the following example shows:

Let

$$R = \left(\left\{\begin{pmatrix} x & y \\ 0 & a \end{pmatrix} | x, y \in \mathbf{Q}, a \in \mathbf{Z}\right\}, +, .\right)$$

and $I \lhd R$. Then every R/I satisfies the a. c. c. on small right ideals, but R does not satisfy the a. c. c. on small right ideals since for each positive integer

$$k, A_k = \left\{\begin{pmatrix} 0 & \frac{m}{2^k} \\ 0 & 0 \end{pmatrix} |m \in \mathbf{Z}\right\}$$

is an infinite ascending chain, $A_1 \subset A_2 \subset \cdots$ of small right ideals in R.

However we have the following:

Properties 2.3

If R/I satisfies the a. c. c. on small ideals for every non-zero small ideal I in R, then R satisfies the a. c. c. on small ideals.

Proof

Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of small ideals in R, then by Lemma 1.2 (a), $I_2|I_1 \subseteq I_3|I_2 \subseteq \cdots$ is an ascending chain of small ideals in R/I_1. But R/I_1 satisfies the a. c. c. on small ideals, then there exists $n \in \mathbf{N}$ such that $I_n = I_m, \forall m \geq n$. Hence R satisfies the a. c. c. on small ideals.

Remark 2.4

(a) If R/I satisfies the d. c. c. on small ideals for every non-zero ideal I in R, then R need not to satisfy the d. c. c. on small ideals as the following example shows:

Let $R = (F[x], +,), F$ is a field and $I \lhd R$. Then every R/I satisfies the d. c. c. on small ideals, but R does not satisfy the d. c. c. on small ideals.

(b) If R satisfies the a. c. c. (resp. d. c. c.) on small right ideals and $I \lhd R$, then I need not to satisfy the a. c. c. (resp. d. c. c.) on small right ideals as the following example shows:

Let

$$R = \left(\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in \mathbf{Q} \right\}, +, . \right),$$

then R satisfies the a. c. c. (resp. d. c. c.) on small right ideals,

$$I = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} | x, y \in \mathbf{Q} \right\}$$

is an ideal in R, but I does not satisfy the a. c. c. (resp. d. c. c.) on small right ideals since for each positive integer k,

$$A_k = \left\{ \begin{pmatrix} x & \frac{m}{2^k} \\ 0 & 0 \end{pmatrix} | m \in \mathbf{Z} \right\}$$

is an infinite ascending chain, $A_1 \subset A_2 \subset \cdots$ of small right ideals in I. While for each positive integer r,

$$A_r = \left\{ \begin{pmatrix} 0 & 2^r m \\ 0 & 0 \end{pmatrix} | m \in \mathbf{Z} \right\}$$

is an infinite descending chain, $A_1 \supset A_2 \supset \cdots$ of small right ideals in I.

(c) If R/I and I satisfies the a. c. c. (resp. d. c. c.) on small right ideals, then R need not to satisfy the a. c. c. (resp. d. c. c.) on small right ideals as the following example shows:

Let

$$R = \left(\left\{ \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} | a, b \in \mathbf{Z}, x \in \mathbf{Q} \right\}, +, . \right)$$

and

$$I = \left\{ \begin{pmatrix} 0 & x \\ 0 & a \end{pmatrix} | x \in \mathbf{Q}, a \in \mathbf{Z} \right\}$$

then I satisfies the a. c. c. (resp. d. c. c.) on small right ideals and

$$R/I = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} | a \in \mathbf{Z} \right\}$$

satisfies the a. c. c. (resp. d. c. c.) on small right ideals, but R does not satisfy the a. c. c. on small right ideals since for each positive integer K,

$$A_k = \left\{ \begin{pmatrix} 0 & \frac{m}{2^k} \\ 0 & 0 \end{pmatrix} | m \in \mathbf{Z} \right\}$$

is an infinite ascending chain, $A_1 \subset A_2 \subset \cdots$ of small right ideals of R. While for each positive integer r,

$$A_r = \left\{ \begin{pmatrix} 0 & 2^r m \\ 0 & 0 \end{pmatrix} | m \in \mathbf{Z} \right\}$$

is an infinite descending chain, $A_1 \supset A_2 \supset \cdots$ of small right ideals of R.

However we have the following

Theorem 2.5

Let R be a commutative ring and I be a small ideal in R. If I and R/I satisfy the a. c. c. (resp. d. c. c.) on small ideals, then R satisfies the a. c. c. (resp. d. c. c.) on small ideals.

Proof

Let $A_1 \subseteq A_2 \subseteq \cdots$ be ascending chain of small ideals of R, then by Lemma 1.2 (a),

$$\frac{A_1 + I}{I} \subseteq \frac{A_2 + I}{I} \subseteq$$

is an ascending chain of small ideals in R/I. But R/I satisfies the a. c. c. on small ideals, hence there exists a positive integer **r** such that

$$\frac{A_r + I}{I} = \frac{A_m + I}{I}, \forall m \geq r$$

therefore $A_r + I = A_m + I, \forall m \geq r$.

Now by Lemma 1.4, $A_1 \cap I \subseteq A_2 \cap I \subseteq \cdots$ is an ascending chain of small ideals in I. But I satisfies the a. c. c. on small ideals, hence there exists a positive integer s such that $A_s \cap I = A_m \cap I, \ \forall m \geq s$. Let $t = \max\{r, s\}$ then $A_t + I = A_m + I$ and $A_t \cap I = A_m \cap I \ \ \forall m \geq t$. Now by modular law we have:

$$A_m = A_m \cap (A_m + I) = A_m \cap (A_t + I) = A_t + (A_m \cap I) = A_t + (A_t \cap I) = A_t, \forall m \geq t$$

Hence R satisfies the a. c. c. on small ideals.

The proof for d. c. c. is similar.

Theorem 2.6

If R_i satisfy the a. c. c. (resp. d. c. c.) on small right ideals for all $i = 1, \cdots, n$ then their direct sum $R_1 \oplus \cdots \oplus R_n$ satisfies the a. c. c. (resp. d. c. c.) on small right ideals.

Proof

By induction on n we need only consider the case $n = 2$. Let $R = R_1 \oplus R_2$. Let I be a small right ideal in R and suppose that I_1 and I_2 be the projection of I in R_1 and R_2 respectively. Since $I = I_1 \oplus I_2$, then by Lemma 1.5 (b) we have I_1 is a small right ideal in R_i.

Now let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of small right ideals in R and let $A_i = \{a_i | (a_i, 0) \in I_i\}$, then each A_i is a small right ideal in R_1. Moreover, $A_1 \subseteq A_2 \subseteq \cdots$ and so by a. c. c. on small right ideals in R_1. Moreover, $A_1 \subseteq A_2 \subseteq \cdots$ and so by a. c. c. on small right ideals in R_1 there exists an integer n such that $A_n = A_{n+1} = \cdots$

Now for $i \geq n$, let $B_i = \{b_i | (0, b_i) \in I_i\}$, then each B_i is a small right ideal in R_2. Moreover, $B_1 \subseteq B_2 \subseteq \cdots$ and so by a. c. c. on small right ideals in R_2, there exists an integer, m such that $B_m = B_{m+1} = \cdots$ We claim that $I_m = I_{m+1} = \cdots$

To show this, let $(a_{m+1}, b_{m+1}) \in I_{m+1}$. Since $A_m = A_{m+1}$ there exists.

$(a_{m+1}, c_{m+1}) \in I_m \subseteq I_{m+1}$. Therefore $(a_{m+1}, c_{m+1}) \in I_{m+1}$ which implies that $(0, b_{m+1} - c_{m+1}) \in I_m$. But $(a_{m+1}, c_{m+1}) \in I_m$, hence $(a_{m+1}, b_{m+1}) \in I_m$ and $I_{m+1} \subseteq I_m$. Therefore $I_m = I_{m+1}$ and R satisfies the a. c. c. on small right ideals.

The proof for d. c. c. is similar.

Remark 2.7

Camillo [2] shows that a commutative ring is Noetherian if and only if every quotient is Goldie. However if R is not commutative this result need not be true as the following example shows:

Let V be a vector space of finite dimension d, where d is a limit cardinal. For example well-order the cardinal as $\mathcal{N}_\alpha$ for ordinal α and let $d = \mathcal{N}_w$.

Let $R = L(V, V)$ be the ring of all linear maps $V \to V$. Then R is a non-commutative ring with identity and R has a unique ascending chain of ideals, $I_0 \subset I_1 \subset \cdots$ where $I_\alpha = \{f : f \in L(V, V)/\text{rank}(f) < \mathcal{N}_\alpha\}$. Hence R is not Noetherian, but in each quotient R/I_α also I_α/I_α and R/I_α are the only annihilator ideals of R/I_α also I_α/I_α and R/I_α are the only complement ideals of R/I_α, hence R/I_α is Goldie.

However we have the following;

Theorem 2.8

A ring R is Noetherian if and only if R/I satisfies a. c. c. on complement and small ideals for all non-zero ideals I of R.

Proof

If R is Noetherian, then R/I is Noetherian for every non-zero ideals I of R. Therefore R/I satisfies a. c. c. on complement and small ideals for every non-zero ideal I of R. To prove the converse suppose that R/I satisfies the a. c. c. on complement and small ideals for every non-zero ideal I of R. Since the sum of two small ideals is small by Lemma 1.5 (a), it follows from the a. c. c. on small ideals in R/I that $rad(R/I)$ is small for all ideal I of R. Since R/I satisfies the a. c. c. on complement, it follows that $soc(R/I)$ is finitely generated. Hence by [4, Theorem 3.8, P. 232], R is Noetherian.

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THE PROPAGATION OF PULSE IN SEA WATER

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Abstract

A propagation of pulse generated by an electric dipole is studied. A general formula for the electric field, which is uniformly valid for large distance and time, is presented. Fourier transform, stationary phase and simple contour integration methods are applied.

1. CURRENT PULSE AND FOURIER TRANSFORM

Electric dipole is excited by a current

$$I(t) = I_0 f(t)$$

where $f(t)$ is a pulse. The Fourier transform of this pulse is defined as

$$\overline{I}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} I(t) dt$$

and inverse Fourier transform is defined as

$$I(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \overline{I}(\omega) d\omega$$

2. THE ELECTRIC FIELD

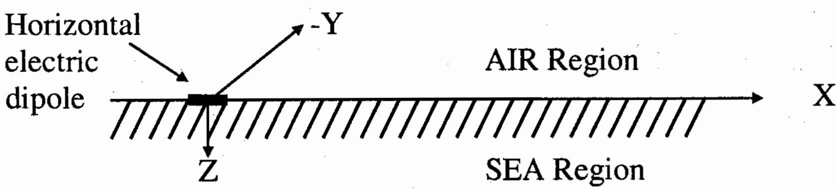
The electric field of an electrically short dipole with its axis along the x-axis and an electric moment $2h_e I_0$ is well known [1] with time dependence $e^{-i\omega t}$, it is

$$\overline{E}(\omega,r)=\frac{2h_e\overline{I}_x(\omega)i\omega\mu_0}{4\pi k^2}\left\{\left(1-\frac{x^2}{r^2}\right)\frac{k^2}{r}+\left(1-\frac{3x^2}{r^2}\right)\frac{ik}{r^2}-\left(1+\frac{3ix^2}{r^2}\right)\frac{i}{r^3}\right\}e^{ikr}, \quad (1)$$

where r is the distance from the center of the dipole, h_e is the effective length of the dipole and k is the complex wave number of the sea water given by [2,9]

$$k=\omega\left[\mu_0\left(\epsilon+\frac{i\sigma}{\omega}\right)\right]^{\frac{1}{2}},$$

where σ is the conductivity of sea region. When a pulse applied to horizontal electric dipole on the surface of the sea an electromagnetic pulse is generated upward into air, downward into the sea and horizontal along the boundary in a surface. Related studies are in [3]-[11] if one is interested to find the field vertically downward along the positive z-axis then one can put $x=0, y=0 \Rightarrow r=z$ in Eq. (1).



By taking inverse Fourier transform of Eq. (1), we can write

$$E(t,r)=\frac{h_e i\mu_0}{4\pi^2}\left\{\left(1-\frac{x^2}{r^2}\right)\frac{I_1}{r}+\left(1-\frac{3x^2}{r^2}\right)\frac{I_2}{r^2}-\left(1+\frac{3ix^2}{r^2}\right)\frac{iI_3}{r^3}\right\},$$

where

$$I_1=\int_{-\infty}^{\infty}\omega\overline{I}(\omega)e^{ikr-i\omega t}d\omega$$

$$I_2=i\int_{-\infty}^{\infty}\omega\frac{1}{k}\overline{I}(\omega)e^{ikr-i\omega t}d\omega$$

$$I_3=-\int_{-\infty}^{\infty}\omega\frac{1}{k^2}\overline{I}(\omega)e^{ikr-i\omega t}d\omega$$

Here $I_1 = -\frac{dI_2}{dr}$ and $I_2 = \frac{dI_3}{dr}$
Therefore to find the electric field we have to find only I_3. By substituting the value of k

$$I_3 = -\int_{-\infty}^{\infty} \frac{\overline{I}(\omega) e^{i\omega\left\{\mu_0\left(\epsilon + \frac{i\sigma}{\omega}\right)\right\}^{1/2} r - i\omega t}}{\mu_0(\epsilon\omega + i\sigma)} d\omega$$

The integral I_3 can be evaluated in the complex ω-plane with a branch cut along the imaginary axis. Therefore,

$$I_3 = \tilde{I}_3 + I_p$$

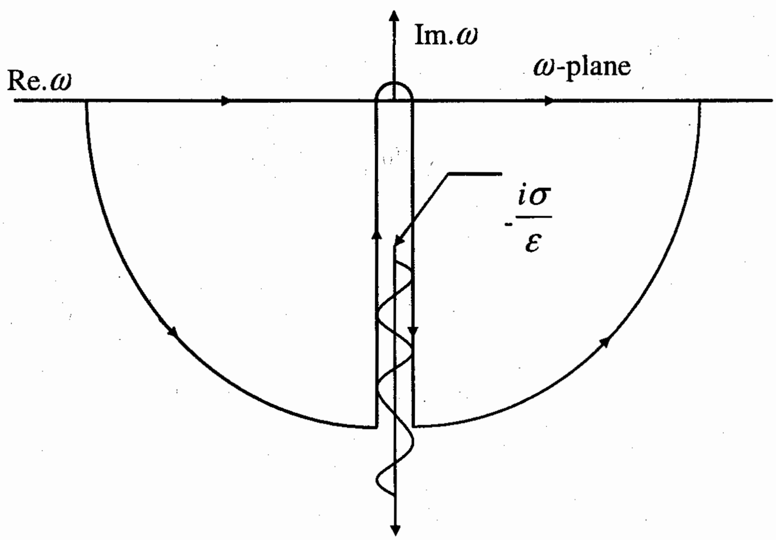
$$\tilde{I}_3 = -\int_{-\infty}^{\infty} \frac{\overline{I}(\omega) e^{i\omega\left\{\mu_0\left(\epsilon + \frac{i\sigma}{\omega}\right)\right\}^{1/2} r - i\omega t}}{\mu_0(\epsilon\omega + i\sigma)} d\omega$$

For $t < 0$, the path of integration may be closed by a large semicircle in the upper half plane. It follows that

$$I_3 = I_p$$

If $\overline{I}(\omega)$ has no singularities in the upper half plane, then

$$I_3 = 0, \qquad t < 0$$



For $t > 0$, the path of integration may be closed in the lower half plane, as shown in the figure, in which contour encloses both sides of the branch cut in the lower half plane. Therefore on the branch cut put

$$\omega = -i\xi$$

when ω in the 4th quadrant
$\Rightarrow \frac{1}{\omega}$ is in Ist. quadrant. $\Rightarrow \frac{i\sigma}{\omega}$ is in 2nd. quadrant.
$\Rightarrow \epsilon + \frac{i\sigma}{\omega}$ increases real part which pulls $\frac{i\sigma}{\omega}$ to the right.
Hence $\epsilon + \frac{i\sigma}{\omega}$ is either in Ist. or in 2nd. quadrant, therefore

$$\begin{aligned}\sqrt{\mu_0\left(\epsilon + \frac{i\sigma}{\omega}\right)} &= \sqrt{\mu_0\left(\epsilon - \frac{\sigma}{\xi}\right)}; \qquad if \frac{\sigma}{\xi} < \epsilon \\ &= i\sqrt{\mu_0\left(\frac{\sigma}{\xi} - \epsilon\right)}; \qquad if \frac{\sigma}{\xi} > \epsilon\end{aligned}$$

similarly when ω in the 3rd quadrant then $\epsilon + \frac{i\sigma}{\omega}$ is either in 3rd or in 4th quadrant, so

$$\begin{aligned}\sqrt{\mu_0\left(\epsilon + \frac{i\sigma}{\omega}\right)} &= \sqrt{\mu_0\left(\epsilon - \frac{\sigma}{\xi}\right)}; \qquad if \frac{\sigma}{\xi} < \epsilon \\ &= -i\sqrt{\mu_0\left(\frac{\sigma}{\xi} - \epsilon\right)}; \qquad if \frac{\sigma}{\xi} > \epsilon\end{aligned}$$

$$\begin{aligned}\tilde{I}_3 = &-\int_{\infty}^{\frac{\sigma}{\epsilon}} \frac{\overline{I}(-i\xi)e^{-\xi t}}{\mu_0(\xi\epsilon - \sigma)} e^{\xi r\sqrt{\mu_0\left(\epsilon - \frac{\sigma}{\xi}\right)}} d\xi - \int_{\frac{\sigma}{\epsilon}}^{0} \frac{\overline{I}(-i\xi)e^{-\xi t}}{\mu_0(\xi\epsilon - \sigma)} e^{i\xi r\sqrt{\mu_0\left(\frac{\sigma}{\xi} - \epsilon\right)}} d\xi \\ &- \int_{0}^{\frac{\sigma}{\omega}} \frac{\overline{I}(-i\xi)e^{-\xi t}}{\mu_0(\xi\epsilon - \sigma)} e^{-i\xi r\sqrt{\mu_0\left(\frac{\sigma}{\xi} - \epsilon\right)}} d\xi - \int_{\frac{\sigma}{\epsilon}}^{\infty} \frac{\overline{I}(-i\xi)e^{-\xi t}}{\mu_0(\xi\epsilon - \sigma)} e^{\xi r\sqrt{\mu_0\left(\epsilon - \frac{\sigma}{\xi}\right)}} d\xi\end{aligned}$$

Ist and last integral will cancel out each other and we are left with

$$\tilde{I}_3 = -\int_{0}^{\frac{\sigma}{\epsilon}} \frac{\overline{I}(-i\xi)e^{-\xi t}}{\mu_0(\xi\epsilon - \sigma)} e^{i\xi r\sqrt{\mu_0\left(\frac{\sigma}{\xi} - \epsilon\right)}} d\xi - \int_{0}^{\frac{\sigma}{\epsilon}} \frac{\overline{I}(-i\xi)e^{-\xi t}}{\mu_0(\xi\epsilon - \sigma)} e^{-i\xi r\sqrt{\mu_0\left(\frac{\sigma}{\xi} - \epsilon\right)}} d\xi$$

Thus by stationary Phase method, which is valid for large value of r

$$\tilde{I}_3 = \frac{-4i\overline{I}\left(\frac{-i\sigma}{2\epsilon}\right)e^{-\frac{\sigma}{2\epsilon}t}}{\mu_0\sigma}\left\{\frac{2\pi}{\epsilon\sqrt{\mu_0\epsilon r}}\right\}^{\frac{1}{2}}\sin\left(\frac{\sigma}{2}\sqrt{\frac{\mu_0}{\epsilon}}r - \frac{\pi}{4}\right)$$

$\tilde{I}_3$ converges to zero for large value of time or distance.
It is interesting to note that if $\overline{I}(\omega)$ has singularities on the negative imaginary axis; and we choose branch cut on negative imaginary axis, then

$$\begin{aligned} I_3 &= I_p \qquad ;t < 0 \\ &= \tilde{I}_3 \qquad ;t > 0 \end{aligned}$$

where I_p is the contribution of the poles due to $\overline{I}(\omega)$ which lies in upper half plane.

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CONVECTION IN A HORIZONTAL VISCOUS FLUID LAYER SANDWICHED BETWEEN TWO POROUS LAYERS

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Abstract

A linear stability analysis is applied to a system consisting of three horizontal layers, a viscous fluid layer sandwiched between two porous medium permeated by the fluid that the lower layer is heated uniformlly from below. Flow in the porous medium is assumed to be governed by Darcy's law. The Beavers-Joseph condition is applied at the interface between any two different layers.Spectral method of Chebysheve polynomials are used to obtain numerical solution for stationary convection and overstability cases.

1. INTRODUCTION

Let $\mathcal{L}_1$,$\mathcal{L}_2$ and $\mathcal{L}_3$ be three horizontal layers that the bottom of three layer $\mathcal{L}_1$ touches the top of the layer $\mathcal{L}_2$ and the bottom of the layer $\mathcal{L}_2$ touches the top of the layer $\mathcal{L}_3$. A right handed system of cartesian coordinates $(x_i, i = 1, 2, 3)$ is chosen so that interface are the plains $x_3 = 0$ and $x_3 = 1$, the top boundary of $\mathcal{L}_1$ is $x_3 = d_{m1}$ and the lower boundary of $\mathcal{L}_3$ is $x_3 = -d_{m2}$.Suppose that the upper layer $\mathcal{L}_1$ and the bottom layer $\mathcal{L}_3$ are occupied by a porous medium permeated by the fluid whereas the middle layer $\mathcal{L}_2$ is filled with an incompressible viscous fluid.Dravity acts in the negative x_3 direction and the porous medium in the third layer is heated at its lower boundary. Convection take place in which temperature driven buoyancy effects are damped by viscous effects. A stationary fluid with a thermal gradient in the x_3 direction (the so called "conduction solution") is one possible solution to this problem and so it is natural to investigate its stability. The question has recently been addressed by Chen[2] who derived the appropriate equations in two layers problem.Briefly, the fluid flow in the porous layers $\mathcal{L}_1$ and $\mathcal{L}_3$, with thickness d_m, is governed by Darcy's, wheres the fluid flow in the middle layer $\mathcal{L}_2$, with thickness d_f, is governed by the Navier-stokes equations. Convection is driven by the temperature dependence of the fluid density. Typically, the Oberbeck-Boussinesq approximation is made in which concepts like local thermal equilibrium, heating from viscous dissipation, radiative effects etc. are ignored as are variations in fluid density expect where they occur in the momentum equation. Let T denote the Kelvin temperature of the fluid and T_0 be a constant reference Kelvin temperature. Then for the purpose of this work, the fluid density ρ_f is related to T by

$$\rho_f = \rho_0[1 - \alpha(T - T_0)] \tag{1.1}$$

where ρ_0 is the density of the fluid at T_0 and α(supposed constant) is the cofficient of volume expansion of the fluid.In many situations(1.1) is inadequate. For example, the description of water[1] around $4°K$.

2. THE GOVERNING EQUATION OF NATURAL CONVECTION

The field equations for this problem are written seoeratly for the fluid layer between two porous layers.The governing equations for porous medium are represented by

$$\begin{aligned} \frac{\rho_0}{\phi}\frac{\partial V_m}{\partial t} &= -\nabla P_m - \frac{\mu}{k}V_m + \rho_f g, \\ (\rho c)_m\frac{\partial T_m}{\partial t} + (\rho c_p)_f V_m . \nabla T_m &= k_m \nabla^2 T_m \end{aligned} \tag{2.2}$$

where T_m is the kelvin temperature of the porous medium, V_m is the solenoidal seepage velocity, $P_m$is the hydrostatic pressure, μ is the dynamic viscosity of the fluid, K is the permeability of the porous substrate, ϕ is its porosity,k_m is

[1]George et.al. [3] describes convection in lakes in which the bottom and the top can be represented by a porous layers which are under-pinned by an impermeable permafrost boundary

the overall Thermal Conductivity of the porous medium $(spc)_f$ is the heat capacity per unit volume of the fluid at constant pressure and $(\rho c)_m$ is the overall heat capacity per unit volume of the porous medium at constant pressure. In fact,

$$(\rho c)_m = \phi(\rho c_p)_f + (1-\phi)(\rho c_p)_m$$

where $(\rho c_p)_m$ is the heat capacity per unit volume of thr porous substrate. The governing equations for the fluid layer are

$$\begin{aligned} \rho_0 \frac{\partial V_f}{\partial t} &= -\bigtriangledown P_f + \mu \bigtriangledown^2 V_f + \rho_f g, \\ (\rho c_p)_f \frac{\partial T_f}{\partial t} &= k_f \bigtriangledown^2 T_f \end{aligned} \tag{2.3}$$

where T_f is the Kelvin temperature of the fluid layer, V_f is the solenoidal fluid velocity, P_f is the hydrostatic pressure and k_f is the thermal conductivity of the fluid.

3. BOUNDARY CONDITIONS

The convection problem is completed by the specification of boundary conditions at the upper surface of porous medium layer, at the first interface between the porous medium and fluid layers and at the second interface between the fluid and the porous medium layers and at the lower boundary of the porous medium layer. Many combination of boundary conditions are possible but for comparison with Chen[2], $x_3 = d_{m1}$is assumed to be impenetrable and held at constant temperature T_u at upper layer, whereas $x_3 = -d_{m2}$ is assumed to be impenetrable and at constant temperature T_l at lower layer. In terms of w_m, w_f and w_m, the axial velocity components of the fluid in $\pounds_1$ and $\pounds_2$ and $\pounds_3$ respectively, these requirements lead to the two conditions:

$$T_m(d_{m1}) = T_u, \quad w_m(d_{m1}) = 0 \tag{3.4}$$

on the top boundary $\pounds_1$ and the two conditions

$$T_m(-d_{m2}) = T_l, \quad w_m(-d_{m2}) = 0 \tag{3.5}$$

on the lower boundary of $\pounds_3$. Strictly speaking, the boundary condition on $x_3 = d_{m1}$ is $v_m = 0$; the format(3.4) specifically uses the fact that v_m is solenoidal (incompressibility constraint). The porous-medium/fluid and the fluid/porous-medium interfaces $x_3 = 1$ and $x_3 = 0$ boundary conditions are based on the assumption that temperature, heat flux, normal fluid velocity and normal stress are continuous so that

$$\begin{aligned} T_m(d) &= T_f(d), \quad k_m \frac{\partial T_m(d)}{\partial x_3} = k_f \frac{\partial T_f(d)}{\partial x_3}, \\ w_m(d) &= w_f(d), \quad -P_f(d) + 2\mu \frac{\partial w_f(d)}{\partial x_3} = -P_m(d) \end{aligned} \tag{3.6}$$

respectively. This leaves one final condition to be specified on the interface. Several possible forms[2] have been proposed for the missing condition but the most popular of these is undoubtedly due to Beavers and Joseph [1] who suggest that

$$\frac{\partial u_f}{\partial x_3} = \frac{\alpha B_j}{\sqrt{K}}(u_f - u_m), \quad \frac{\partial v_f}{\partial x_3} = \frac{\alpha B_j}{\sqrt{K}}(v_f - v_m) \tag{3.7}$$

where u_f, v_f are the limiting tangential components of the fluid velocity as the interface is approached from the fluid layer $\mathcal{L}_2$, whereas u_m, v_m are the same limiting components of tangential fluid velocity as the interface approached from the porous layer $\mathcal{L}_1$ and $\mathcal{L}_3$. Clearly, discontinuities in shear velocity across the interface are inherent in this specification of the last boundary condition. Equation (2.2), (2.3) together with boundary condition (3.4), (3.5), (3.6), (3.7) possess a static (equilibrium) solution in which the fluid is stationary everywhere and heat is conducted across the layers in accordance with the thermal boundary conditions. Specifically,

$$V_f = 0, \quad V_m = 0$$

and the static temperature and hydrostatic pressure fields satisfy the equations

$$-\nabla P_m + \rho_f g = 0, \quad -\nabla P_f + \rho_f g = 0, \quad \nabla^2 T_m = \nabla^2 T_f = 0 \tag{3.8}$$

together with the exterior boundary conditions

$$T_m(d_{m1}) = T_u, \quad T_m(-d_{m2}) = T_l \tag{3.9}$$

[2]Jones [4] proposes continuity of shear stress at the interface. In truth, the nature of this boundary condition has little impact on results under most circumstances.

and the interfacial conditions

$$T_m(d) = T_f(d), \quad k_m \frac{\partial T_m(d)}{\partial x_3} = k_f \frac{\partial T_f(d)}{\partial x_3}, \quad P_f(d) = P_m(d) \tag{3.10}$$

In conclusion, it follows almost immediately that the equilibrium temperature fields in the fluid and porous medium are respectively

$$\begin{aligned} T_m &= T_1 - (T_1 - T_u)\frac{x_3}{d_m}, \quad 1 \leq x_3 \leq d_m, \\ T_f &= T_0 - (T_0 - T_1)\frac{x_3}{d_f}, \quad 0 \leq x_3 \leq 1, \\ T_m &= T_0 - (T_l - T_0)\frac{x_3}{d_m}, \quad -d_m \leq x_3 \leq d_0 \end{aligned} \tag{3.11}$$

where d_m is the depth of the porous layer, d_f is the depth of the fluid layer and T_1 and T_0 is the temperature on $\pounds$ and is determined by the continuity of heat flux across $x_3 = 1$ and $x_3 = 0$ respectively.

4. PERTURBED EQUATIONS

Suppose that the static equilibrium solution is now perturbed so that the velocity, pressure and temperature fields in the fluid and porous layers are respectively

$$v_f, \quad P_f + p_f, \quad T_0 - (T_0 - T_u)\frac{x_3}{d_f} + \theta_f \tag{4.12}$$

and

$$v_m, \quad P_m + p_m, \quad T_0 - (T_1 - T_0)\frac{x_3}{d_m} + \theta_m \tag{4.13}$$

Taking account of the properties of the equilibrium solution, it follows from the general field equations (2.3) and (2.2) that v_f, p_f and θ_f satisfy

$$\begin{aligned} \rho_0(\frac{\partial v_f}{\partial t} + v_f . \nabla v_f) &= - \nabla p_f + \mu \nabla^2 v_f \quad - \quad \rho_0 \alpha \theta_f g, \\ (\rho c_p)_f [\frac{\partial \theta_m}{\partial t} + v_f . (\nabla \theta_f - \frac{(T_0 - T_u)}{d_f} e_3)] \quad &= \quad k_f \nabla^2 \theta_f \end{aligned} \tag{4.14}$$

where v_f is solenoidal whereas v_m, p_m and θ_m satisfy

$$\begin{aligned} \frac{\rho_0}{\phi}\frac{\partial v_m}{\partial t} &= - \nabla p_m + \frac{\mu}{k} v_m \quad - \quad \rho_0 \alpha \theta_m g, \\ (\rho c_p)\frac{\partial \theta_m}{\partial t} + (\rho c_p)_f v_m . (\nabla \theta_m - \frac{(T_1 - T_0)}{d_m} e_3)] \quad &= \quad k_m \nabla^2 \theta_m \end{aligned}$$

with v_m solenoidal. The modified boundary conditions on the upper boundary of the porous layer ($x_3 = d_{m1}$), the porous/fluid and fluid/porous interfaces ($x_3 = 0$,) and the lower boundary of the porous layer ($x_3 = -d_{m2}$) are respectively

$$\begin{aligned}
\theta_m(d_{m1}) &= 0, \quad w_m(d_{m1}) = 0, \\
\theta_f(d) &= \theta_m(d), \quad k_f \frac{\partial \theta_f(d)}{\partial x_3} = k_m \frac{\partial \theta_m(d)}{\partial x_3}, \\
-p_f(d) + 2\mu \frac{\partial w_f(d)}{\partial x_3} &= -p_m(d), \quad w_f(d) = w_m(d), \\
\frac{\partial u_f(d)}{\partial x_3} = \frac{\alpha_{BJ}}{\sqrt{K}}(u_f(d) &- u_m(d)), \quad \frac{\partial v_f(d)}{\partial x_3} = \frac{\alpha_{BJ}}{\sqrt{K}}(v_f(d) - v_m(d)), \\
\theta_m(-d_{m2}) &= 0, \quad w_m(-d_{m2}) = 0
\end{aligned} \tag{4.16}$$

5. NON-DIMENSSIONLISATION

The non-dimensionlisation of (4.14), (4.15) and the boundary conditions (4.16) is technical but routine. Nield[5] presents a detailed description of the procedure. Most importantly, each layer has a different length and time scale. Using the scaling suggested by Chen and Chen[2], non dimensional coordinate $\hat{x}_f$, time $\hat{t}_f$, perturbed velocity $\hat{v}_f$, perturbed velocity $\hat{v}_f$, pressure $\hat{p}_f$ and temperature $\hat{\theta}_f$ in the upper (fluid) layer are introduced by the definitions

$$\begin{aligned}
&x = d_f \hat{x}_f, \quad t_f = \frac{d_f^2}{\lambda_f}\hat{t}_f, \quad v_f = \frac{\lambda_f}{d_f}\hat{v}_f \\
&\quad p_f = \frac{\mu\lambda_f}{d_f^2}\hat{p}_f, \quad \theta_f = |T_0 - T_u|\hat{\theta}_f
\end{aligned} \tag{5.17}$$

here λ_f is the thermal diffusivity of the fluid phase and is defined by $\lambda_f = k_f/(\rho c_p)_f$. With this change of variables, the equation(4.14) describing the motion of the fluid layer now assume the non-dimensional form

$$\begin{aligned}
&\frac{\partial \hat{v}_f}{\partial \hat{t}_f} + \hat{v}_f . \nabla_f \hat{v}_f = P_{rf}[- \nabla_f \hat{P}_f + \nabla_f^2 \hat{v}_f + Ra_f \hat{\theta}_f e_3] \\
&\quad \frac{\partial \hat{\theta}_f}{\partial \hat{t}_f} + \hat{v}_f .(\nabla_f \hat{\theta}_f - sign(T_0 - T_u)e_3) = \nabla_f^2 \hat{\theta}_f
\end{aligned} \tag{5.18}$$

where Pr_F and Ra_f denote respectively the Prandtl number and Rayleigh number of the fluid layer and are defined by

$$Pr_f = \frac{\mu}{\rho_0 \alpha_f}, \quad Ra_f = \frac{g\alpha d_f^3 |T_0 - T_u|}{\mu \lambda_f} \tag{5.19}$$

A similar procedure is applied to a porous medium in which non-dimensional spatial coordinates $\hat{x}_m$, time $\hat{t}_m$, pressure $\hat{p}_m$ and temperature $\hat{\theta}_m$ are introduce by the definitions

$$\begin{aligned} & x = d_m \hat{x}_m, \quad t_m = \frac{d_m^2}{\lambda_m}\hat{t}_m, \quad v_m = \frac{\lambda_m}{d_m}\hat{v}_m \\ & p_m = \frac{\mu\lambda_m}{K}\hat{p}_m, \quad \theta_m = |T_1 - T_0|\hat{\theta}_m \end{aligned} \tag{5.20}$$

Here λ_m is the thermal diffusivity of the porous medium and is defined by $\lambda_m = k_m/(\rho c_p)_f$.With this change of variables, the equations (4.15) governing the motion of the fluid in the porous layer now become

$$\begin{aligned} \frac{Da}{\phi}\frac{\partial \hat{v}_m}{\partial \hat{t}_m} &= -Pr_m \hat{v}_m + Pr_m Ra_m \hat{\theta}_m e_3, \\ G_m \frac{\partial \hat{\theta}_m}{\partial \hat{t}_m} + \hat{v}_m.(\nabla_m \hat{\theta}_m - sign(T_l - T_0)e_3) &= \nabla_m^2 \hat{\theta}_m \end{aligned} \tag{5.21}$$

where $G_m = (\rho c_p)_m/(\rho c_p)_f$ and Pr_m, D_m and Ra_m denote respectively the Prandtl number, Darcy number and Rayleigh number of the porous layer and are defined by

$$Pr_m = \frac{\mu}{\rho_0 \lambda_m}, \quad D_a = \frac{K}{d_m^2}, \quad Ra_m = \frac{g\rho_0 \alpha K d_m |T_l - T_0|}{\mu \lambda_m} \tag{5.22}$$

The scaling (5.17) and (5.20) are now used to non-dimensionalise the boundary conditions (4.16). The procedure is straightforward and yields

$$\begin{aligned} \hat{\theta}_m(2) &= 0, \quad \hat{w}_m(2) = 0, \\ \hat{\theta}_f(d) &= \epsilon_T \hat{\theta}_m(d), \quad \frac{\partial \hat{\theta}_f(d)}{\partial x_3} = \frac{\partial \theta_m(d)}{\partial x_3}, \end{aligned}$$

$$\begin{aligned}
\epsilon_T \hat{d} D_a(\hat{p}_f - 2\frac{\partial \hat{w}_f}{\partial x_3}) &= \hat{p}_m, \quad \epsilon_T \hat{w}_f(d) = \hat{w}_m(d), \\
\epsilon_T \frac{\partial \hat{u}_f}{\partial x_3} = \frac{\alpha_{BJ}}{\hat{d}\sqrt{Da}}(\epsilon_T \hat{u}_f &- \hat{u}_m), \quad \epsilon_T \frac{\partial \hat{v}_f}{\partial x_3} = \frac{\alpha_{BJ}}{\hat{d}\sqrt{Da}}(\epsilon_T \hat{v}_f - \hat{v}_m), \\
\hat{\theta}_m(-1) &= 0, \quad \hat{w}_m(-1) = 0.
\end{aligned} \tag{5.23}$$

where the parameters $\epsilon_T, \hat{d}$ and $\hat{k}$ are defined by

$$\epsilon_T = \frac{\hat{d}}{\hat{k}}, \quad \hat{d} = \frac{d_m}{d_f}, \quad \hat{k} = \frac{k_m}{k_f}$$

6. LINEARISATION OF PROBLEM

Until this point, no approximations have been made in the derivation of the perturbation equations. All subsequent analyses in this chapter are based on the linearised version of equation(5.18) and (5.21), obtained from them by ignoring all "product terms". For the Fluid layer, w_f and θ_f satisfy

$$\begin{aligned}
\frac{\partial v_f}{\partial t_f} &= P_{rf}[-\nabla_f P_f + \nabla_f^2 v_f + Ra_f \theta_f e_3], \\
\frac{\partial \theta_f}{\partial t_f} - H w_f &= \nabla_f^2 \theta_f
\end{aligned} \tag{6.24}$$

and for the porous layer, w_m and θ_m satisfy

$$\begin{aligned}
\frac{Da}{\phi}\frac{\partial v_m}{\partial t_m} &= -Pr_m v_m + Pr_m Ra_m \theta_m e_3, \\
G_m \frac{\partial \theta_m}{\partial t_m} - H w_m &= \nabla_m^2 \theta_m
\end{aligned} \tag{6.25}$$

Where $H = sign(T_l - T_u) = sign(T_0 - T_l) = sign(T_l - T_0)$ and the "hat" superscript has been dropped although all variables are non-dimensional. Since the boundary conditions (5.23) are already linear, no further action is required here except to remove superscripts. By taking the double curl of the momentum equation in each layer, the hydrostatic pressure are suppressed. The specification of the final problem is completed by taking the third component of the reworked momentum equation in each layer together with the appropriate energy equation, In the fluid layer

$$\begin{aligned}\frac{1}{Pr_f}\frac{\partial}{\partial t}\nabla^2 w_f &= \nabla^4 w_f + Ra_f \triangle_2 \theta_f,\\ \frac{\partial \theta_f}{\partial t} - H w_f &= \nabla^2 \theta_f, \end{aligned} \tag{6.26}$$

and in the porous layer

$$\begin{aligned}\frac{1}{Pr_m}\frac{Da}{\phi}\frac{\partial}{\partial t}\nabla^2 w_m &= \nabla^4 w_m + Ra_f \triangle_2 \theta_m,\\ G_m\frac{\partial \theta_m}{\partial t} - H w_m &= \nabla^2 \theta_m, \end{aligned} \tag{6.27}$$

The Beaver-Joseph and normal stress interfacial boundary conditions must be reworked to eliminate pressure and horizontal component of velocity. Hydrostatic pressure term are removed by computing the two-dimensional Laplacian of the boundary condition and by using the divergence of the respective momentum equation to eliminate the Laplacian of pressure. Similarly, both Beaver-Joseph conditions can be combined together by constructing the two-dimensional divergence of the tangential components of the fluid velocity. The upshot of these considerations is that these boundary conditions are transformed to

$$\hat{d}^3\epsilon_T Da\frac{\partial}{\partial x_3}(\nabla^2 w_f - \frac{1}{Pr_f}\frac{\partial w_f}{\partial t} + 2\triangle_2 w_f) = -(\frac{Da}{Pr_m\phi}\frac{\partial}{\partial t} + 1)\frac{\partial w_m}{\partial x_3}, \tag{6.28}$$

$$\epsilon_T \hat{d}\frac{\partial}{\partial x_3}(w_f - \frac{\hat{d}\sqrt{Da}}{\alpha_{BJ}}\frac{\partial w_f}{\partial x_3}) = \frac{\partial w_m}{\partial x_3}. \tag{6.29}$$

7. LINEARISATION OF EQUATIONS

The linearisation of the equation (6.27)and(6.26) and the related boundary conditions is the resultant vector obtained by applying combining the relationships:

$$\begin{aligned} w_m(t,X) &= w_m(x_3)exp[i(p_m x + q_m y) + \sigma_m t],\\ \theta_m(t,X) &= \theta_m(x_3)exp[i(p_m x + q_m y) + \sigma_m t],\\ w_m(t,X) &= w_m(x_3)exp[i(p_m x + q_m y) + \sigma_m t],\\ \theta_f(t,X) &= \theta_f(x_3)exp[i(p_f x + q_f y) + \sigma_f t]. \end{aligned} \tag{7.30}$$

The governing equation of two layers can be represented as a system of equations, called the basic equation. Linearisation quantities are applied (7.30) to all x_3-components of the basic equations, (6.27) and (6.26), to obtain

$$\begin{aligned} \frac{\sigma_f}{Pr_f}(D_f^2 - a_f^2)w_f &= (D_f^2 - a_f^2)^2 e_f - Ra_f a_f^2 \theta_f, \\ \sigma_f \theta_f &= w_f + (D_f^2 - a_f^2)\theta_f, \\ -\frac{Da}{\phi}\frac{\sigma_m}{P_r m}(D_m^2 - a_m^2)w_m &= (D_m^2 - a_m^2)w_m + Ra_m a_m^2 \theta_m, \\ G_m \sigma_m \theta_m &= w_m + (D_m^2 - a_m^2)\theta_m. \end{aligned} \tag{7.31}$$

Where $a_m^2 = p_m^2 + q_m^2$, $a_f^2 = p_f^2 + q_f^2$ are non-dimensionalised wave numbers in the porous medium and fluid respectively. For a given set of physical parameters and a_m, Ra_m is determined by the condition that the real part of σ_f and σ_m are zero. However, in this particular problem it is non-trivial fact that σ_f and σ_m are always real; in fact, there is a principle of exchange of stabilities[3]. Hence for a given a_m, Ra_m is computed when $\sigma_f = \sigma_m = 0$. The eigenvalue problem for σ_m and σ_f is completed by specification of boundary conditions at $x_3 = 2, x_3 = 1, x_3 = 0$ and $x_3 = -1$. Pressures are computed from the two-dimensional divergence of the momentum equations whereas non-axial velocity component are eliminated by judicious use of the incompressibility constraints. Using these ideas, it can be verified from (5.23), (6.28) and (6.29) that the final boundary conditions are:- Upper boundary $x_3 = 2$

$$w_m = 0, \quad \theta_m = 0, \tag{7.32}$$

Middle boundaries $x_3 = 1$ and $x_3 = 0$

$$\begin{aligned} \theta_f = \epsilon_T \theta_m, \quad D_f \theta_f = D_m \theta_m, \quad w_m &= \epsilon_T \theta_m \\ \epsilon_T \hat{d}(D_f w_f - \frac{\hat{d}\sqrt{Da}}{\alpha_{BJ}} D_f^2 w_f) &= D_m w_m, \\ \hat{d}^3 \epsilon_T Da(D_f^3 w_f - 3a_f^2 D_f w_f - \frac{\sigma_f}{Pr_f} D_f w_f) &= -(\frac{Da}{\phi}\frac{\sigma_m}{Pr_m} + 1) D_m w_m, \end{aligned} \tag{7.33}$$

[3]As a working rule, stationary convection is usually the only destabilising mechanism when two effects are competing (viscosity and thermal here) but once another stabilising effects such as magnetic field come into play, overstability now becomes possible, that is stationary eigenvalue are fully complex

Lower boundary $x_3 = -1$

$$w_m = 0, \quad \theta_m = 0. \tag{7.34}$$

$$D_m\psi = \frac{d\psi}{dx_3}(-1 < x_3 < 0) and (1 < x_3 < 2), \quad a_f = \hat{d} a_m,$$

$$D_f\psi = \frac{d\psi}{dx_3}(0 < x_3 < 1), \quad \sigma_f = \frac{\hat{d}^2}{\hat{k}}\sigma_m, \tag{7.35}$$

where $D_f = D_m$, since the two layers have the same width from 0 to 1.

8. RESULTS AND REMARKS

The marginal stability curves are computed for a thermally-driven convection of a viscous fluid layer sandwiched symmetrically between two porous layers heated from below for isothermal rigid boundaries, with reciprocal thermal conductivity ratio $\hat{k}^{-1} = 0.7$, Darcy number $\delta = 4 \times 10^{-6}$, Beavers-Joseph constant $\alpha_{BJ} = 0.1$ and for a variety of reciprocal depth ratios ranging from 0.01 to 0.2. The results are illustrated in figures which show the relation between Rayleigh number Ra_m and wave number a_m for different values of depth ratio. To draw the figures showing separated curves, we used two different range for Rayleigh number as shown in figure 2 and 1. The table express the value of Rayleigh number Ra_m versus wave number a_m for different value of depth ratios as mentioned before. The three layers problem, the case of a porous medium layer sandwiched symmetrically

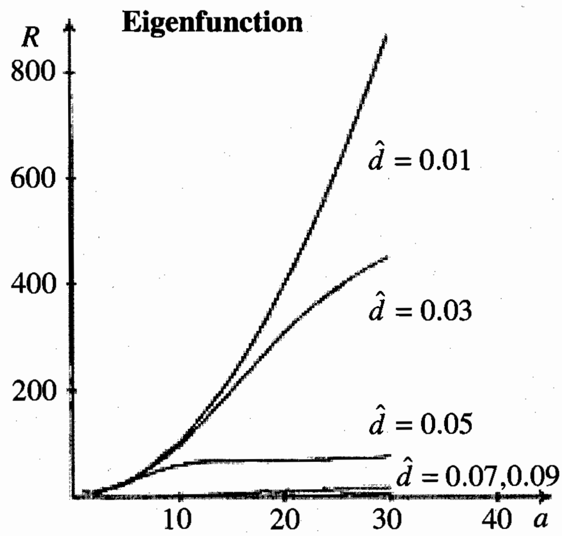

Figure 1: R_{a_m} versus a_m.

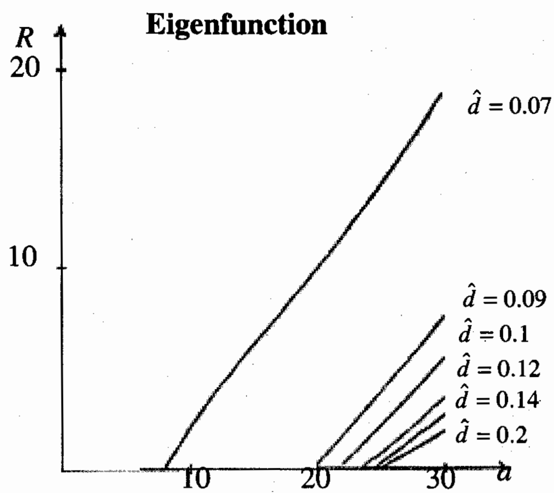

Figure 2: R_{a_m} versus a_m.

Wave No. a	Critical Rayleigh No. R_c									
	d=	0.01	0.03	0.05	0.07	0.09	0.1	0.12	0.14	0.2
1.0			11.24	10.51						
2.0		0.57	9.84	9.26						
3.0		14.34	13.78	12.88						
4.0		20.93	20.10	18.50						
5.0		29.63	28.43	25.53						
6.0		40.36	38.62	33.47						
7.0		53.08	50.62	41.64						
8.0		71.65	64.36	49.24	0.06					
9.0			79.75	55.56	1.06					
10.0		103.01	96.68	60.32	2.07					
11.0		123.56	115.01	63.63	2.97					
12.0		146.05	134.58	65.78	3.81					
13.0		170.48	155.16	67.11	3.81					
14.0		196.84	176.53	67.88	5.36					
15.0		225.11	198.39	68.31	6.11					
16.0		255.29	220.46	68.53	6.85					
17.0		287.37	242.49	68.66	7.59					
18.0		321.34	264.21	68.75	8.33					
19.0		357.18	285.38	68.85	9.09					
20.0		394.88	305.82	69.00	9.85	0.21				
21.0		434.42	325.35	69.22	10.63	0.92				
22.0		475.79	343.83	69.51	11.43	1.62	0.21			
23.0		518.96	361.22	69.88	12.25	2.32	0.84	0.00		
24.0		563.92	377.45	70.35	13.09	3.02	1.46	0.22	0.00	
25.0		610.64	392.51	70.90	13.95	3.74	2.10	0.73	0.28	0.02
26.0		659.10	406.43	71.55	14.84	4.46	2.74	1.26	0.73	0.38
27.0		709.28	419.25	72.29	15.76	5.19	3.39	1.79	1.19	0.74
28.0		761.15	431.02	73.13	16.71	5.94	4.05	2.33	1.66	1.10
29.0		814.68Z	441.83	74.05	17.69	6.70	4.73	2.88	2.13	1.47
30.0		869.85	451.75	75.07	18.70	7.49	5.43	3.46	2.62	1.85

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INTRANSITIVE ACTION OF THE GROUP PSL(2, Z) ON A SUBSET $Q^*(\sqrt{k^2m})$ OF $Q(\sqrt{m})$

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Abstract

It is well-known that $PSL(2, Z)$ is the group generated by the linear-fractional transformations $x : z \rightarrow -\frac{1}{z}$ and $y : z \rightarrow \frac{z-1}{z}$, which satisfy the relations $x^2 = y^3 = 1$. We denote this modular group by $G = \langle x, y : x^2 = y^3 = 1 \rangle$. Let $n = k^2m$, where m is a square-free positive integer and k is any non zero integer. Then $Q^*(\sqrt{n}) = \{\frac{a+\sqrt{n}}{c} : a, c$ and $b = \frac{a^2-n}{c}$ are integers and $(a, b, c) = 1\}$ is a G-subset of the real quadratic field $Q(\sqrt{m})$ for all k [8].

In this paper we show that for each non-square positive integer $n > 2$, the action of the group G on $Q^*(\sqrt{n})$ is intransitive.

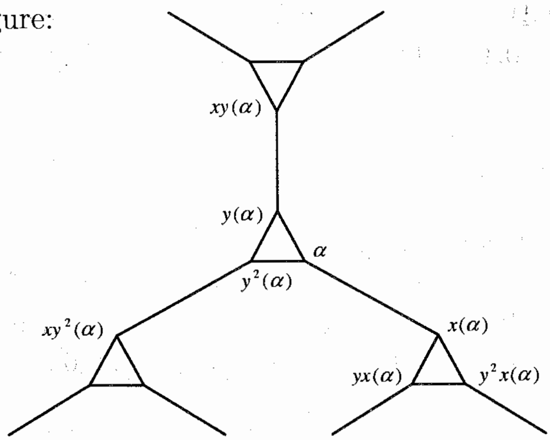
1. INTRODUCTION

Let $PSL(2, Z)$ be the modular group, that is, the group of linear-fractional transformations $z \rightarrow \frac{az+b}{cz+d}$, where a, b, c, d are in Z and $ad - bc = 1$.
It is well-known that $PSL(2, Z)$ is the group generated by the linear-fractional transformations $x : z \rightarrow -\frac{1}{z}$ and $y : z \rightarrow \frac{z-1}{z}$, which satisfy the relations $x^2 = y^3 = 1$. For brevity we denote this modular group by $G = \langle x, y : x^2 = y^3 = 1 \rangle$. Note that G is the free product of two cyclic groups.
Let $Q(\sqrt{m}) = \{a + b\sqrt{m} : a, b \in Q\}$. An element $a + b\sqrt{m}$, $b \neq 0$, of $Q(\sqrt{m})$ is called a real quadratic irrational number. It has been shown in [8] that every real quadratic irrational number can be uniquely represented as $\frac{a+\sqrt{n}}{c}$, where n is a non-square positive integer and a, $\frac{a^2-n}{c}$ and c are relatively prime integers. Let $n = k^2m$, where m is a square-free positive integer and k is any non zero integer. Then
$Q^*(\sqrt{n}) = \{\frac{a+\sqrt{n}}{c} : a, c \text{ and } b = \frac{a^2-n}{c} \text{ are integers and } (a, b, c) = 1\}$ is a subset of the real quadratic field $Q(\sqrt{m})$ for all k.
For $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$; α and its conjugate $\bar{\alpha} = \frac{a-\sqrt{n}}{c}$ may or may not have the same sign. If α and $\bar{\alpha}$ have different signs, then α is called an ambiguous number [8].
A set Ω with an action of some group G on it, is said to be a G-set. In our case the set $Q(\sqrt{m})$ will be a G-set for $G = \langle x, y : x^2 = y^3 = 1 \rangle$. Let Ω be a G-set. Then we say that Ω is a transitive G-set or that G acts on Ω transitively if Ω is non empty and for any p, q in Ω there exists a g in G such that $p^g = q$.
A subset Ω' of a G-set is called a G-subset if $g \in G \Rightarrow \omega^g \in \Omega'$ for each $\omega \in \Omega'$.
An action of G on $Q(\sqrt{m})$ has been considered in [8]. It has been shown there that the ambiguous numbers in $Q^*(\sqrt{n})$ are finite in number and that part of the coset diagram containing these numbers forms a single closed path. It has also been shown that the set $Q^*(\sqrt{n})$ is invariant under the action of G.
The exact number of ambiguous numbers in $Q^*(\sqrt{n})$ has been determined in [3], as a function of n. In [4], [5] it has been proved, with the aid of coset diagrams, that for each prime $p > 2$, the action of the group G on the subset $Q^*(\sqrt{p})$ of the real quadratic field $Q(\sqrt{p})$ is intransitive whereas the action of the group G on $Q^*(\sqrt{2})$ is transitive.
In this paper we generalize these results and prove that for each non-square

positive integer $n > 2$, the action of the group G on a subset $Q^*(\sqrt{n})$ of the real quadratic field $Q(\sqrt{m})$ is intransitive.
In Section 3, we describe the classification of the elements of $Q^*(\sqrt{n})$ for each non-square positive integer n by using the notion of congruence. With the help of this classification, we explore the G-subsets of $Q^*(\sqrt{n})$ in Section 4. These G-subsets of $Q^*(\sqrt{n})$ are used to determine disjoint G-orbits of $Q^*(\sqrt{n})$. In particular, we deduce that for each non-square positive integer $n > 2$. The action of the group G on a subset $Q^*(\sqrt{n})$ of the real quadratic field $Q(\sqrt{m})$ is intransitive.

2. COSET DIAGRAMS

We use coset diagrams to study the action of the group G on the real quadratic fields $Q(\sqrt{m})$. In this case a coset diagram is just a graphical representation of a permutation action of the group G. In the case of y, there is a need to distinguish y from y^{-1}. Since y is of order 3, the 3-cycles of y are represented by small triangles, with the convention that y permutes their vertices counter-clockwise. Fixed points of x and y, if they exist, are denoted by heavy dots. Then the geometry of the figure obviously makes distinction between x-edges and y-edges.
Thus in the case of G the diagram consists of a set of small triangles representing the action of $C_3 = \langle y : y^3 = 1\rangle$ and a set of edges representing the action of $C_2 = \langle x : x^2 = 1\rangle$.
This graph can be interpreted as a coset diagram, with the vertices identified with the cosets of $Stab_\alpha(G)$, the stabilizer of some vertex α of the graph, or as 1-skeleton of the cover of the fundamental complex of the presentation which corresponds to the subgroup $Stab_\alpha(G)$. A general fragment of the coset diagram for the action of G on $Q(\sqrt{m})$, where $\alpha \in Q(\sqrt{m})$, will look as shown in the following figure:



3. A CLASSIFICATION OF THE ELEMENTS OF $\mathbf{Q}^*(\sqrt{\mathbf{n}})$

A classification of the elements $\frac{a+\sqrt{p}}{c}, b = \frac{a^2-p}{c}$, of $Q^*(\sqrt{p})$, p an odd prime, with respect to odd-even nature of a, b, c has been given in [2]. These results have been generalized in [6], [7] by using the notion of congruence.
Before we proceed further, we have the following basic definition.
Let s be a fixed positive integer. An element $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ is said to be of the form $[u, v, w]$, where u, v, w are the integers if $u \equiv a(mod\ s)$, $v \equiv b(mod\ s)$, and $w \equiv c(mod\ s)$.
Two elements $\alpha = \frac{a+\sqrt{n}}{c}, b = \frac{a^2-n}{c}$ and $\alpha' = \frac{a'+\sqrt{n}}{c'}, b' = \frac{(a')^2-n}{c'}$ in $Q^*(\sqrt{n})$ are said to be of the same form if $a \equiv a'(mod\ s)$, $b \equiv b'(mod\ s)$, $c \equiv c'(mod\ s)$. If α, α' in $Q^*(\sqrt{n})$ are of the same form then we write $\alpha \sim \alpha'$.
It is easy to see that the relation $\alpha \sim \alpha'$, where $\alpha, \alpha' \in Q^*(\sqrt{n})$ is an equivalence relation. For each value of $s > 1$, we get different equivalence classes of elements of $Q^*(\sqrt{n})$ so we discuss these cases separately. It is interesting to note that the classification obtained for $s = 2$ is actually the classification of the elements of $Q^*(\sqrt{n})$ with respect to odd-even character of a, b, c. Moreover, the equivalence classes of elements of $Q^*(\sqrt{n})$ obtained for $s = 2$ do not give any useful information except that if $n \equiv 1(mod\ 4)$ then the set consisting of all elements of $Q^*(\sqrt{n})$ of the form $[1, 0, 0]$ is invariant under the action of the group G.
In [7] we have classified the elements of $Q^*(\sqrt{n})$ with respect to congruence modulo $s = 4, 8$. These classes of the elements of $Q^*(\sqrt{n})$ are very useful for the study of the action of this modular group $G = \langle x, y : x^2 = y^3 = 1\rangle$ on $Q^*(\sqrt{n})$.
We here enlist these classifications for use in the sequel.

1. If $n \equiv 0(mod\ 4)$ then the equivalence classes of elements of $Q^*(\sqrt{n})$ are [0,0,1], [0,0,3], [0,1,0], [0,3,0], [1,1,1], [1,3,3], [2,0,1], [2,0,3], [2,1,0], [2,3,0], [3,3,3] and [3,1,1].

2. If $n \equiv 1(mod\ 4)$ then the equivalence classes of elements of $Q^*(\sqrt{n})$ are [0,3,1], [0,1,3], [1,0,1], [1,1,0], [1,0,3], [1,3,0], [2,3,1], [2,1,3], [3,0,3], [3,3,0], [3,0,1], [3,1,0], [1,2,2] and [3,2,2].
   These are the only equivalence classes of elements of $Q^*(\sqrt{n})$ if $n \equiv 5(mod\ 8)$. If $n \equiv 1(mod\ 8)$ then the equivalence classes of elements of $Q^*(\sqrt{n})$ are [1,2,0], [1,0,2], [3,2,0], [3,0,2], [3,0,0], [1,0,0], [0,3,1], [0,1,3],

[1,0,1], [1,1,0], [1,0,3], [1,3,0], [2,3,1], [2,1,3], [3,0,3], [3,3,0], [3,0,1], [3,1,0] and there are no further classes.

3. If $n \equiv 2(mod\ 4)$ then the equivalence classes of elements of $Q^*(\sqrt{n})$ are [0,2,1], [0,1,2], [0,2,3], [0,3,2], [1,1,3], [1,3,1], [2,2,1], [2,2,3], [2,1,2], [2,3,2], [3,1,3] and [3,3,1]

4. If $n \equiv 3(mod\ 4)$ then the equivalence classes of elements of $Q^*(\sqrt{n})$ are [0,1,1], [0,3,3], [1,2,1], [1,1,2], [1,2,3], [1,3,2], [2,1,1], [2,3,3], [3,2,3], [3,3,2], [3,2,1] and [3,1,2].

4. G-SUBSETS OF $\mathbf{Q^*(\sqrt{n})}$

In this section we determine the G-subsets of $Q^*(\sqrt{n})$ with the help of the classification of the elements of $Q^*(\sqrt{n})$ with respect to congruence modulo 4.

Theorem 4.1

Let $n \equiv 0$ *or* $3(mod\ 4)$, be a non-square positive integer. Let
$A = \{\frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ with $c \equiv 1(mod\ 4)\}$,
$B = \{\frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ with $c \equiv 3(mod\ 4)\}$. Then $A \cup \{\frac{-1}{\alpha} : \alpha \in A\}$ and $B \cup \{\frac{-1}{\alpha} : \alpha \in B\}$ are both G-subsets of $Q^*(\sqrt{n})$.

Proof

Keeping in view the classification of the elements of $Q^*(\sqrt{n})$ with respect to congruence modulo 4, we consider the following two cases.

Case I If $n \equiv 0(mod\ 4)$ then the set $A = \{\frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ with $c \equiv 1(mod\ 4)\}$ consists of elements of the form [0,0,1], [1,1,1], [2,0,1] and [3,1,1] only. Also if $\alpha = \frac{a+\sqrt{n}}{c}, b = \frac{a^2-n}{c}$, then $x(\alpha) = \frac{-a+\sqrt{n}}{b} = \frac{a_1+\sqrt{n}}{c_1}$ with $a_1 = -a, b_1 = c, c_1 = b$.
Thus x maps elements of the form [0,0,1], [1,1,1], [2,0,1] and [3,1,1] onto the elements of the form [0,1,0], [3,1,1], [2,1,0] and [1,1,1] respectively.
Hence $A \cup \{\frac{-1}{\alpha} : \alpha \in A\}$ consists of elements of $Q^*(\sqrt{n})$ which are of the form [0,0,1], [0,1,0], [1,1,1], [2,0,1], [2,1,0] and [3,1,1] only.
Next, since $y(\alpha) = \frac{-a+b+\sqrt{n}}{b} = \frac{a_2+\sqrt{n}}{c_2}$ with $a_2 = -a+b, b_2 = -2a+b+c, c_2 = b$.
So y maps elements of the form [0,0,1], [0,1,0], [1,1,1], [2,0,1], [2,1,0] and [3,1,1]

in $Q^*(\sqrt{n})$ onto the elements of $Q^*(\sqrt{n})$ of the form [0,1,0],[1,1,1], [0,0,1], [2,1,0], [3,1,1] and [2,0,1] respectively.
Since $G = \langle x, y : x^2 = y^3 = 1\rangle$ is generated by $x(z) = \frac{-1}{z}, y(z) = \frac{z-1}{z}$ each $g \in G$ is a word in x, y or y^2. So it is enough to show that if $\alpha \in Q^*(\sqrt{n})$ is from any one of the forms [0,0,1], [0,1,0], [1,1,1], [2,0,1], [2,1,0] and [3,1,1] then $x(\alpha)$ and $y(\alpha)$ are also from these forms.
This shows that the set $A \cup \{\frac{-1}{\alpha} : \alpha \in A\}$ is a G-subset of $Q^*(\sqrt{n})$.
Similarly we can prove that the set $B \cup \{\frac{-1}{\alpha} : \alpha \in B\}$, where
$B = \{\frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ with $c \equiv 3(mod\ 4)\}$, consisting of all elements of the forms [0,0,3], [0,3,0], [1,3,3], [2,0,3], [2,3,0] and [3,3,3] is also a G-subset of $Q^*(\sqrt{n})$.
Case II Let $n \equiv 3(mod\ 4)$. Then it is easy to prove that the set $A \cup \{\frac{-1}{\alpha} : \alpha \in A\}$ consisting of all elements of the forms [0,1,1], [1,2,1], [1,1,2], [2,1,1], [3,2,1] and [3,1,2] is a G-subset of $Q^*(\sqrt{n})$. Similarly the set
$B \cup \{\frac{-1}{\alpha} : \alpha \in B\}$ consisting of all elements of the forms [0,3,3], [1,2,3], [1,3,2], [2,3,3], [3,2,3] and [3,3,2] is also a G-subset of $Q^*(\sqrt{n})$. Hence the proof is completed.

Theorem 4.2

Let $n \equiv 1(mod\ 4)$. Then the set of elements of $Q^*(\sqrt{n})$ of the form [0,3,1], [0,1,3], [1,0,1], [1,1,0], [1,0,3], [1,3,0], [2,3,1], [2,1,3], [3,0,3], [3,3,0], [3,0,1] and [3,1,0] is a G-subset of $Q^*(\sqrt{n})$. Similarly the set of elements of $Q^*(\sqrt{n})$, where $n \equiv 5(mod\ 8)$, of the forms [1,2,2], [3,2,2] is a G-subset of $Q^*(\sqrt{n})$. However, if $n \equiv 1(mod\ 8)$, then the set of elements of $Q^*(\sqrt{n})$ of the forms [3,2,0], [3,0,2],[3,0,0], [1,0,0], [1,2,0] and [1,0,2] is also a G-subset of $Q^*(\sqrt{n})$.

Proof

Proof is analogous to the proof of Theorem 4.1.
Finally, we conclude this paper with the following remarks.

Remarks 4.3

1. $Q'(\sqrt{n}) = \{\frac{a+\sqrt{n}}{c} : \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n}), a$ is odd , $\frac{a^2-n}{c}$ and c are both even integers $\}$ and $Q^*(\sqrt{n})\backslash Q'(\sqrt{n})$, where $n \equiv 1(mod\ 4)$, are both proper G-subsets of a G-set $Q^*(\sqrt{n})$.

2. If $\alpha \in Q'(\sqrt{n})$, then $\alpha^G \subseteq Q'(\sqrt{n})$.

3. If $\alpha \in (Q^*(\sqrt{n})\backslash Q'(\sqrt{n}))$, then the orbit $\alpha^G \subseteq (Q^*(\sqrt{n})\backslash Q'(\sqrt{n}))$.

4. $(\sqrt{n})^G$ and $(-\sqrt{n})^G$ are disjoint orbits of $Q^*(\sqrt{n})$ for all non square positive integers $n \equiv 0$ or $3(mod\ 4)$, where if $n \equiv 1(mod\ 4)$, then $(\sqrt{n})^G$ and $(\frac{1+\sqrt{n}}{2})^G$ are disjoint orbits of $Q^*(\sqrt{n})$.

5. In the case $n \equiv 2(mod\ 4), n \neq 2$, we have:
   For $c \equiv 1(mod\ 8)$ and $c' \equiv 5(mod\ 8)$, $(\frac{a+\sqrt{n}}{c})^G$ and $(\frac{a+\sqrt{n}}{c'})^G$ are two of the disjoint orbits of $Q^*(\sqrt{n})$.

6. In [4], it was proved that the action of the group G on $Q^*(\sqrt{2})$ is transitive. The intransitivity of the action of G on $Q^*(\sqrt{n})$, $n \neq 2$, follows from Theorems 4.1, 4.2 and Remark 4.3(5).

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ON THE EXISTENCE OF COMMON FIXED POINT FOR COMPATIBLE MAPPINGS

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Abstract

We establish a common fixed point theorem for two pairs of compatible mappings under a new contractive condition, which is independent of the known contractive definitions. This theorem improves various known results.

Keywords: fixed point, complete metric space, compatible maps.

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1. INTRODUCTION

The study of common fixed point of mappings satisfying contractive type conditions has been a very active field of research activity during the last two decades. The most general of the common fixed point theorems pertain to four mappings, say A, B, S and T of a metric space (X, d), and use either a Banach type contractive condition of the form,

$$d(Ax, By) \leq hm(x, y), \quad 0 \leq h < 1, \tag{1}$$

where

$$m(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), [d(Sx, By) + d(Ax, Ty)]/2\},$$

or, a Meir-Keeler type (ϵ, δ)-contractive condition of the form, given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\epsilon \leq m(x, y) < \epsilon + \delta \Rightarrow d(Ax, By) < \epsilon \tag{2}$$

or, a ϕ-contractive condition of the form

$$d(Ax, By) \leq \phi(m(x, y)), \tag{3}$$

involving a contractive gauge function $\phi : R_+ \to R_+$ is such that $\phi(t) < t$ for each $t > 0$.
Clearly, condition (1) is a special case of both conditions (2) and (3). A ϕ-contractive condition (3) does not guarantee the existence of a fixed point unless some additional condition is assumed. Therefore, to ensure the existence of common fixed point under the contractive condition (3), the following conditions on the function ϕ have been introduced and used by various authors.

(I) $\phi(t)$ is non decreasing and $t/(t - f(t))$ is non increasing ([2]),

(II) $\phi(t)$ is non decreasing and $\lim_n \phi^n(t) = 0$ for each $t > 0$([3], [8]),

(III) ϕ is upper semi continuous ([1], [3], [9], [13], [14]), or equivalently,

(IV) ϕ is non decreasing and continuous from right ([23]).

It is now known that (e.g. [3], [16])if any of the conditions (I), (II), (III) or (IV) is assumed on ϕ, then a ϕ-contractive condition (3) implies an analogous (ϵ, δ)-contractive condition (2) and both the contractive conditions hold simultaneously. Similarly, a Meir-Keeler type (ϵ, δ)-contractive condition does not ensure the existence of a fixed point. The following example illustrates that an (ϵ, δ)-contractive condition of type (2) neither ensures the existence of a fixed point nor implies an analogous ϕ-contractive condition (3).

Example 1

([16]): Let $X = [0, 2]$ and d be the Euclidean metric on X. Define $f : X \to X$ by $fx = (1 + x)/2$ if $x < 1$; $fx = 0$ if $x \geq 1$. Then, it satisfies the contractive condition

$$\epsilon \leq \max\{d(x, y), d(x, fx), d(y, fy), [d(x, fy) + d(fx, y)]/2\} < \epsilon + \delta$$

$\Rightarrow d(fx, fy) < \epsilon$, with $\delta(\epsilon) = 1$ for $\epsilon \geq 1$ and $\delta(\epsilon) = 1 - \epsilon$ for $\epsilon < 1$ but f does not have a fixed point . Also, f does not satisfy the contractive condition

$$d(fx, fy) \leq \phi(\max\{d(x, y), d(x, fx), d(y, fy), [d(x, fy) + d(fx, y)]/2\}),$$

since the desired function $\phi(t)$ cannot be defined at $t = 1$.

Hence, the two type of contractive conditions (2) and (3) are independent of each other. Thus, to ensure the existence of common fixed point under the contractive condition (2), the following conditions on the function δ have been introduced and used by various authors.

(V) δ is non decreasing ([12], [13]),

(VI) δ is lower semi-continuous ([7], [8]).

Jachymski [3] has shown that the (ϵ, δ)-contractive condition (2) with a non-decreasing δ implies a ϕ-contractive condition (3). Also, Pant *et al.*[16] have shown that the (ϵ, δ)-contractive condition (2) with a lower semi continuous δ, implies a ϕ-contractive condition (3). Thus, we see that if additional conditions are assumed on δ then the $(\epsilon, \delta)-$ contractive condition (2) implies an analogous ϕ-contractive condition (3) and both the contractive conditions hold simultaneously.
It is thus clear that contractive conditions (2) and (3) hold simultaneously whenever (2) or (3) is assumed with additional condition on δ or ϕ respectively. It follows, therefore, that the known common fixed point theorems can be extended and generalized if instead of assuming one of the contractive condition (2) or (3) with additional conditions on δ and ϕ, we assume contractive condition (2) together with the following condition of the form.

$$d(Ax,By) < \max\{k_1[d(Sx,Ty) + d(Ax,Sx) + d(By,Ty)],$$
$$k_2[d(Sx,By) + d(Ax,Ty)]/2\} \text{ for } 0 \le k_1 < 1, \ \ 1 \le k_2 < 2.$$

We prove a common fixed point theorem for four mappings adopting this approach in this paper. This gives a new approach of ensuring the existence of fixed points under an (ϵ, δ)-contractive condition consists of assuming additional conditions which are independent of the ϕ-contractive condition implied by (V) and (VI).
Two self-mappings A and S of a metric space (X,d) are said to be *compatible* (see Jungck [7]) if, $lim_n d(ASx_n, SAx_n) = 0$, whenever x_n is a sequence in X such that $lim_n Ax_n = lim_n Sx_n = t$ for some t in X. It is easy to see that compatible mappings commute at their coincidence points.

To prove our theorem, we shall use the following lemma of Jachymski [3]:

Lemma (2.2 **of** [3]):

Let A, B, S and T be self mappings of a metric space (X, d) such that $AX \subset TX$, $BX \subset SX$. Assume further that given $\epsilon > 0$ there exists $\delta > 0$ such that for all x, y in X

$$\epsilon < M(x,y) < \epsilon + \delta \Rightarrow d(Ax, By) \le \epsilon \tag{4}$$

and

$$d(Ax, By) < M(x,y), \quad \text{whenever } M(x,y) > 0, \tag{5}$$

where $M(x,y) = \max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), [d(Sx,By)+d(Ax,Ty)]/2\}$
Then for each x_0 in X, the sequence y_n in X defined by the rule

$$y_{2n} = Ax_{2n} = Tx_{2n+1}; y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$$

is a Cauchy sequence.

Theorem 1

Let (A,S) and (B,T) be compatible pairs of self mappings of a complete metric space (X,d) such that

(i) $AX \subset TX$, $BX \subset SX$,

(ii) given $\epsilon > 0$ there exists $\delta > 0$ such that for all x, y in X
$\epsilon \leq M(x,y) < \epsilon + \delta \Rightarrow d(Ax, By) < \epsilon$,

(iii) $d(Ax, By) < \max\{k_1[d(Sx,Ty) + d(Ax,Sx) + d(By,Ty)],$
$k_2[d(Sx,By) + d(Ax,Ty)]/2\}$, for $0 \leq k_1 < 1$, $1 \leq k_2 < 2$.

If one of the mappings A, B, S and T is continuous then A, B, S and T have unique common fixed point.

Proof

Let x_0 be any point in X. Define sequences x_n and y_n in X given by the rule

$$y_{2n} = Ax_{2n} = Tx_{2n+1}; y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}. \tag{6}$$

This can be done by virtue of (i). Since the contractive condition (ii) of this theorem implies the contractive conditions (4) and (5) of above lemma (2.2 of Jachymski), so using this Lemma, we conclude that y_n is a Cauchy sequence in X. But X is complete, so there exists a point z in X such that $y_n \to z$. Also, using (6), we have

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \to z; y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \to z. \tag{7}$$

Suppose that S is continuous. Then $SSx_{2n} \to Sz$, $SAx_{2n} \to Sz$ and compatibility of A and S implies that $ASx_{2n} \to Sz$. Also, since $AX \subset TX$, corresponding to each value of n, there exists z_{2n} in X such that $ASx_{2n} = Tz_{2n}$. Thus we have $ASx_{2n} = Tz_{2n} \to Sz$ and $SSx_{2n} \to Sz$. We show that $lim_n Bz_{2n} = Sz$. If not, then there exists a subsequence $\{Bz_{2m}\}$ of $\{Bz_{2n}\}$, a number $r > 0$ and a positive integer N such that for each $m \geq N$, we have $d(ASx_{2m}, Bz_{2m}) \geq r$, $d(Sz, Bz_{2m}) \geq r$ and in view of (iii), we get

$$d(ASx_{2m}, Bz_{2m}) < \max\{k_1[d(SSx_{2m}, Tz_{2m}) + d(ASx_{2m}, SSx_{2m}) + d(Bz_{2m}, Tz_{2m})],$$
$$k_2[d(SSx_{2m}, Bz_{2m}) + d(ASx_{2m}, Tz_{2m})]/2\},$$

which, on letting $m \to \infty$, yields $d(Sz, Bz_{2m}) < d(Sz, Bz_{2m})$, a contradiction. Hence, we get $lim_n Bz_{2n} = Sz$. Also, we claim that $Az = Sz$.
If $Az \neq Sz$, then by virtue of (iii), for sufficiently large values of n, we get

$$d(Az, Bz_{2n}) < \max\{k_1[d(Sz, Tz_{2n}) + d(Az, Sz) + d(Bz_{2n}, Tz_{2n})],$$

$$k_2[d(Sz, Bz_{2n}) + d(Az, Tz_{2n})]/2\},$$

which, on letting $n \to \infty$, yields $d(Az, Sz) < d(Az, Sz)$, a contradiction. Hence, we have $Az = Sz$. Again, since $AX \subset TX$, there exists a point w in X such that $Az = Tw$. If $Bw \neq Tw$, then applying the condition (iii), we get

$$d(Az, Bw) < \max\{k_1[d(Sz, Tw) + d(Az, Sz) + d(Bw, Tw)],$$

$$k_2[d(Sz, Bw) + d(Az, Tw)]/2\},$$

which yields, $d(Az, Bw) < d(Az, Bw)$, a contradiction. Hence, we have $Az = Bw$ and so, $Sz = Az = Tw = Bw$.

Since compatible maps commute at their coincidence points, we get $ASz = SAz$ and $BTw = TBw$. Moreover, $AAz = ASz = SAz = SSz$ and $BBw = BTw = TBw = TTw$. If $Az \neq AAz$, then using condition (ii), we obtain

$$d(Az, AAz) = d(AAz, Bw) < \max\{d(SAz, Tw), d(AAz, SAz), d(Bw, Tw),$$

$$[d(SAz, Bw) + d(AAz, Tw)]/2\},$$

which yields $d(Az, AAz) < d(AAz, Bw)$, a contradiction. Hence, we have $Az = AAz = SAz$ and so Az is a common fixed point of A and S. Similarly, $Bw(= Az)$ is a common fixed point of B and T. The uniqueness of the common fixed point follows from (ii). The proof is similar when T is assumed to be continuous in place of S. Moreover, since $AX \subset TX$ and $BX \subset SX$, the proof follows on the similar lines when A or B is assumed to be continuous. This establishes the theorem.

We now give an example to illustrate the above theorem.

Example 2

Let $X = [2, 20]$ and d be the Euclidean metric on X. Define A, B, S and $T : X \to X$ as follows:
$Ax = 2$ for each x,
$Bx = 2 \;\; if \;\; x < 4 \;\; or \;\; \geq 5, \; Bx = 3 + x \;\; if \;\; 4 \leq x < 5;$
$Sx = x \;\; if \;\; x \leq 8, \;\; Sx = 8 \;\; if \;\; x > 8;$

$Tx = 2 \;\; if \;\; x < 4 \;\; or \;\; \geq 5, \;\; Tx = 9 + x \;\; if \;\; 4 \leq x < 5.$

Then A, B, S and T satisfy all the conditions of the above theorem and have a unique common fixed point $x = 2$. It can be seen in this example that A, B, S and T satisfy the condition (ii) when $\delta(\epsilon) = 1$ if $\epsilon \geq 6$ and $\delta(\epsilon) = 6 - \epsilon$ if $\epsilon < 6$. Thus, $\delta(\epsilon)$ is neither non decreasing nor lower semi continuous. It can also be verified that the mappings A, B, S and T do satisfy the contractive condition (iii) with $k_1 = 1/2$ and $k_2 = 1$. However, A, B, S and T do not satisfy the ϕ-contractive condition (3) since the required function $\phi(t)$ can not be defined at $t = 6$. Hence we see that the present example does not satisfy the conditions of any previously known common fixed point theorem for contractive type mappings, since neither the mappings satisfy a ϕ-contractive condition nor δ is lower semi continuous or non decreasing.

Remarks

Pant [21] has shown that condition (iii) of the above theorem 1 is independent of ϕ-contractive conditions. The result of Pant is a particular case of this Theorem 1 when $k_1 = 0$. So our result extends the result of Pant [21] and gives a generalization of Meir-Keeler type common fixed point theorem. Also, as various assumptions either on ϕ or on δ have been considered to ensure the existence of common fixed points under contractive conditions, so our Theorem 1 improves various results *viz*, [5, 6, 18, 20] and all other similar results for fixed points.

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DIFFERENTIAL EQUATIONS WITH A SINGULAR MATRIX AND CONVERGENCE OF APPROXIMATE SOLUTIONS

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Abstract

The method of quasilinearization coupled with the method of lower and upper solutions is applied to systems of nonlinear differential equations with a singular matrix. It generates sequences of approximate solutions which are convergent to the solution and the convergence is quadratic or semiquadratic.

1. INTRODUCTION

Let $y_0, z_0 \in C^1(J, I\!R^m)$ with $y_0(t) \leq z_0(t)$, $y_0'(t) \leq z_0'(t)$ on J and define the following set

$$\Omega = \{(t, u, v): \; y_0(t) \leq u \leq z_0(t), \;\; y_0'(t) \leq v \leq z_0'(t), \quad t \in J, \; u, v \in I\!R^m\}.$$

In this paper, the vectorial inequalities mean that the same inequalities hold between their corresponding components. We consider the system of differential equations

$$Ax'(t) = f(t, x(t), x'(t)), \;\; t \in J = [0, b] \tag{1.1}$$

with the initial condition

$$x(0) = x_0 \in I\!R^m \tag{1.2}$$

assuming that A is a singular square matrix of order m and $f \in C(\Omega, I\!R^m)$. The method of quasilinearization is described by Bellman [1], Bellman and

Kalaba [2], and has recently been generalized in recent years by Lakshmikantham and various coauthors to apply to a wide variety of problems, (see, for example, [5]–[9] and [3]). In this paper we apply this method to problems of type (1.1–1.2). If we replace f by the sum of convex and concave functions then corresponding iterations converge to the solution of problem (1.1–1.2) and this convergence is quadratic or semiquadratic. This paper generalizes the results of [4]. Note that if f does not depend on the third variable with a unit matrix in the place of A, then problem (1.1–1.2) is considered in [7].

2. ASSUMPTIONS

Consider the system of differential equations of the form:

$$\begin{cases} Ax'(t) &= f(t,x(t),x'(t))+g(t,x(t),x'(t)) \equiv F(t,x,x'),\ t\in J=[0,b], \\ x(0) &= x_0 \in I\!R^m, \end{cases} \tag{2.1}$$

where $f, g \in C(J \times I\!R^m \times I\!R^m, I\!R^m)$. Note that problem (2.1) is identical with the following

$$\begin{cases} x'(t) &= (A+B)^{-1}[F(t,x,x')+Bx'(t)],\quad t\in J, \\ x(0) &= x_0 \end{cases}$$

provided that B is an $m\times m$ matrix such that $(A+B)^{-1}$ exists. A function $v \in C^1(J, I\!R^m)$ is said to be a lower solution of problem (2.1) if

$$v'(t) \leq (A+B)^{-1}[F(t,v(t),v'(t))+Bv'(t)],\quad t\in J,\quad v(0)\leq x_0,$$

and an upper solution of (2.1) if the inequalities are reversed.
Let us introduce the following assumptions:
H_1. There exists a square matrix B of order m such that the matrix $A+B$ is nonsingular and $(A+B)^{-1}B \geq 0$; moreover $F:\Omega \to I\!R^m$ satisfies the Lipschitz condition with respect to the last variable, so for $u, \alpha, \bar\alpha \in I\!R^m$ the condition

$$|(A+B)^{-1}[F(t,u,\alpha)-F(t,u,\bar\alpha)| \leq (A+B)^{-1}B|\alpha-\bar\alpha|$$

holds for $y_0(t)\leq u\leq z_0(t),\ y_0'(t)\leq \alpha,\bar\alpha \leq z_0'(t)$ on J, where $|\alpha| = [|\alpha_1|,\cdots,|\alpha_m|]^T$ for $\alpha \in I\!R^m$.
H_2. f_x, g_x, f_{xx}, g_{xx} exist, are continuous, $(A+B)^{-1}f_x$ is non increasing in the third variable, $(A+B)^{-1}g_x$ is nondecreasing in the third variable, and

$(A+B)^{-1}f_{xx} \geq 0, \;\; (A+B)^{-1}g_{xx} \leq 0.$
H_3. $(A+B)^{-1}[f_x(t, y_0, z_0') + g_x(t, z_0, y_0')] \geq 0$ for $t \in J$.
H_4. There exist $m \times m$ matrices $C \geq 0$ and $D \geq 0$ such that

$$\begin{aligned} |(A+B)^{-1}\left[f_x(t,u,v) - f_x(t,u,\bar{v})\right]| &\leq C\sum_{i=1}^{m}|v_i - \bar{v}_i| \\ |(A+B)^{-1}\left[g_x(t,u,v) - g_x(t,u,\bar{v})\right]| &\leq D\sum_{i=1}^{m}|v_i - \bar{v}_i| \end{aligned}$$

for $y_0(t) \leq u \leq z_0(t), \;\; y_0'(t) \leq v, \bar{v} \leq z_0'(t), \;\; t \in J$ with $v = [v_1, \cdots, v_m]^T$, $\bar{v} = [\bar{v}_1, \cdots, \bar{v}_m]^T$.

3. MAIN RESULTS

The next lemma is a special case of Theorem 1.1.4 [7].

Lemma 1

Assume that $s_{ij}(t) \geq 0, \;\; t \in J$ for $i \neq j$, where $S = [s_{ij}]$ is a continuous square matrix of order m. Let

$$\begin{cases} p'(t) \leq S(t)p(t), \;\; t \in J, \\ p(0) \leq 0 = [\underbrace{0, \cdots, 0}_{m}]^T. \end{cases}$$

Then $p(t) \leq 0$ on J.

Now we are in a position to prove the following result:

Theorem 1

Assume that $f, g \in C(\Omega, I\!R^m)$, and

(1) $y_0, z_0 \in C^1(J, I\!R^m)$ are lower and upper solutions of problem (2.1) and moreover $y_0(t) \leq z_0(t)$ and $y_0'(t) \leq z_0'(t)$ on J,

(2) the assumptions H_1–H_4 hold,

(3) problem (2.1) has at most one solution.

Then, there exist monotone sequences $\{y_n\}$, $\{z_n\}$ which converge uniformly to the unique solution x of problem (2.1). Moreover, the convergence is quadratic with respect to u and it is semiquadratic with respect to u' for $u = y_n$ and $u = z_n$.

Proof. Let $u, v, \alpha, \beta \in I\!R^m$, and $y_0(t) \le v \le u \le z_0(t)$, $\; y_0'(t) \le \beta \le \alpha \le z_0'(t)$, $t \in J$.

Using the integral mean value theorem we obtain

$$\begin{aligned}
&(A+B)^{-1}[F(t,u,\alpha) - F(t,v,\beta)] \\
&= (A+B)^{-1}[F(t,u,\alpha) - F(t,v,\alpha) + F(t,v,\alpha) - F(t,v,\beta)] \\
&= (A+B)^{-1}\left\{\left[\int_0^1 F_x(t, su+(1-s)v, \alpha)ds\right](u-v) + F(t,v,\alpha) - F(t,v,\beta)\right\}.
\end{aligned}$$

Hence

$$\begin{aligned}
(A &+B)^{-1}[F(t,u,\alpha) - F(t,v,\beta)] \\
&\ge (A+B)^{-1}\{[f_x(t,v,\alpha) + g_x(t,u,\alpha)](u-v) - B(\alpha-\beta)\},
\end{aligned} \tag{3.1}$$

by conditions H_1 and H_2.

Let y_{n+1}, z_{n+1} be the solutions of linear IVP's:

$$\begin{cases} y_{n+1}'(t) = (A+B)^{-1}\{F(t,y_n,y_n') + By_n'(t) + V(t,y_n,z_n)[y_{n+1}(t) - y_n(t)]\}, \\ y_{n+1}(0) = x_0 \end{cases}$$

and

$$\begin{cases} z_{n+1}'(t) = (A+B)^{-1}\{F(t,z_n,z_n') + Bz_n'(t) + V(t,y_n,z_n)[z_{n+1}(t) - z_n(t)]\}, \\ z_{n+1}(0) = x_0 \end{cases}$$

for $n = 0, 1, \cdots$, where $V(t,y,z) = f_x(t,y,z') + g_x(t,z,y')$.

First of all, we shall prove that

$$\begin{cases} y_0(t) \le y_1(t) \le z_1(t) \le z_0(t), \quad t \in J, \\ y_0'(t) \le y_1'(t) \le z_1'(t) \le z_0'(t), \quad t \in J. \end{cases} \tag{3.2}$$

Put $p = y_0 - y_1$, so $p(0) \le 0$. Then we see that

$$\begin{aligned}
p'(t) &\le (A+B)^{-1}\{F(t,y_0,y_0') + By_0'(t) - F(t,y_0,y_0') - By_0'(t) \\
&- V(t,y_0,z_0)[y_1(t) - y_0(t)]\} \\
&= (A+B)^{-1}[f_x(t,y_0,z_0') + g_x(t,z_0,y_0')]p(t), \quad t \in J.
\end{aligned}$$

Assumption H_3 and Lemma 1 yield $p(t) \le 0$ on J proving that $y_0(t) \le y_1(t)$ on J. Since $(A+B)^{-1}[f_x(t,y_0,z_0') + g_x(t,z_0,y_0')] \ge 0$, and $p(t) \le 0$ on J, then $p'(t) \le 0$, so $y_0'(t) \le y_1'(t)$ on J. By the same way we can show that $z_1(t) \le z_0(t)$ and $z_1'(t) \le z_0'(t), \quad t \in J$.
Put $p = y_1 - z_1$. Then, by (3.1) and Assumption H_2, we have

$$\begin{aligned} p'(t) \; &= (A+B)^{-1}\{F(t,y_0,y_0') - F(t,z_0,z_0') + B[y_0'(t) - z_0'(t)] \\ &+V(t,y_0,z_0)[y_1(t) - y_0(t) - z_1(t) + z_0(t)] \\ &\le (A+B)^{-1}\{-[f_x(t,y_0,z_0') + g_x(t,z_0,z_0')][z_0(t) - y_0(t)] \\ &+B[z_0'(t) - y_0'(t)] + V(t,y_0,z_0)[y_1(t) - y_0(t) - z_1(t) + z_0(t)] \\ &+B[y_0'(t) - z_0'(t)]\} \\ &= (A+B)^{-1}\{[f_x(t,y_0,z_0') + g_x(t,z_0,y_0')]p(t) \\ &+[g_x(t,z_0,y_0') - g_x(t,z_0,z_0')][z_0(t) - y_0(t)]\} \\ &\le (A+B)^{-1}[f_x(t,y_0,z_0') + g_x(t,z_0,y_0')]p(t), \quad p(0) = 0. \end{aligned}$$

Hence, we have $p(t) \le 0$, and then $p'(t) \le 0$ on J showing that $y_1(t) \le z_1(t)$, $y_1'(t) \le z_1'(t)$, $t \in J$. It means that (3.2) holds.
In the next step we need to show that y_1 and z_1 are lower and upper solutions of problem (2.1), respectively. Then, by (3.1) and assumption H_2, we obtain

$$\begin{aligned} y_1'(t) \; &= (A+B)^{-1}\{F(t,y_0,y_0') + By_0'(t) + V(t,y_0,z_0)[y_1(t) - y_0(t)]\} \\ &\le (A+B)^{-1}\{F(t,y_1,y_1') + By_1'(t) \\ &-[f_x(t,y_0,y_1') + g_x(t,y_1,y_1')][y_1(t) - y_0(t)] + V(t,y_0,z_0)[y_1(t) - y_0(t)]\} \\ &= (A+B)^{-1}\{F(t,y_1,y_1') + By_1'(t) + [f_x(t,y_0,z_0') - f_x(t,y_0,y_1') \\ &+g_x(t,z_0,y_0') - g_x(t,y_1,y_1')][y_1(t) - y_0(t)]\} \\ &\le (A+B)^{-1}[F(t,y_1,y_1') + By_1'(t)], \end{aligned}$$

and

$$\begin{aligned} z_1'(t) \; &= (A+B)^{-1}\{F(t,z_0,z_0') + Bz_0'(t) + V(t,y_0,z_0)[z_1(t) - z_0(t)]\} \\ &\ge (A+B)^{-1}\{F(t,z_1,z_1') + Bz_1'(t) \\ &+[f_x(t,z_1,z_0') + g_x(t,z_0,z_0')][z_0(t) - z_1(t)] + V(t,y_0,z_0)[z_1(t) - z_0(t)]\} \\ &= (A+B)^{-1}\{F(t,z_1,z_1') + Bz_1'(t) + [f_x(t,z_1,z_0') - f_x(t,y_0,z_0') \\ &+g_x(t,z_0,z_0') - g_x(t,z_0,y_0')][z_0(t) - z_1(t)]\} \\ &\ge (A+B)^{-1}[F(t,z_1,z_1') + Bz_1'(t)] \end{aligned}$$

showing that y_1, z_1 are lower and upper solutions of problem (2.1), respectively.
Let us assume that

$$\begin{cases} y_{k-1}(t) \le y_k(t) \le z_k(t) \le z_{k-1}(t), & t \in J, \\ y_{k-1}'(t) \le y_k'(t) \le z_k'(t) \le z_{k-1}'(t), & t \in J, \end{cases}$$

and let y_k, z_k be lower and upper solutions of problem (2.1) for some $k \geq 1$.
We shall prove that:

$$\begin{cases} y_k(t) \leq y_{k+1}(t) \leq z_{k+1}(t) \leq z_k(t), & t \in J, \\ y_k'(t) \leq y_{k+1}'(t) \leq z_{k+1}'(t) \leq z_k'(t), & t \in J. \end{cases} \tag{3.3}$$

Put $p = y_k - y_{k+1}$. Then

$$\begin{aligned} p'(t) &\leq (A+B)^{-1}\{F(t,y_k,y_k') + By_k'(t) - F(t,y_k,y_k') - By_k'(t) \\ &-V(t,y_k,z_k)[y_{k+1}(t) - y_k(t)]\} = (A+B)^{-1}V(t,y_k,z_k)p(t) \end{aligned}$$

with $p(0) = 0$. Note that

$$\begin{aligned} (A+B)^{-1}V(t,y_k,z_k) &= (A+B)^{-1}[f_x(t,y_k,z_k') + g_x(t,z_k,y_k')] \\ &\geq (A+B)^{-1}[f_x(t,y_0,z_0') + g_x(t,z_0,y_0')] \geq 0, \quad t \in J, \end{aligned}$$

by assumptions H_2 and H_3. Hence, by Lemma 1, $p(t) \leq 0$, $p'(t) \leq 0$, $t \in J$ showing that $y_k(t) \leq y_{k+1}(t)$ and $y_k'(t) \leq y_{k+1}'(t)$, $t \in J$. Using the same argument we can prove that $z_{k+1}(t) \leq z_k(t)$, $z_{k+1}'(t) \leq z_k'(t)$, $t \in J$.
Let $p = y_{k+1} - z_{k+1}$. Then $p(0) = 0$. Using relation (3.1) and Assumption H_2 we get

$$\begin{aligned} p'(t) &= (A+B)^{-1}\{F(t,y_k,y_k') + By_k'(t) + V(t,y_k,z_k)[y_{k+1}(t) - y_k(t)] \\ &-F(t,z_k,z_k') - Bz_k'(t) - V(t,y_k,z_k)[z_{k+1}(t) - z_k(t)]\} \\ &\leq (A+B)^{-1}\{-[f_x(t,y_k,z_k') + g_x(t,z_k,z_k')][z_k(t) - y_k(t)] \\ &+V(t,y_k,z_k)[y_{k+1}(t) - y_k(t) - z_{k+1}(t) + z_k(t)]\} \\ &= (A+B)^{-1}\{[g_x(t,z_k,y_k') - g_x(t,z_k,z_k')][z_k(t) - y_k(t)] \\ &+V(t,y_k,z_k)p(t)\} \\ &\leq (A+B)^{-1}V(t,y_k,z_k)p(t), \quad t \in J. \end{aligned}$$

It proves that $y_{k+1}(t) \leq z_{k+1}(t)$, and $y_{k+1}'(t) \leq z_{k+1}'(t)$, $t \in J$, so relation (3.3) holds.
Hence, by induction, we have

$$\begin{cases} y_0(t) \leq y_1(t) \leq \cdots \leq y_n(t) \leq z_n(t) \leq \cdots \leq z_1(t) \leq z_0(t), & t \in J, \\ y_0'(t) \leq y_1'(t) \leq \cdots \leq y_n'(t) \leq z_n'(t) \leq \cdots \leq z_1'(t) \leq z_0'(t), & t \in J \end{cases}$$

for all n.
Employing standard techniques, it can be shown that $y_n \to y$, $y_n' \to y'$, $z_n \to z$, $z_n' \to z'$, $y, z \in C^1(J, \mathbb{R}^m)$. Lebesgue theorem yields that

$$\begin{aligned} y(t) &= x_0 + (A+B)^{-1}\left\{\int_0^t [F(s,y(s),y'(s)) + By'(s)]\,ds\right\}, \quad t \in J, \\ z(t) &= x_0 + (A+B)^{-1}\left\{\int_0^t [F(s,z(s),z'(s)) + Bz'(s)]\,ds\right\}, \quad t \in J \end{aligned}$$

showing that y and z are solutions of problem (2.1). Hence, by Assumption (2.1), we have $y = z = x$ on J is the unique solution of (2.1).
The order of convergence of sequences $\{y_n\}$, $\{z_n\}$, $\{y'_n\}$, $\{z'_n\}$ is considered in the next part of our considerations. For this purpose, we put

$$p_{n+1} = x - y_{n+1} \geq 0, \qquad q_{n+1} = z_{n+1} - x \geq 0 \quad \text{on} \quad J,$$

and note that $p_{n+1}(0) = q_{n+1}(0) = 0$ for $n \geq 0$. Using the integral mean value theorem and Assumptions H_1, H_2, H_4, we get

$$\begin{aligned}
p'_{n+1}(t) &= (A+B)^{-1}\{F(t,x,x') + Bx'(t) - F(t,y_n,x') + F(t,y_n,x') \\
&-F(t,y_n,y'_n) - V(t,y_n,z_n)[y_{n+1}(t) - x(t) + x(t) - y_n(t)] - By'_n(t)\} \\
&\leq (A+B)^{-1}\left\{\left[\int_0^1 F_x(t, sx + (1-s)y_n, x')ds\right] p_n(t) + 2B|p'_n(t)|\right. \\
&\left. +V(t,y_n,z_n)[p_{n+1}(t) - p_n(t)]\right\} \\
&\leq (A+B)^{-1}\{[f_x(t,x,x') - f_x(t,y_n,x') + f_x(t,y_n,x') - f_x(t,y_n,z'_n) \\
&+g_x(t,y_n,x') - g_x(t,z_n,x') + g_x(t,z_n,x') - g_x(t,z_n,y'_n)]p_n(t) \\
&+V(t,y_n,z_n)p_{n+1}(t) + 2B|p'_n(t)|\} \\
&\leq (A+B)^{-1}\left\{p_n^T(t)\left[\int_0^1 f_{xx}(t, sx + (1-s)y_n, x')ds\right] p_n(t)\right. \\
&-[p_n(t) + q_n(t)]^T\left[\int_0^1 g_{xx}(t, sy_n + (1-s)z_n, x')ds\right] p_n(t) \\
&\left. +V(t,y_n,z_n)p_{n+1}(t) + 2B|p'_n(t)|\right\} \\
&+\sum_{i=1}^{m} |q'_{i,n}(t)|Cp_n(t) + \sum_{i=1}^{m} |p'_{i,n}(t)|Dp_n(t).
\end{aligned}$$

This implies

$$\begin{aligned} p'_{n+1}(t) &\leq (L_1+L_2)p_n^2(t)+\tfrac{1}{2}L_2[p_n^2(t)+q_n^2(t)]+(K_1+K_2)p_{n+1}(t) \\ &+\tfrac{m}{2}Cp_n^2(t)+\tfrac{1}{2}CW|q'_n(t)|^2+\tfrac{m}{2}Dp_n^2(t)+\tfrac{1}{2}DW|p'_n(t)|^2 \\ &+2(A+B)^{-1}B|p'_n(t)| \\ &= Kp_{n+1}(t)+L_3p_n^2(t)+\tfrac{1}{2}L_2q_n^2(t)+\tfrac{1}{2}DW|p'_n(t)|^2+\tfrac{1}{2}CW|q'_n(t)|^2 \\ &+2(A+B)^{-1}B|p'_n(t)| \end{aligned} \tag{3.4}$$

with $p_n^2=[p_{1,n}^2,\cdots,p_{m,n}^2]^T$, and

$$|(A+B)^{-1}|\sum_{i=1}^{m}|f_{xx}^i|\leq L_1,\quad |(A+B)^{-1}|\sum_{i=1}^{m}|g_{xx}^i|\leq L_2,$$

$$L_3=L_1+\tfrac{3}{2}L_2+\tfrac{m}{2}(C+D),\quad (A+B)^{-1}f_x\leq K_1,$$

$$(A+B)^{-1}g_x\leq K_2,\ \ W=[w_{ij}],\ \ w_{ij}=1,\ \ K=K_1+K_2,$$

where L_1,L_2,K_1,K_2 and W are $m\times m$ matrices, and $L_1,L_2,K\geq 0$. There is no loss of generality assuming that K^{-1} exists such that $k_{ij}\geq 0$, where k_{ij} represents the components of this matrix.
Hence

$$\begin{aligned} p_{n+1}(t) &\leq \int_0^t e^{K(t-s)}\Big[L_3p_n^2(s)+\tfrac{1}{2}L_2q_n^2(s)+\tfrac{1}{2}DW|p'_n(s)|^2+\tfrac{1}{2}CW|q'_n(s)|^2 \\ &+2(A+B)^{-1}B|p'_n(s)|\Big]ds,\ t\in J. \end{aligned}$$

This implies

$$\begin{aligned} \max_{t\in J}\|p_{n+1}(t)\| &\leq A_1\max_{t\in J}\|p_n(t)\|^2+A_2\max_{t\in J}\|q_n(t)\|^2+A_3\max_{t\in J}\|p'_n(t)\|^2 \\ &+A_4\max_{t\in J}\|q'_n(t)\|^2+A_5\max_{t\in J}\|p'_n(t)\|, \end{aligned} \tag{3.5}$$

where $\|w\|^2=[|w_1|^2,\cdots,|w_m|^2]^T$, $w\in I\!R^m$, and

$$A_0=K^{-1}e^{Kb},\quad A_1=A_0L_3,\quad A_2=\frac{1}{2}A_0L_2,\quad A_3=\frac{1}{2}A_0DW,$$

$$A_4 = \frac{1}{2} A_0 C W, \quad A_5 = 2A_0 (A+B)^{-1} B.$$

Combining (3.4) and (3.5) we obtain

$$\begin{aligned}
\max_{t \in J} \|p'_{n+1}(t)\| \leq & (KA_1 + L_3) \max_{t \in J} \|p_n(t)\|^2 \\
& + \left(KA_2 + \tfrac{1}{2} L_2\right) \max_{t \in J} \|q_n(t)\|^2 \\
& + \left(KA_3 + \tfrac{1}{2} DW\right) \max_{t \in J} \|p'_n(t)\|^2 \\
& + \left(KA_4 + \tfrac{1}{2} CW\right) \max_{t \in J} \|q'_n(t)\|^2 \\
& + (KA_5 + 2(A+B)^{-1} B) \max_{t \in J} \|p'_n(t)\|.
\end{aligned}$$

Using the similar argument we get

$$\begin{aligned}
q'_{n+1}(t) & = (A+B)^{-1}\{F(t, z_n, z'_n) - F(t, x, z'_n) + F(t, x, z'_n) - F(t, x, x') \\
& +B[z'_n(t) - x'(t)] + V(t, y_n, z_n)[z_{n+1}(t) - z_n(t)]\} \\
& \leq (A+B)^{-1} \left\{ \left[\int_0^1 F_x(t, sz_n + (1-s)x, z'_n) ds\right] q_n(t) + 2B|q'_n(t)| \right. \\
& \left. +V(t, y_n, z_n)[q_{n+1}(t) - q_n(t)] \right\} \\
& \leq (A+B)^{-1}\{[f_x(t, z_n, z'_n) - f_x(t, y_n, z'_n) + g_x(t, x, z'_n) - g_x(t, z_n, z'_n) \\
& +g_x(t, z_n, z'_n) - g_x(t, z_n, y'_n)]q_n(t) + 2B|q'_n(t)| + V(t, y_n, z_n) q_{n+1}(t)\} \\
& \leq (A+B)^{-1} \left\{ [q_n(t) + p_n(t)]^T \left[\int_0^1 f_{xx}(t, sz_n + (1-s)y_n, z'_n) ds\right] q_n(t) \right. \\
& \left. -q_n^T \left[\int_0^1 g_{xx}(t, sx + (1-s)z_n, z'_n) ds\right] q_n(t) + 2B|q'_n(t)| \right\} \\
& +D \sum_{i=1}^{m} |q'_{i,n}(t) + p'_{i,n}(t)| q_n(t) + K q_{n+1}(t) \\
& \leq K q_{n+1}(t) + \tfrac{1}{2} L_1 p_n^2(t) + \left[\tfrac{3}{2} L_1 + L_2 + mD\right] q_n^2(t) + \tfrac{1}{2} DW |p'_n(t)|^2 \\
& + \tfrac{1}{2} DW |q'_n(t)|^2 + 2(A+B)^{-1} B |q'_n(t)|.
\end{aligned}$$

Hence

$$\max_{t\in J}\|q_{n+1}(t)\| \le B_1\max_{t\in J}\|p_n(t)\|^2 + B_2\max_{t\in J}\|q_n(t)\|^2 + B_3\max_{t\in J}\|p_n'(t)\|^2$$
$$+B_3\max_{t\in J}\|q_n'(t)\|^2 + A_5\max_{t\in J}\|q_n'(t)\|,$$

where

$$B_1 = \frac{1}{2}A_0L_1, \quad B_2 = A_0\left[\frac{3}{2}L_1 + L_2 + mD\right], \quad B_3 = \frac{1}{2}A_0DW.$$

Combining this with the last relation for q'_{n+1} we get

$$\max_{t\in J}\|q'_{n+1}(t)\| \le B_4\max_{t\in J}\|p_n(t)\|^2 + B_5\max_{t\in J}\|q_n(t)\|^2 + B_6\max_{t\in J}\|p_n'(t)\|^2$$
$$+B_6\max_{t\in J}\|q_n'(t)\|^2 + B_7\max_{t\in J}\|q_n'(t)\|,$$

where

$$B_4 = KB_1 + \tfrac{1}{2}L_1, \quad B_5 = KB_2 + \tfrac{3}{2}L_1 + L_2 + mD,$$
$$B_6 = KB_3 + \tfrac{1}{2}DW, \quad B_7 = KA_5 + 2(A+B)^{-1}B.$$

It ends the proof.

Remark 1. If $(A+B)^{-1}$ and B are positive, then we can omit the matrix $(A+B)^{-1}$ in the assumptions of Theorem 1; for example, the Lipschitz condition in Assumption H_1 takes now the form

$$|F(t,u,v) - F(t,u,\bar{v})| \le B|v - \bar{v}|.$$

Under such modified Assumptions $H_1 - H_4$, the assertion of Theorem 1 holds.

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ON FUZZY d-ALGEBRAS

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Abstract

In this paper we introduce the notions of fuzzy subalgebras and d-ideals in d-algebras, and investigate some of their results.

1. INTRODUCTION

Y. Imai and K. Iseki [1, 2] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the notion of BCI-algebras is a generalization of BCK-algebras. J. Neggers and H. S. Kim [8] introduced the class of d-algebras which is another generalization of BCK-algebras, and investigated relations between d-algebras and BCK-algebras.
L. A. Zadeh [6] introduced the notion of fuzzy sets, and A. Rosenfeld [11] introduced the notion of fuzzy group. Following the idea of fuzzy groups, O.

G. Xi [5] introduced the notion of fuzzy BCK-algebras. After that, Y. B. Jun and J. Meng [12] studied fuzzy BCK-algebras. B. Ahmad [10] fuzzified BCI-algebras. In this paper we fuzzify d-algebras.

2. PRELIMINARIES

In this section we cite the fundamental definitions that will be used in the sequel:

Definition 2.1

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCK-algebra* if it satisfies the following conditions:

(1) $((x * y) * (x * z)) * (z * y) = 0$,

(2) $(x * (x * y)) * y = 0$,

(3) $x * x = 0$,

(4) $x * y = 0, y * x = 0 \Rightarrow x = y$,

(5) $0 * x = 0$

for all $x, y, z \in X$.

Definition 2.2

Let X be a BCK-algebra and I be a subset of X, then I is called *an ideal* of X if

(I1) $0 \in I$,

(I2) y and $x * y \in I \Rightarrow x \in I$

for all $x, y \in X$.

Definition 2.3[8]

A nonempty set X with a constant 0 and a binary operation $*$ is called a *d-algebra* , if it satisfies the following axioms:

(d1) $x * x = 0$,

(d2) $0 * x = 0$,

(d3) $x * y = 0$ and $y * x = 0 \Rightarrow x = y$

for all x, $y \in X$.

Definition 2.4[9]

Let S be a non-empty subset of a d-algebra X, then S is called *subalgebra* of X if $x * y \in S$, for all $x, y \in S$.

Definition 2.5[9]

Let X be a d-algebra and I be a subset of X, then I is called *d-ideal* of X if it satisfies following conditions:

(Id1) $0 \in I$,

(Id2) $x * y \in I$ and $y \in I \Rightarrow x \in I$,

(Id3) $x \in I$ and $y \in X \Rightarrow x * y \in I$, i.e., $I \times X \subseteq I$.

Definition 2.6

A mapping $f : X \to Y$ of d-algebras is called *a homomorphism* if $f(x * y) = f(x) * f(y)$, for all x, $y \in X$.

Note that if $f : X \to Y$ is homomorphism of d-algebras, then $f(0)= \grave{0}$.

Definition 2.7

Let X be a non-empty set. A *fuzzy (sub)set* μ of the set X is a mapping $\mu : X \to [0, 1]$.

Definition 2.8

Let μ be the fuzzy set of a set X. For a fixed $s \in [0, 1]$, the set $\mu_s = \{x \in X : \mu(x) \geq s\}$ is called an *upper level* of μ.

3. FUZZY SUBALGEBRAS

Definition 3.1

A fuzzy set μ in d-algebra X is called a *fuzzy subalgebra* of X if it satisfies $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$, for all x, $y \in X$.

Example 3.2[8]

Let $X := \{0, 1, 2\}$ be a set given by the following Cayley table:

*	0	1	2
0	0	0	0
1	2	0	2
2	1	1	0

Then $(X; *, 0)$ is a d-algebra, but not a BCK-algebra, since $(2 * (2 * 2)) * 2 = (2 * 0) * 2 = 1 * 2 = 2 \neq 0$. We define a fuzzy set $\mu : X \longrightarrow [0, 1]$ by $\mu(0) = 0.7$, $\mu(x) = 0.02$, where for all $x \neq 0$. It is easy to see that μ is a fuzzy subalgebra of X.

Example 3.3

Let $X = \{0, 1, 2, \cdots\}$ be a set and the operation $*$ be defined as follows:

$$x * y := \begin{cases} 0 & \text{if x} \leq \text{y} \\ x - y & \text{if y} < \text{x} \end{cases}$$

Then $(X; *, 0)$ is an infinite d-algebra. If we define a fuzzy set $\mu : X \longrightarrow [0, 1]$ by $\mu(0) = t_1$, $\mu(x) = t_2$ for all $x \neq 0$, where $t_1 > t_2$. Then μ is a fuzzy subalgebra of X.

Example 3.4

Let $X = \{0, 1, 2, \cdots\}$ be a set and the operation $*$ be defined as follows:

$$x * y := \begin{cases} 0 & \text{if x} \leq \text{y} \\ 1 & \text{otherwise} \end{cases}$$

Then $(X; *, 0)$ is an infinite d-algebra, but not BCK-algebras, since $(2 * (2 * 0)) * 0 = (2 * 1) * 0 = 1 * 0 = 1 \neq 0$.

Example 3.5

Let $X = [0, a] \subset [0, 1]$, a being a fixed number, and the operation $*$ be defined as follows:

$$x * y = min(x, max(x, y) - min(x, y)), \quad \forall x, y \in X.$$

Then $(X; *, 0)$ is an infinite d-algebra.

Proposition 3.6

A fuzzy set μ of a d-algebra X is a fuzzy subalgebra if and only if for every $t \in [0, 1]$ the upper level μ_t is either empty or a subalgebra of X.

Proof

Suppose that μ is a fuzzy subalgebra of a d-algebra X and $\mu_t \neq 0$, then for any $x, y \in \mu_t$, we have $\mu(x * y) \geq \min \{\mu(x), \mu(y)\} \geq t$ which implies $x * y \in \mu_t$, and hence μ_t is a subalgebra of X.

Conversely, take $t = \min \{\mu(x), \mu(y)\}$, for any x, $y \in X$. Then by assumption, μ_t is a subalgebra of X, which implies $x * y \in \mu_t$, so that $\mu(x * y) \geq t = \min \{\mu(x), \mu(y)\}$. Hence μ is a fuzzy subalgebra of X.

Proposition 3.7

Any subalgebra of a d-algebra X can be realized as a level subalgebra of some fuzzy subalgebra of X.

Proof

Let A be a subalgebra of a d-algebra X and μ be a fuzzy set in X defined by

$$\mu(x) := \begin{cases} t, & \text{if x} \in \text{A}; \\ 0, & \text{otherwise.} \end{cases}$$

where $t \in (0, 1)$. It is clear that $\mu_t = A$. Let x, $y \in X$. If x, $y \in A$, then $x * y \in A$. So $\mu(x) = \mu(y) = \mu(x * y) = t$, and $\mu(x * y) \geq \min \{\mu(x), \mu(y)\}$. If x, $y \notin A$, then $\mu(x) = \mu(y) = 0$. Thus $\mu(x * y) \geq \min \{\mu(x), \mu(y)\} = 0$. If at most one of x, $y \in A$, then at least one of $\mu(x)$ and $\mu(y)$ is equal to 0. Therefore, $\min\{\mu(x), \mu(y)\} = 0$ and $\mu(x * y) \geq 0$ which completes the proof.

Corollary 3.8

Let A be a subset of X. Then the characteristic function χ_A is a fuzzy subalgebra of X if and only if A is a subalgebra of X.

Lemma 3.9

Let μ be a fuzzy subalgebra with finite image. If $\mu_s = \mu_t$, for some s, $t \in Im(\mu)$, then s=t.

Lemma 3.10

Let μ and λ be two fuzzy subalgebras of X with identical family of level subalgebras. If $Im(\mu) = \{t_1, t_2, \cdots, t_n\}$ and $Im(\lambda)$= $\{s_1, s_2, \cdots, s_m\}$, where $t_1 \geq t_2 \geq \cdots t_n$ and $s_1 \geq s_2 \geq \cdots s_m$. Then

(1) m=n.

(2) $\mu_{t_i} = \lambda_{s_i}$, for i= 1, 2, $\cdots$, n.

(3) If $\mu(x) = s_i$, then $\lambda(x) = s_i$, for all $x \in X$ and i=1, 2, $\cdots$, n.

Proposition 3.11

Let μ and λ be two fuzzy subalgebras of X with identical family of level subalgebras. Then $\mu = \lambda \Rightarrow Im(\mu) = Im(\lambda)$.

Proof

Let $Im(\mu) = Im(\lambda) = \{s_1, ..., s_n\}$ and $s_1 > ... > s_n$. By lemma 3.10, for any $x \in X$, there exists s_i such that $\mu(x) = s_i = \lambda(x)$. Thus $\mu(x) = \lambda(x), \forall\, x \in X$. This completes the proof.

Proposition 3.12

Two level subalgebras μ_s and μ_t, $(s < t)$ of a fuzzy subalgebra are equal if and only if there is no $x \in X$ such that $s \leq \mu(x) < t$.

Proof

Suppose that $\mu_s = \mu_t$, for some $s < t$. If there exists $x \in X$ such that $s \leq \mu(x) < t$, then μ_t is a proper subset of μ_s, which is contradicting the hypothesis.
Conversely, suppose that there is no $x \in X$ such that $s \leq \mu(x) < t$. If $x \in \mu_s$, then $\mu(x) \geq s$ and so $\mu(x) \geq t$, since $\mu(x)$ does not lie between s and t. Thus $x \in \mu_t$, which gives $\mu_s \subseteq \mu_t$. The converse inclusion is obvious since $s < t$. Therefore, $\mu_s = \mu_t$.

4. FUZZY d - IDEALS

Definition 4.1

A fuzzy set μ in X is called *fuzzy BCK-ideal* of X if it satisfies the following inequalities:

(1) $\mu(0) \geq \mu(x)$,

(2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$, for all $x, y \in X$.

Definition 4.2

A fuzzy set μ in X is called *fuzzy d-ideal* of X if it satisfies the following inequalities:

(Fd1) $\mu(0) \geq \mu(x)$,

(Fd2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$,

(Fd3) $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$

for all $x, y \in X$.

Example 4.3[11]

Let $X := \{0, 1, 2, 3\}$ be a d-algebra with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	0
3	3	3	3	0

Then $(X; *, 0)$ is not BCK-algebra since $\{(1*3)*(1*2)\}*(2*3) = 1 \neq 0$. We define fuzzy set μ in X by $\mu(0) = 0.8$ and $\mu(x) = 0.01$, for all $x \neq 0$ in X. Then it is easy to show that μ is a d-ideal of X.

We can easily observe the following propositions:

- In a d-algebra every fuzzy d-ideal is a fuzzy BCK-ideal, and every fuzzy BCK-ideal is a fuzzy d-subalgebra.
- Every fuzzy d-ideal of a d-algebra is a fuzzy d-subalgebra.

Theorem 4.4

If each non-empty level subset $U(\mu; t)$ of μ is a fuzzy ideal of X then μ is a fuzzy d-ideal of X, where $t \in [0, 1]$.

Proof

Assume that each non-empty level subset $U(\mu; s)$ of μ is a d-ideal. Then it is easy to show that μ satisfies $(Fd_1)$and (Fd_2). Assume that $\mu(x*y) <$ min $\{\mu(x), \mu(y)\}$, for some x, $y \in X$. Take $t_0 := \frac{1}{2}\{\mu(x*y) +$ min $(\mu(x), \mu(y))\}$, then x, $y \in U(\mu; t_0)$. Since μ is a d-ideal of X, $x*y \in U(\mu; t_0)$, therefore, $\mu(x*y) \geq t_0$, a contradiction. Hence assumption is wrong. This completes the proof.

Definition 4.5[4]

Let λ and μ be the fuzzy sets in a set X. The cartesian product $\lambda \times \mu : X \times X \longrightarrow [0, 1]$ is defined by $(\lambda \times \mu)(x, y) = \min\{\lambda(x), \mu(x)\}$, $\forall\ x, y \in X$.

Theorem 4.6

If λ and μ are fuzzy d-ideals of a d-algebra X. Then $\lambda \times \mu$ is a fuzzy d-ideal of $X \times X$.

Proof

For any $(x, y) \in X \times X$, we have

$(\lambda \times \mu)(0, 0)= \min \{\lambda(0), \mu(0)\} \geq \min \{\lambda(x), \mu(y)\} = (\lambda \times \mu)(x, y)$

Let (x_1, x_2) and $(y_1, y_2) \in X \times X$. Then

$(\lambda \times \mu)((x_1, x_2))= \min \{\lambda(x_1), \mu(x_2)\}$

$\geq \min \{ \min \{\lambda(x_1 * y_1), \lambda(y_1)\}, \min \{\mu(x_2 * y_2), \mu(y_2)\}\}$

$= \min \{ \min \{\lambda(x_1 * y_1), \mu(x_2 * y_2)\}, \min \{\lambda(y_1), \mu(y_2)\}\}$

$= \min \{(\lambda \times \mu)((x_1 * y_1, x_2 * y_2)), (\lambda \times \mu)((y_1, y_2))\}$

$= \min \{(\lambda \times \mu)((x_1, x_2) * (y_1, y_2)), (\lambda \times \mu)((y_1, y_2))\}$

and $(\lambda \times \mu)((x_1, x_2) * (y_1, y_2))= (\lambda \times \mu)((x_1 * y_1, x_2 * y_2))$
$= \min \{\lambda(x_1 * y_1), \mu(x_2 * y_2)\}$

$\geq \min \{ \min \{\lambda(x_1), \lambda(y_1)\}, \min \{\mu(x_2), \mu(y_2)\}\}$

$= \min \{ \min \{\lambda(x_1), \mu(x_2)\}, \min \{\lambda(y_1), \mu(y_2)\}\}$

$= \min \{(\lambda \times \mu)((x_1, x_2)), (\lambda \times \mu)((y_1, y_2))\}$.

Hence $\lambda \times \mu$ is a fuzzy $d-$ideal of $X \times X$.

Theorem 4.7

Let λ and μ be fuzzy sets in a d-algebra X such that $\lambda \times \mu$ is a fuzzy d-ideal of $X \times X$. Then

(i) either $\lambda(0) \geq \lambda(x)$ or $\mu(0) \geq \mu(x)$, $\forall x \in X$.

(ii) If $\lambda(0) \geq \lambda(x)$, $\forall x \in X$, then either $\mu(0) \geq \lambda(x)$ or $\mu(0) \geq \mu(x)$.

(iii) If $\mu(0) \geq \mu(x)$, $\forall x \in X$, then either $\lambda(0) \geq \lambda(x)$ or $\lambda(0) \geq \mu(x)$.

Proof

(i). We prove it using reductio ad absurdum.

Assume $\lambda(x) > \lambda(0)$ and $\mu(y) > \mu(0)$, for some x, y $\in X$. Then
$(\lambda \times \mu)(x, y) = \min\ \{\lambda(x), \mu(y)\} > \min\ \{\lambda(0), \mu(0)\} = (\lambda \times \mu)(0, 0)$

$\Rightarrow (\lambda \times \mu)(x, y) > (\lambda \times \mu)(0, 0),\ \forall\ x,\ y\ \in X.$

which is a contradiction. Hence (i) is proved.

(ii) Again, we use reduction to absurdity.

Assume $\mu(0) < \lambda(x)$ and $\mu(0) < \mu(y),\ \forall\ x,\ y \in X$. Then,
$(\lambda \times \mu)(0, 0) = \min\ \{\lambda(0), \mu(0)\} = \mu(0)$

and $(\lambda \times \mu)(x,\ y) = \min\ \{\lambda(x),\ \mu(y)\} > \mu(0) = (\lambda \times \mu)(0,\ 0)$

$\Rightarrow (\lambda \times \mu)(x,\ y) > (\lambda \times \mu)(0,\ 0).$

which is a contradiction. Hence (ii) is proved.

(iii) The proof is similar to (ii).

The partial converse of the Theorem 4.6 is the following.

Theorem 4.8

If $\lambda \times \mu$ is a fuzzy d-ideal of $X \times X$, then λ or μ is a fuzzy d-ideal of X.

Proof

By Theorem 4.7(i), without loss of generality we assume that $\mu(0) \geq \mu(x),\ \forall\ x \in X$.

It follows from Theorem 4.7(iii) that either $\lambda(0) \geq \lambda(x)$ or $\lambda(0) \geq \mu(x)$. If $\lambda(0) \geq \mu(x),\ \forall\ x \in X$. Then $(\lambda \times \mu)(0, x) = \min\ \{\lambda(0), \mu(x)\} = \mu(x)$ $--(A)$

Since $\lambda \times \mu$ is a fuzzy d-ideal of $X \times X$, therefore,

$(\lambda \times \mu)(x_1, x_2) \geq \min\ \{(\lambda \times \mu)((x_1, x_2) * (y_1, y_2)), (\lambda \times \mu)(y_1, y_2)\}$

and $(\lambda \times \mu)((x_1, x_2) * (y_1, y_2)) \geq \min\ \{(\lambda \times \mu)((x_1, x_2), (\lambda \times \mu)((y_1, y_2)\}$

It implies that $(\lambda \times \mu)(x_1, x_2) \geq \min\ \{(\lambda \times \mu)((x_1 * y_1, x_2 * y_2)), (\lambda \times \mu)(y_1, y_2)\}$.

and $(\lambda \times \mu)((x_1 * y_1, x_2 * y_2)) \geq \min\ \{(\lambda \times \mu)((x_1, x_2), (\lambda \times \mu)((y_1, y_2)\}$.

Putting x_1=y_1=0, we have

$(\lambda \times \mu)(0, x_2) \geq \min\ \{(\lambda \times \mu)((0, x_2 * y_2)), (\lambda \times \mu)(0, y_2)\}$

and $(\lambda \times \mu)((0, x_2 * y_2)) \geq \min\ \{(\lambda \times \mu)((0, x_2), (\lambda \times \mu)((0, y_2)\}$

using equation(A), we have

$\mu(x_2) \geq \min\{\mu(x_2 * y_2), \mu(y_2)\}$

and $\mu(x_2 * y_2) \geq \min\{\mu(x_2), \mu(y_2)\}$.

This proves that μ is a fuzzy d-ideal of X. The second part is similar. This completes the proof.

Definition 4.9[4]

Let A be a fuzzy set in a set S, *the strongest fuzzy relation* on S that is fuzzy relation on A is μ_A given by $\mu_A(x, y)$= $\min\{A(x), A(y)\}$, for all x, $y \in S$.

Theorem 4.10

Let A be a fuzzy set in a d-algebra X and μ_A be the strongest fuzzy relation on X. Then A is a fuzzy d-ideal of X if and only if μ_A is a fuzzy d-ideal of $X \times X$.

Proof

Suppose that A is a fuzzy d-ideal of X. Then

$\mu_A(0, 0)$= $\min\{A(0), A(0)\} \geq \min\ \{A(x), A(y)\}$= $\mu_A(x, y)$, $\forall\ (x, y) \in X \times X$.

For any $x = (x_1, x_2)$ and $y = (y_1, y_2) \in X \times X$, we have

$\mu_A(x) = \mu_A(x_1, x_2)$= $\min\ \{A(x_1), A(x_2)\}$

$\geq \min\ \{\ \min\ \{A(x_1 * y_1), A(y_1)\}, \min\ \{A(x_2 * y_2), A(y_2)\}\}$

$= \min\ \{\ \min\ \{A(x_1 * y_1), A(x_2 * y_2)\}, \min\ \{A(y_1), A(y_2)\}\}$

$=\min\ \{\mu_A(x_1 * y_1, x_2 * y_2), \mu_A(y_1, y_2)\}$

$= \min\ \{\mu_A((x_1, x_2) * (y_1 * y_2)), \mu_A(y_1, y_2)\}$

$= \min\ \{\mu_A(x * y), \mu_A(y)\}$

and $\mu_A(x * y) = \mu_A((x_1, x_2) * (y_1, y_2))$
$= \mu_A((x_1 * y_1, x_2 * y_2))$
$= \min \{A(x_1 * y_1), A(x_2 * y_2)\}$
$\geq \min \{\min \{A(x_1), A(y_1)\}, \min \{A(x_2), A(y_2)\}\}$
$= \min \{ \min \{A(x_1), A(x_2)\}, \min \{A(y_1), A(y_2)\}\}$
$= \min \{\mu_A((x_1, x_2), \mu_A(y_1, y_2)\}$
$= \min \{\mu_A(x), \mu_A(y)\}$.
Hence μ_A is a fuzzy d-ideal of $X \times X$.
Conversely, suppose that μ_A is a fuzzy d-ideal of $X \times X$. Then
$\min\{A(0), A(0)\} = \mu_A(0,0) \geq \mu_A(x,y) = \min \{A(x), A(y)\}, \forall (x,y) \in X \times X$.
It follows that $A(x) \leq A(0), \forall x \in X$
For any $x = (x_1, x_2)$, $y = (y_1, y_2) \in X \times X$, we have
$\min\{A(x_1), A(x_2)\} = \mu_A(x_1, x_2)$
$\geq \min \{\mu_A((x_1, x_2) * (y_1, y_2)), \mu_A(y_1, y_2)\}$
$= \min \{\mu_A(x_1 * y_1, x_2 * y_2), \mu_A(y_1, y_2)\}$
$= \min \{\min\{A(x_1 * y_1), A(x_2 * y_2)\}, \min\{A(y_1), A(y_2)\}\}$
$= \min \{\min\{A(x_1 * y_1), A(y_1)\}, \min \{A(x_2 * y_2), A(y_2)\}\}$.
Putting $x_2 = y_2 = 0$, we have
$\mu_A(x_1) \geq \min \{\mu_A(x_1 * y_1), \mu_A(y_1)\}$.
Likewise, $\mu_A(x_1 * y_1) \geq \min \{\mu_A(x_1), \mu_A(y_1)\}$.
Hence A is fuzzy d-ideal of X.

Definition 4.11

Let $f : X \longrightarrow Y$ be a mapping of d-algebras and μ be a fuzzy set of Y. The map μ^f is the *pre-image* of μ under f, if $\mu^f(x) = \mu(f(x)), \forall\, x \in X$.

Theorem 4.12

Let $f : X \longrightarrow Y$ be a homomorphism of d-algebras. If μ is a fuzzy d-ideal of

Y, then μ^f is a fuzzy d-ideal of X.

Proof

For any $x \in X$, we have

$\mu^f(x) = \mu(f(x)) \leq \mu(\acute{0}) = \mu(f(0)) = \mu^f(0)$

Let $x, y \in X$. Then

$\min\ \{\mu^f(x * y), \mu^f(y)\}$

$= \min\ \{\mu(f(x * y)), \mu(f(y))\}$

$=\min\ \{\mu(f(x) * f(y)), \mu(f(y))\}$

$\leq \mu(f(x)) = \mu^f(x)$

and $\min\ \{\mu^f(x), \mu^f(y)\}$

$= \min\ \{\mu(f(x)), \mu(f(y))\}$

$=\min\ \{\mu(f(x)), \mu(f(y))\}$

$\leq \mu(f(x) * f(y)) = \mu(f(x * y)) = \mu^f(x * y)$.

Hence μ^f is a fuzzy d- ideal of X.

Theorem 4.13

Let $f : X \longrightarrow Y$ be an epimorphism of d-algebras. If μ^f is a fuzzy d-ideal of X, then μ is a fuzzy d-ideal of Y.

Proof

Let $y \in Y$, there exists $x \in X$ such that $f(x) = y$. Then

$\mu(y) = \mu(f(x)) = \mu^f(x) \leq \mu^f(0) = \mu(f(0)) = \mu(\acute{0})$

Let $x, y \in Y$. Then there exist $a, b \in X$ such that $f(a) = x$ and $f(b) = y$. It follows that $\mu(x) = \mu(f(a)) = \mu^f(a)$

$\geq \min\ \{\mu^f(a * b), \mu^f(b)\}$

$= \min\ \{\mu(f(a * b)), \mu(f(b))\}$

$= \min\ \{\mu(f(a) * f(b)), \mu(f(b))\}$

$= \min\ \{\mu(x * y), \mu(y)\}$

and $\mu(x * y) = \mu(f(a) * f(b)) = \mu^f(a * b)$
$\geq \min \{\mu^f(a), \mu^f(b)\}$
$= \min \{\mu(f(a)), \mu(f(b))\}$
$= \min \{\mu(x), \mu(y)\}.$

Hence μ is a fuzzy d- ideal of Y.

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STOKES-BITSADZE PROBLEM - I

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Abstract

The stream function, Airy-stress function formulation of the classical Stokes problem in the plane has not been exhaustively studied as an elliptic boundary value problem. Bitsadze appears to have been the first to question the well-posedness of this formulation subject to certain boundary conditions. The div-curl formulation [Tahir, 1999b] has enabled us to present existence and uniqueness results for the solution of Stoke-Bitsadze Problem. Stoke-Bitsadze Problem-I corresponds to the boundary conditions prescribed by Vanmaele et al [1994].

1. INTRODUCTION

The classical Stokes problem has played a key role in the computer solution of incompressible viscous flows for over three decades. In this paper we shall be concerned with the two-dimensional Stokes problem as an elliptic boundary value problem in the plane. There are numerous formulations of the Stokes equations in two-dimensions, each deriving from the equations governing creeping incompressible flows

$$div\sigma = \mathbf{0} \text{ (conservation of momentum)}, \tag{1.1}$$

$$div\ \mathbf{u} = 0 \text{ (incompressibility condition)}. \tag{1.2}$$

We have assumed that there are no body forces present; $\mathbf{u} = (u, v)$ is the velocity of the fluid, while σ denotes the Cauchy stress tensor. It is possible to split the stress σ into an isotropic part and an anisotropic part

$$\sigma = -p\mathbf{I} + \mathbf{T} \tag{1.3}$$

where, after assuming a scaling with respect to density, p is the kinematic pressure and $\mathbf{T}$ is the extra-stress tensor. Both σ and $\mathbf{T}$ are symmetric (conservation of angular momentum). For a Newtonian fluid, $\mathbf{T}$ is related to the velocity gradient and is given by

$$\mathbf{T} = \eta\{\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathbf{T}}\}, \tag{1.4}$$

where η is the (constant) kinematic viscosity. The first formulation of the Stokes equation is therefore the first-order system in the variables $(\mathbf{u}, p, \mathbf{T})$:

$$\begin{aligned} -\nabla p + div\mathbf{T} &= 0, \\ T - \eta\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathbf{T}} &= 0, \\ div\ \mathbf{u} &= 0. \end{aligned} \tag{1.5}$$

In cartesians, equations (1.5) constitute an elliptic system of six equations in the six unknown variables $(u, v, p, T^{xx}, T^{xy}, T^{yy})$.

The most common formulation of the Stokes equations is in terms of the primitive variables $(\mathbf{u}, p)$:

$$\begin{aligned} -\nabla p + \eta\Delta\mathbf{u} &= \mathbf{0}, \\ div\ \mathbf{u} &= 0, \end{aligned} \tag{1.6}$$

where (1.6a) is obtained from the substitution of (1.5b) into (1.5a). In cartesian coordinates, equation (1.6) constitute an elliptic system of three equations in the three unknowns (u, v, p), equations (1.6a) being a second-order in the velocity.

Introducing the stream-function ψ such that

$$u = \psi_y\ ,\ \nu = -\psi_{\mathbf{x}}, \tag{1.7}$$

the incompressibility condition (1.2) is automatically satisfied given continuity of the second-order derivatives of ψ. Moreover the pressure may be eliminated from the two equations (1.6a) to obtain

$$\Delta^2\psi = 0, \tag{1.8}$$

where $\triangle^2$ is the biharmonic operator. This is a single fourth-order equation in the single variable ψ. Alternatively, in terms of vorticity

$$\omega = \ curl\ \mathbf{u} \equiv \nu_x - u_y, \tag{1.9}$$

the equations (1.6) may be written in stream-function vorticity formulation:

$$\begin{aligned} \triangle\omega &= 0, \\ \triangle\psi &= -\omega. \end{aligned} \tag{1.10}$$

There is also the velocity-vorticity-pressure formulation

$$\begin{aligned} \eta\mathbf{curl}\omega - \nabla p &= \mathbf{0}, \\ curl\ \mathbf{u} - \omega &= 0, \\ div\ \mathbf{u} &= 0. \end{aligned} \tag{1.11}$$

with **curl** $\omega = (-\omega_y, \omega_x)$.

The researchers have also been interested in the stream function/stress function formulation, see for example [Coleman, 1981], [Davis & Devlin, 1993], [Owen & Phillips, 1994], [Cassidy, 1996] and [Thatcher, 1997].

Let the components of the velocity be given in terms of stream-function $\psi(x, y)$ by (1.7) and the components of extra stress $\mathbf{T}$ be given, in terms of the Airy stress function $\phi(x, y)$ and pressure p, by

$$\begin{aligned} \sigma^{xx} &= -p + T^{xx} = \phi_{yy}, \\ \sigma^{xy} &= T^{xy} = \phi_{xy}, \\ \sigma^{yy} &= -p + T^{yy} = \phi_{xx}, \end{aligned} \tag{1.12}$$

where upper indices denote stress components while the lower indices denote the second derivatives. Then the momentum and mass balance equations (1.1) and (1.2) are satisfied by continuity. The tensor $\mathbf{T}$ can thus be expressed as

$$\mathbf{T} = \begin{bmatrix} p + \phi_{yy} & -\phi_{xy} \\ -\phi_{xy} & p + \phi_{xx} \end{bmatrix} \tag{1.13}$$

Using the equation (1.13) the following equations are obtained from (1.4) and (1.7)

$$\begin{aligned} p + \phi_{yy} &= 2\eta\psi_{xy}, \\ -\phi_{xy} &= \eta(\psi_{yy} - \psi_{xx}), \\ p + \phi_{xx} &= -2\eta\psi_{xy}, \end{aligned} \tag{1.14}$$

The pressure p can then be eliminated between (1.14a) and (1.14c) giving the following second order elliptic system in ϕ and ψ

$$\begin{aligned} \phi_{xx} - \phi_{yy} &= -4\eta\psi_{xy}, \\ -\phi_{xy} &= \eta(\psi_{xy} - \psi_{xx}), \end{aligned} \tag{1.15}$$

or the stream function/stress function formulation of the Stokes equations.

The closure of each system (1.5), (1.6), (1.8), (1.10), (1.11) and (1.15) by suitable boundary conditions to guarantee the well-posedness of the resulting elliptic boundary value problem has not been exhaustively studied, except for the primitive variable system (1.6) and the biharmonic equation (1.8). For the system (1.6) we refer to [Girault & Raviart, 1986] and [Temam, 1977]. For a polygon domain this problem has been discussed by [Kellog & Osborn, 1976] and [Grisvard, 1985]. The biharmonic operator (1.8) has been studied in [Tikhonov & Samarskii, 1963], [Kondrat'ev & Oleinik, 1983], [Kolodorkina, 1972], [Girault & Raviart, 1986] and [Grisvard, 1985]. The smoothness of the solution of the first boundary value problem for the biharmonic equation in a rectangle has been discussed in [Koval'chuk, 1969]. The well-posedness of the boundary value problems involving the Stokes operator in velocity-vorticity-pressure form has been investigated by Bochev [1977].

In the stream function stress function formulation (1.15), the researchers have been suffering a difficulty concerning the appropriate boundary conditions, see for example [Cassidy, 1996]. Owen and Phillips, [1994] embed the system (1.15) in biharmonic equations and determine the appropriate boundary conditions for the higher order system.

2. THE CAUCHY-RIEMANN SYSTEM

The Cauchey-Riemann system (or the div-curl system) is of special interest to us. The double div-curl formulation [Tahir, 1999b], of the stream function stress function formulation plays a key role for the study of boundary value problems for the Stokes flow.

The Cauchey-Riemann system $\psi_y = -\phi_x$, $\phi_y = \psi_x$ is the simplest first order elliptic system of partial differential equations. Every pair of sufficiently smooth functions ϕ and ψ satisfying this system also satisfies the Laplace equation. The inhomogeneous Cauchey-Riemann system which, in planar cartesians appear as below

$$\begin{aligned} div(\phi, \psi) &= f_1, \\ curl(\phi, \psi) &= f_2, \end{aligned} \tag{2.1}$$

has been of interest for the researchers in the last two decades, see for example [Tahir, 1999a], [Borzi et al. 1997], [Chang & Gunzburger, 1990], [Ghil & Balgovind, 1979], [Hafez & Phillips, 1985], [Lomax & Martin, 1974], [Neittaanmäki & Saranen, 1981], [Nicolaides, 1992][Rose, 1981], and [Vanmaele et al 1994].

Collectively the Cauchey-Riemann system (2.1) is elliptic while individually both the partial differential equations are hyperbolic. If ϕ and ψ are twice continuously differentiable and $f_1 = f_2 = 0$ then ϕ and ψ are harmonic. For the ellipsity of the system we refer to [Wendlend, 1979]. The following result for the Cauchy-Riemann system is of great importance.

Theorem 2.1 [Vanmaele et al 1994]

Let $f_1,\ f_2 \epsilon L_2(\Omega)$. In a square domain $\Omega = (0,1) \times (0,1)$ with boundary Γ, the Cauchy-Riemann system (2.1) with the boundary conditions

$$\begin{aligned} \phi \epsilon H^{\frac{1}{2}}(\Gamma_1) \text{ is known on } \Gamma_1 &= 0 \times (0,1), \\ \psi \epsilon H^{\frac{1}{2}}(\Gamma_2) \text{ is known on } \Gamma_2 &= \Gamma \backslash \Gamma_1, \end{aligned} \tag{2.2}$$

possesses a unique solution $(\phi, \psi) \epsilon H^1(\Omega) \times H^1(\Omega)$.

It is remarked that the result is valid for any rectangular domain.

3. THE STOKES-BITSADZE SYSTEM

We re-scale the dependent variable in (1.15) as follows; $2\eta\psi \to \psi, \phi \to \phi$ and the system then reduces to

$$\begin{aligned} \phi_{xx} - \phi_{yy} + 2\psi_{xy} &= 0, \\ \psi_{xx} - \psi_{yy} - 2\phi_{xy} &= 0. \end{aligned} \tag{3.1}$$

The system (3.1) is the second order elliptical system; the Bitsadze system [Nakhushev, 1988] and identified as Stokes-Bitsadze System [Tahir, 1999b]. The ellipticity of Stokes-Bitsadze System (SBS), in the sense of Petrovskii [1946], is proved by Thatcher [1997].

Bitsadze [1964] shows that Dirichlet problem for the SBS (3.1) has the infinite set of linearly independent solutions. He also concludes that the Dirichlet problem for the SBS (3.1) is neither Fredholmian nor Noetherian[4]. For the details on Fredholm and Noether problems we refer to [Bitsadze, 1968 & 1988] and [Mikhlin, 1970]. Bitsadze [1988] shows that the Fredholmian character of the Neumann problem is also violated for the SBS[5]. Wendland [1979] considers the Dirichlet problem for the system (3.1) and proves the violation of Lopatinski condition to show the problem to be non-Fredholm. For the details on Lopatinski condition we refer to [Wendland, 1979].

3.1 The Div-Curl Formulation of Stokes-Bitsadze System [Tahir, 1999b]

The Stokes-Bitsadze system (3.1) can be written as

$$\begin{aligned} \partial_x(\psi_y + \phi_x) + \partial_y(\psi_x + \phi_y) = 0, \\ \partial_x(\psi_x + \phi_y) - \partial_y(\psi_y + \phi_x) = 0. \end{aligned} \tag{3.2}$$

We introduce $\Phi(x, y)$ and $\Psi(x, y)$ which are defined as

$$\begin{aligned} \Phi(x, y) \equiv div(\phi, \psi) = \psi_y + \phi_x, \\ \Psi(x, y) \equiv curl(\phi, \psi) = \psi_x - \phi_y. \end{aligned} \tag{3.3}$$

It follows immediately that (3.1) has the double div-curl formulation

$$\begin{aligned} div(\Phi, \Psi) &= 0, \\ curl(\Phi, \Psi) &= 0. \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} \Phi(x, y) &= div(\phi, \psi), \\ \Psi(x, y) &= curl(\phi, \psi). \end{aligned} \tag{3.5}$$

3.2 Remark

It is easy to see that SBS (3.1) remains unchanged either (ϕ, ψ) is replaced by $(-\psi, \phi)$ or (Φ, Ψ) is replaced by $(-\Psi, \Phi)$. We will see that this is an important

[4]The situation contrasts greatly with a system of a single elliptic equation, see for details [Kuz'min, 1967] and [Bitsadze, 1968].

[5]Similar facts can also be observed when a number of independent variables is more than two. For some multidimensional analogs of Bitsadze systems we refer to [Yanushauskas, 1995], [Treneva, 1985] and [Kuz'min, 1967].

property which is useful in prescribing the appropriate boundary conditions to the Stokes-Bitsadze problem.

3.3 Boundary Value Problem of Poincaré

The Stokes-Bitsadze system (3.1) can also be expressed as

$$ß\aleph = A\aleph_{xx} + 2B\aleph_{xy} + C\aleph_{YY} = 0, \tag{3.6}$$

where $= (\phi, \psi)^T$ is the required real vector and

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.7}$$

An extensive class of problems for second order elliptic equations is covered by the linear boundary value problem of Poincaré. In the domain $\Omega \subset R^2$ with boundary Γ the Poincaré problem for SBS is formulated as follows: to seek a solution $= (\phi, \psi)^T$ for the system (3.6) subject to the boundary conditions

$$p^1\aleph_x + p^2\aleph_y + q\aleph = \Im(x, y), \quad (x, y)\epsilon\Gamma, \tag{3.8}$$

where p^1, p^2 and q are real 2×2 matrices given on the boundary Γ and $\Im$ a real vector given on Γ. For a detailed study on the Poincaré problem, for second order elliptic systems in the plane, we refer to [Bitsadze, 1968].

4. THE STOKES-BITSADZE PROBLEM

We, in the Stokes-Bitsadze problem, prescribe the boundary conditions of Poincare for the SBS in the double div-curl formulation and prove the uniqueness and existence of a solution. We are concerned with various kinds of Stokes-Bitsadze problem but in this paper we are going to discuss only the Stokes-Bitsadze problem-I which prescribes the boundary conditions like in [Vanmaele et al, 1994]. The other problems will be discussed somewhere else.

4.1 Stokes-Bitsadze problem-I

Let us recall the Stokes-Bitsadze system (SBS) in the div-curl formulation

$$\begin{aligned} \operatorname{div}(\Phi, \Psi) &= 0, \\ \operatorname{curl}(\Phi, \Psi) &= 0, \end{aligned} \tag{4.1}$$

where Φ and Ψ are defined by

$$\begin{aligned}\Phi &\equiv \text{div}(\phi, \psi) = \psi_{\text{y}} + \phi_{\text{x}}, \\ \Psi &\equiv \text{curl}(\phi, \psi) = \psi_{\text{x}} - \phi_{\text{y}}.\end{aligned} \tag{4.2}$$

We consider the SBS in the square domain $\Omega = (0,1) \times (0,1)$ with boundary Γ and prescribe the following boundary conditions of Poincaré[6].

$$\left.\begin{array}{rcl}\Phi &=& f_1 \\ \phi &=& g_1\end{array}\right\} \quad \text{on } \Gamma_1 = \{0\} \times (0,1), \tag{4.3}$$

$$\left.\begin{array}{rcl}\Psi &=& f_2 \\ \psi &=& g_2\end{array}\right\} \quad \text{on } \Gamma_2 = \Gamma \backslash \Gamma_1. \tag{4.4}$$

4.2 Theorem

For $f_i,\ g_i \,\epsilon H^{\frac{1}{2}}(\Gamma_i)$ where $i = 1, 2$, there exists a unique solution $(\phi, \psi)\ \epsilon\ H^1(\Omega) \times H^1(\Omega)$ to the Stokes-Bitsadze problem (4.1)-(4.4).

Proof

For $f_i\ \epsilon\ H^{\frac{1}{2}}(\Gamma_i)$ the div-curl system (4.1) considered with (4.3a), (4.4a) possesses a unique solution $(\Phi, \Psi)\ \epsilon\ H^1(\Omega) \times H^1(\Omega)$, see Theorem 2.1, which obviously implies that $\Phi,\ \Psi\ \epsilon L_2(\Omega)$. Knowing Φ, Ψ uniquely in Ω we have

$$\left.\begin{array}{rcl}\psi_y + \phi_x &=& \Phi \\ \psi_x - \phi_y &=& \Psi\end{array}\right\} \quad \text{in } \Omega. \tag{4.5}$$

Now for $g_i\ \epsilon\ H^{\frac{1}{2}}(\Gamma_i)$, the Cauchy-Riemann system (4.5) considered with the boundary conditions (4.3b), (4.4b) possesses a unique solution (ϕ, ψ) in $H^1(\Omega) \times H^1(\Omega)$, see Theorem 2.1, and the proof is complete.
An obvious corollary, which is indeed the modified form of the boundary conditions of the Theorem 4.2, can be stated as follows.

[6]The matrices on the boundary Γ_1 are $p^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $p^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $q = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and the matrices on the boundary Γ_2 are $p^1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $p^2 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$, $q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

4.3 Corollary

For $f_i,\ g_i \in H^{\frac{1}{2}}(\Gamma_i)$ where $i = 1, ..., 4$, there exists a unique solution $(\phi, \psi) \epsilon H^1(\Omega)\times H^1(\Omega)$ for the SBS with the following boundary conditions of Poincaré

$$\left.\begin{array}{rcl}\Phi &=& f_1\\ \phi &=& g_1\end{array}\right\} \quad \text{on } \Gamma_1 = \{0\} \times (0,1),$$
$$\left.\begin{array}{rcl}\Psi &=& f_2\\ \psi &=& g_2\end{array}\right\} \quad \text{on } \Gamma_2 = \{1\} \times (0,1),$$
$$\left.\begin{array}{rcl}\phi_n &=& f_3\\ \psi &=& g_3\end{array}\right\} \quad \text{on } \Gamma_3 = (0,1) \times \{0\},$$
$$\left.\begin{array}{rcl}\phi_n &=& f_4\\ \psi &=& g_4\end{array}\right\} \quad \text{on } \Gamma_4 = (0,1) \times \{1\}.$$

where the subscript n denotes the derivative with respect to the outward normal on the boundary.

As a consequence of Remark 2.3 we interchange the boundary conditions for Φ with Ψ, and ϕ with ψ which gives the following corollary to the Theorem 4.2.

4.4 Corollary

For $f,\ g_i \in H^{\frac{1}{2}}(\Gamma_i)$ there exists a unique solution $(\phi, \psi) \in H^1(\Omega) \times H^1(\Omega)$ for the SBS with the following Poincaré conditions on the boundary $\Gamma = \cup_{i=1}^{2}\Gamma_i$

$$\left.\begin{array}{rcl}\Psi &=& f_1\\ \psi &=& g_1\end{array}\right\} \quad \text{on } \Gamma_1 = \{0\} \times (0,1),$$
$$\left.\begin{array}{rcl}\Phi &=& f_2\\ \phi &=& g_2\end{array}\right\} \quad \text{on } \Gamma_2 = \Gamma\backslash\Gamma_1.$$

We can further modify Corollary 4.4 to obtain the following result.

4.5 Corollary

For $f_i,\ g_i \in H^{\frac{1}{2}}(\Gamma_i)$ there exists a unique solution $(\phi, \psi) \in H^1(\Omega) \times H^1(\Omega)$ for the SBS with the following Poincaré conditions on the boundary $\cup_{i=1}^{4}\Gamma_i$

$$\left.\begin{array}{rcl}\Psi &=& f_1\\ \psi &=& g_1\end{array}\right\} \quad \text{on } \Gamma_1 = \{0\} \times (0,1),$$

$$\left.\begin{array}{rcl}\Phi &=& f_2\\ \phi &=& g_2\end{array}\right\} \quad \text{on } \Gamma_2 = \{1\} \times (0,1),$$

$$\left.\begin{array}{rcl}\psi_n &=& f_3\\ \phi &=& g_3\end{array}\right\} \quad \text{on } \Gamma_3 = (0,1) \times \{0\},$$

$$\left.\begin{array}{rcl}\psi_n &=& f_4\\ \phi &=& g_4\end{array}\right\} \quad \text{on } \Gamma_4 = (0,1) \times \{1\}.$$

For the domain $\Omega = (0,1) \times (0,1)$ the following theorem can be proved on the same lines as in Theorem 4.2.

4.6 Corollary

For $f,\ g_i \in H^{\frac{1}{2}}(\Gamma_i)$ there exists a unique solution $(\phi, \psi) \in H^1(\Omega) \times H^1(\Omega)$ for the SBS with the following Poincaré conditions on the boundary $\Gamma = \cup_{i=1}^{2}\Gamma_i$

$$\left.\begin{array}{rcl}\Phi &=& f_1\\ \psi &=& g_1\end{array}\right\} \quad \text{on } \Gamma_1 = \{0\} \times (0,1),$$

$$\left.\begin{array}{rcl}\Psi &=& f_2\\ \phi &=& g_2\end{array}\right\} \quad \text{on } \Gamma_2 = \Gamma \backslash \Gamma_1.$$

4.7 Corollary

For $f_i,\ g_i \in H^{\frac{1}{2}}(\Gamma_i)$ there exists a unique solution $(\phi, \psi) \in H^1(\Omega) \times H^1(\Omega)$ for the SBS with the following Poincaré conditions on the boundary $\cup_{i=1}^{4}\Gamma_i$

$$\left.\begin{array}{rcl}\phi_n &=& f_1\\ \psi &=& g_1\end{array}\right\} \quad \text{on } \Gamma_1 = \{0\} \times (0,1),$$

$$\left.\begin{array}{rcl}\psi_n &=& f_2\\ \phi &=& g_2\end{array}\right\} \quad \text{on } \Gamma_2 = \{1\} \times (0,1),$$

$$\left.\begin{array}{rcl}\Psi &=& f_3\\ \phi &=& g_3\end{array}\right\} \quad \text{on } \Gamma_3 = (0,1) \times \{0\},$$

$$\left.\begin{array}{rcl}\Psi &=& f_4\\ \phi &=& g_4\end{array}\right\} \quad \text{on } \Gamma_4 = (0,1) \times \{1\}.$$

The interchange of Φ with Ψ, and ϕ with ψ allows us to state the following corollary to Theorem 4.6.

4.8 Corollary

For $f,\ g_i \,\epsilon\, H^{\frac{1}{2}}(\Gamma_i)$ there exists a unique solution $(\phi,\psi)\,\epsilon\, H^1(\Omega)\times H^1(\Omega)$ for the SBS with the following Poincaré conditions on the boundary $\Gamma=\cup_{i=1}^{2}\Gamma_i$

$$\left.\begin{aligned}\Psi &= f_1\\ \phi &= g_1\end{aligned}\right\}\qquad \text{on } \Gamma_1=\{0\}\times(0,1),$$

$$\left.\begin{aligned}\Phi &= f_2\\ \psi &= g_2\end{aligned}\right\}\qquad \text{on } \Gamma_2=\Gamma\backslash\Gamma_1.$$

We can further modify Corollary 4.8 to obtain the following result.

4.9 Corollary

For $f,\ g_i \,\epsilon\, H^{\frac{1}{2}}(\Gamma_i)$ there exists a unique solution $(\phi,\psi)\,\epsilon\, H^1(\Omega)\times H^1(\Omega)$ for the SBS with the following Poincaré conditions on the boundary $\cup_{i=1}^{4}\Gamma_i$

$$\left.\begin{aligned}\psi_n &= f_1\\ \phi &= g_1\end{aligned}\right\}\qquad \text{on } \Gamma_1=\{0\}\times(0,1),$$

$$\left.\begin{aligned}\phi_n &= f_2\\ \phi &= g_2\end{aligned}\right\}\qquad \text{on } \Gamma_2=\{1\}\times(0,1),$$

$$\left.\begin{aligned}\Phi &= f_3\\ \psi &= g_3\end{aligned}\right\}\qquad \text{on } \Gamma_3=(0,1)\times\{0\},$$

$$\left.\begin{aligned}\Phi &= f_4\\ \psi &= g_4\end{aligned}\right\}\qquad \text{on } \Gamma_4=(0,1)\times\{1\}.$$

5. CONCLUSION

The double div-curl formulation of the Stokes-Bitsadze System has enabled us to present existence and uniqueness results for the solution of Stokes-Bitsadze Problem-I.

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ON CERTAIN SUBCLASS OF P-VALENT FUNCTIONS DEFINED BY RUSCHEWEYH DERIVATIVE

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Abstract

The main object of this paper is to obtain a number of sharp results involving a sufficient condition in terms of coefficients, coefficient bounds, maximization theorem concerning coefficients, distortion theorem and closure theorem for certain subclass $R_{n,p}(b, B)(b \neq 0$, complex, $0 < \beta \leq 1, p \in N = (1, 2, \cdots), n > -p)$ of analytic and p-valent functions defined by the $(n + p - 1)$-th order Ruscheweyh derivative. We shall also prove that a subclass of p-valent analytic functions is closed order convolution.

1. INTRODUCTION

Let $S(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in N = \{1, 2, \cdots\}) \tag{1.1}$$

which are analytic and p-valent in the unit disc $U = \{z : |z| < 1\}$. We use Ω to denote the class of bounded analytic functions $w(z)$ in U satisfies the conditions $w(0) = 0$ and $|w(z)| \leq |z|$ for $z \in U$.

We denote by $(f * g)(z)$ the Hadamard product (convolution) of two functions $f(z)$ and $g(z)$ is given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} (p \in N) \tag{1.2}$$

Thus

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k} \tag{1.3}$$

The $(n+p-1)$-th order Ruscheweyth derivative $D^{n+p-1}f(z)$ of a function $f(z)$ in $S(p)$ is defined by

$$D^{n+p-1}f(z) = \frac{z^p(z^{n-1}f(z))^{(n+p-1)}}{(n+p-1)!} \tag{1.4}$$

where n is any integer such that $n > -p$. It is easy to see from (1.3) and (1.4) that

$$D^{n+p-1}f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z) \tag{1.5}$$

$$= z^p + \sum_{k=1}^{\infty} \delta(n,k) a_{p+k} z^{p+k} \tag{1.6}$$

where (and throughout the paper)

$$\delta(n,k) = \binom{n+p+k-1}{n+p-1} \quad (k \in N) \tag{1.7}$$

Particularly, the symbol $D^n f(z)$ was named the n-th order Ruscheweyh derivative of $f(z)$ by Al-Amiri [3].

Let $R_{n,p}(b,\beta)(b \neq 0$, complex, $0 < \beta \leq 1, p \in N$ and $n > -p)$ denote the class of functions $f(z) \in S(p)$ satisfy the condition.

$$\left| \frac{\frac{(D^{n+p-1}f(z))'}{p^{z^{p-1}}} - 1}{2\beta\left(\frac{(D^{n+p-1}f(z))'}{p^{z^{p-1}}} - 1 + b\right) - \left(\frac{(D^{n+p-1}f(z))'}{p^{z^{p-1}}} - 1\right)} \right| < 1, z \in U \tag{1.8}$$

It is easily seen that for $f(z) \in R_{n,p}(b,\beta)$, the values

$$\frac{(D^{n+p-1}f(z))'}{p^{z^{p-1}}}$$

lie inside the circle in the right half-plane with center at

$$\frac{1-(2\beta-1)(2\beta-1-2\beta b)r^2}{1-(2\beta-1)^2r^2}$$

and radius

$$\frac{2\beta|b|r}{1-(2\beta-1)^2r^2}$$

Further, it follows from Schwarz's Lemma [9] that if $f(z) \in R_{n,p}(b,\beta)$, then

$$\frac{(D^{n+p-1}f(z))'}{p^{z^{p-1}}} = \frac{1+(2\beta-1-2\beta b)w(z)}{1+(2\beta-1)w(z)}, w(z) \in \Omega \tag{1.9}$$

We note that

1. $R_{1-p,p}((1-\alpha)\cos\lambda e^{-i\lambda},\beta) = R_p^\lambda(\alpha,\beta)(|\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1, 0 < \beta \leq 1, p \in N)$ (Mogra [8]),

2. $R_{0,1}((1-\alpha)\cos\lambda e^{-i\lambda},\beta) = R^\lambda(\alpha,\beta)(|\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1, 0 < \beta \leq 1)$ (Ahuja [2]),

3. $R_{0,1}\left(\frac{2\beta(1-\alpha)}{1+\beta}\cos\lambda e^{-i\lambda}, \frac{1+\beta}{2}\right) = R^\lambda_{\alpha,\beta}(|\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1, 0 < \beta \leq 1)$ (Makówka [7]) ,

4. $R_{0,1}\left(\frac{2\beta(1-\alpha)}{1+\beta}, \frac{1+\beta}{2}\right) = R_{\alpha,\beta}(0 \leq \alpha < 1, 0 < \beta \leq 1)$ (Juneja and Morga [5]),

5. $R_{0,1}\left(\frac{2\beta}{1+\beta}, \frac{1+\beta}{2}\right) = R_{(\beta)}(0 < \beta \le 1)$ (Padmanabhan [10] and Caplinger and Causey [4]),

6. $R_{0,1}\left(1, \frac{1}{2}\right) = R$(MacGregor[6]),

7. $R_{1-p,p}\left(1, \frac{2M-1}{2M}\right) = S_p(M)(M > 1/2)$ (Sohi [11]),

8. $R_{0,1}(b, 1) = R(b)$ (Halim [1]),

We further, observe that for special choice of the parameters b, β, n and p our class $R_{n,p}(b, \beta)$ give rise the following new subclasses of p-valent analytic functions,

1. $$R_{n,p}\left(\left(1 - \frac{\alpha}{p}\right)\cos\lambda e^{-i\lambda}, 1\right) = R^{\lambda}_{n,p}(\alpha)$$
$$= \left\{f(z) \in S(p) : \quad Re\ e^{i\lambda}\frac{(D^{n+p-1}f(z))'}{z^{p-1}} > \alpha\cos\lambda, |\lambda| < \frac{\pi}{2}, 0 \le \alpha < p,\right.$$
$$\left. p \in N, z \in U\right\}$$

2. $$R_{n,p}(b, 1) = R_{n,p}(b)$$
$$= \left\{f(z) \in S(p) : \quad Re\left\{p + \frac{1}{b}\frac{(D^{n+p-1}f(z))'}{z^{p-1}} - p)\right\} > 0,\right.$$
$$\left. b \ne 0, \quad \text{complex}, \quad n > -p, p \in N, z \in U\right\}$$

3. $$R_{n,p}\left(\left(1, \frac{\alpha}{p}\right)\cos\lambda e^{-i\lambda}, \frac{2M-1}{2M}\right) = R^{*\lambda}_{n,p,M}(\alpha)$$
$$= \left\{f(z) \in S(p) : \left|\frac{e^{i\lambda}\frac{(D^{n+p-1}f(z))'}{z^{p-1}} - \alpha\cos\lambda - ip\sin\lambda}{(p-\alpha)\cos\lambda} - M\right| < M,\right.$$
$$\left. |\lambda| < \frac{\pi}{2}, 0 \le \alpha < p, n > -p, p \in N, M > \frac{1}{2}, z \in U\right\}$$

4. $$R_{n,p}\left(b, \frac{2M-1}{2M}\right) = R_{n,p,M}(b)$$
$$= \left\{f(z) \in S(p) : \left|\frac{\frac{(D^{n+p-1}f(z))'}{pz^{p-1}} - 1 + b}{b} - M\right| < M,\right.$$
$$\left. b \ne 0, \quad \text{complex}, \ n > -p, p \in N, M > \frac{1}{2}, z \in U\right\},$$

5. $$R_{n,p}\left(\cos\lambda e^{-i\lambda},\frac{2-\cos\lambda}{2}\right)=R^{*\lambda}_{n,p}$$
$$=\{f(z)\in S(p):\left|e^{i\lambda}\frac{(D^{n+p-1}f(z))'}{pz^{p-1}}-(1+i\sin\lambda)\right|<1,$$
$$|\lambda|<\frac{\pi}{2},n>-p,p\in N,z\in U\}$$

6. $$R_{n,p}\left(\left(1-\frac{\alpha}{p}\right)\cos\lambda e^{-i\lambda},1-\rho\right)=R^{*\lambda}_{n,p}(\rho)$$
$$=\{f(z)\in S(p):\left|\frac{e^{i\lambda}\frac{(D^{n+p-1}f(z))'}{z^{p-1}}-\alpha\cos\lambda-ip\sin\lambda}{(p-\alpha)\cos\lambda}-\frac{1}{2\rho}\right|<\frac{1}{2\rho},$$
$$|\lambda|<\frac{\pi}{2},0\le\alpha<p,n>-p,p\in N,0\le\rho<1,z\in U\}$$

7. $$R_{n,p}\left(\left(1-\frac{\alpha}{p}\right)\sigma\cos\lambda e^{-i\lambda},\frac{1}{2}\right)=[R^{\lambda}_{n,p}(\alpha)]^{\sigma}$$
$$=\{f(z)\in S(p):\left|\frac{e^{i\lambda}\frac{(D^{n+p-1}f(z))'}{z^{p-1}}-\alpha\cos\lambda-ip\sin\lambda}{(p-\alpha)\cos\lambda}-1\right|<\sigma,$$
$$|\lambda|<\frac{\pi}{2},0\le\alpha<p,n>-p,p\in N,0<\sigma\le1,z\in U\}$$

8. $$R_{n,p}\left(\frac{2\beta}{1+\beta},\frac{1+\beta}{2}\right)=R_{n,p}(\beta)$$
$$=\{f(z)\in S(p):\left|\frac{\frac{(D^{n+p-1}f(z))'}{z^{p-1}}-p}{\frac{(D^{n+p-1}f(z))'}{z^{p-1}}+p}\right|<\beta,0<\beta\le1,$$
$$n>-p,p\in N,z\in U\},$$

9. $$R_{n,p}\left(\frac{2\beta\left(1-\frac{\alpha}{p}\right)}{1+\beta},\frac{1+\beta}{2}\right)=R_{n,p}(\alpha,\beta)$$
$$=\{f(z)\in S(p):\left|\frac{\frac{(D^{n+p-1}f(z))'}{z^{p-1}}-p}{\frac{(D^{n+p-1}f(z))'}{z^{p-1}}+p-2\alpha}\right|<\beta,0\le\alpha<p,$$
$$0<\beta\le1,n>-p,p\in N,z\in U\},$$

10. $$R_{n,p}\left(\frac{2\beta\left(1-\frac{\alpha}{p}\right)\cos e^{-i\lambda}}{1+\beta},\frac{1+\beta}{2}\right)=R^{\lambda}_{n,p}(\alpha,\beta)$$
$$=\{f(z)\in S(p):\left|\frac{\frac{(D^{n+p-1}f(z))'}{z^{p-1}}-p}{\frac{(D^{n+p-1}f(z))'}{z^{p-1}}-p+2(p-\alpha)\cos\lambda e^{-i\lambda}}\right|$$
$$<\beta,|\lambda|<\frac{\pi}{2},0\leq\alpha<p,0<\beta\leq1,n>-p,p\in N,z\in U\},$$

11. $$R_{n,p}\left(\left(1-\frac{\alpha}{p}\right)\cos\lambda e^{-i\lambda},\beta\right)=R^{\lambda}_{n,p,\beta}(\alpha)$$
$$=\{f(z)\in S(p):\left|\frac{\frac{(D^{n+p-1}f(z))'}{z^{p-1}}-p}{2\beta\left(\frac{(D^{n+p-1}f(z))'}{z^{p-1}}-p+(p-\alpha)\cos\lambda e^{-i\lambda}\right)-\left(\frac{(D^{n+p-1}f(z))'}{z^{p-1}}-p\right)}\right|$$
$$1.<1,|\lambda|<\frac{\pi}{2},0\leq\alpha<p,0<\beta\leq1,n>-p,p\in N,z\in U\},$$

12. $$R_{n,p}\left(1-a-d,2\frac{1-a+d}{2d}\right)=R_{n,p}(a,d)$$
$$=\{f(z)\in S(p):\left|\frac{(D^{n+p-1}f(z))'}{p^{z^{p-1}}}-a\right|<d,a+d\geq1,d\leq a\leq d+1,$$
$$n>-p,p\in N,z\in U\},$$

13. $$R_{n,p}(1-a-d)\left(1-\frac{\alpha}{p}\right)\cos\lambda e^{-i\lambda},\frac{1-a+d}{2d}\Big)=R^{\lambda}_{n,p}(a,d,\alpha)$$
$$=\{f(z)\in S(p):\left|\frac{e^{i\lambda}\frac{(D^{n+p-1}f(z))'}{z^{p-1}}-\alpha\cos\lambda-ip\sin\lambda}{(p-\alpha)\cos\lambda}-a\right|<d,$$
$$a+d\geq1,d\leq a\leq d+1,|\lambda|<\frac{\pi}{2},0\leq\alpha<p,n>-p,p\in N,z\in U\},$$

14. $$R_{n,p}\left((1-m-M)\left(1-\frac{\alpha}{p}\right)\cos\lambda e^{-i\lambda},\frac{1-m+M}{2M}\right)\Big)=R^{\lambda}_{n,p}(m,M,\alpha)$$
$$=\{f(z)\in S(p):\left|\frac{e^{i\lambda}\frac{(D^{n+p-1}f(z))'}{z^{p-1}}-\alpha\cos\lambda-ip\sin\lambda}{(p-\alpha)\cos\lambda}-m\right|<M,$$
$$|\lambda|<\frac{\pi}{2},0\leq\alpha<p,|m-1|<M\leq m,m>\frac{1}{2},z\in U\}$$

As noticed above the class $R_{n,p}(b,\beta)$ includes the various subclasses of p-valent analytic functions, a study of its properties will lead to a unified study of these classes. In the present paper, we determine a sufficient condition, coefficient esstimates, maximization of $|a_{p+2}-\mu a_{p+1}^2|$ over the class $R_{n,p}(b,\beta)$, and distortion theorem for $f(z)\in R_{n,p}(b,\beta)$. We shall further prove that the subclass $R_{n,p,\delta}(b)$ of $S(p)$ is closed under convalution.

2. A SUFFICIENT CONDITION

Theorem 1

Let the function $f(z)$ defined by (1.1) be analytic in U. Then $f(z)\in R_{n,p}(b,\beta)$ if, for some b, n and $p(b\neq 0$, complex, $n>-p, p\in N)$.

$$\sum_{k=1}^{\alpha}\frac{(p+k)}{p}\delta(n,k)|a_{p+k}|\leq\frac{\beta|b|}{(1-\beta)} \tag{2.1}$$

whenever $0<\beta\leq\frac{1}{2}$, and

$$\sum_{k=1}^{\alpha}\frac{(p+k)}{p}\delta(n,k)|a_{p+k}|\leq b \tag{2.2}$$

Whenever $\frac{1}{2}\leq\beta\leq 1$

Proof

Suppose that (2.1) holds for $0<\beta\leq\frac{1}{2}$ and that $f(z)=z^p+\sum_{k=1}^{\infty}a_{p+k}z^{p+k}$, then for $z\in U$.

$$|(D^{n+p-1}f(z))'-Pz^{p-1}|-|2\beta[(D^{n+p-1}f(z))'-pz^{p-1}+pbz^{p-1}]-$$
$$[(D^{n+p-1}f(z))'-pz^{p-1}]|$$

$$=|\sum_{k=1}^{\infty}(p+k)\delta(n,k)a_{p+k}z^{p+k-1}|-|2\beta pbz^{p-1}-\sum_{k=1}^{\infty}(1-2\beta)(p+k)\delta(n,k)a_{p+k}.z^{p+k-1}|$$

$$\leq|\sum_{k=1}^{\infty}(p+k)\delta(n,k)|a_{p+k}|r^{p+k-1}-2\beta p|b|r^{p-1}+\sum_{k=1}^{\infty}(1-2\beta)(p+k)\delta(n,k)|a_{p+k}|r^{p+k-1}$$

$$< 2\left\{\sum_{k=1}^{\infty}(1-\beta)(p+k)\delta(n,k)|a_{p+k}| - \beta p|b|\right\} r^{p-1}$$

The last inequality is nonpositive by (2.1), so that $f(z) \in R_{n,p}(b,\beta)$.

For the second part, we assume that (2.2) holds true for $\frac{1}{2} \leq \beta \leq 1$. In this case, we observe that

$$|(D^{n+p-1}f(z))' - pz^{p-1}| - |2\beta[(D^{n+p-1}f(z))' - pz^{p-1} + pbz^{p-1}]$$
$$-[(D^{n+p-1}f(z))' - pz^{p-1}]|$$

$$= |\sum_{k=1}^{\infty}(p+k)\delta(n,k)a_{p+k}z^{p+k-1}| - |2\beta pbz^{p-1} + \sum_{k=1}^{\infty}(1-2\beta)(p+k)\delta(n,k)a_{p+k}z^{p+k-1}|$$
$$< 2\beta\left\{\sum_{k=1}^{\infty}(p+k)\delta(n,k)|a_{p+k}| - p|b|\right\} r^{p-1}$$
$$\leq 0, \quad \text{by (2.2)}$$

This proves that $f(z) \in R_{n,p}(b,\beta)$. Hence the theorem.

We note that

$$D^{n+p-1}f(z) = z^p - \frac{\beta b}{\frac{(p+k)}{p}\delta(n,k)(1-\beta)} z^{p+k}$$

is an external function with respect to the first part of the theorem and

$$D^{n+p-1}f(z) = z^p - \frac{b}{\frac{(p+k)}{p}\delta(n,k)} z^{p+k}$$

is an external function with respect to the second part of the theorem since

$$\left|\frac{\frac{(D^{n+p-1}f(z))'}{z^{p-1}} - 1}{2\beta\left[\frac{(D^{n+p-1}f(z))'}{pz^{p-1}} - 1\right] - \left[\frac{(D^{n+p-1}f(z))'}{pz^{p-1}} - 1\right]}\right| = 1$$

for $z = 1$, $b \neq 0$, complex, $0 < \beta \leq 1$, $n > -p, p \in N$, and $k = 1, 2, \cdots$.

We also observe that the converse of the above theorem may not be true. For example, consider the function $f(z)$ given by

$$\frac{(D^{n+p-1}f(z))'}{z^{p-1}} = \frac{1-(2\beta-1-2\beta b)z}{1-(2\beta-1)z}$$

It is easily seen that $f(z) \in R_{n,p}(b,\beta)$ but

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{\frac{(p+k)}{p}\delta(n,k)(1-\beta)}{\beta|b|}|a_{p+k}| \\ = &\sum_{k=1}^{\infty} \frac{\frac{(p+k)}{p}\delta(n,k)(1-\beta)}{\beta|b|} \cdot \frac{2\beta|b|}{\frac{(p+k)}{p}\delta(n,k)}(2\beta-1)^{k-1} \\ = &\sum_{k=1}^{\infty} 2(1-\beta)(2\beta-1)^{k-1} > 1 \end{aligned}$$

for $b \neq 0$, complex, $0 < \beta \leq \frac{1}{2}, n > -p$ and $p \in N$, and also

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{\frac{(p+k)}{p}\delta(n,k)}{|b|}|a_{p+k}| \\ = &\sum_{k=1}^{\infty} \frac{\frac{(p+k)}{p}\delta(n,k)}{|b|} \cdot \frac{2\beta|b|}{\frac{(p+k)}{p}\delta(n,k)}(2\beta-1)^{k-1} \\ = &\sum_{k=1}^{\infty} 2\beta(2\beta-1)^{k-1} > 1 \end{aligned}$$

for $b \neq 0$, complex, $\frac{1}{2} \leq \beta \leq 1, n > -p, p \in N$, and $z \in U$.

Corollary 1

Let the function $f(z)$ defined by (1.1) be analytic in U. If for $b \neq 0$, complex,

$$\sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right) \delta(n,k)|a_{p+k}| \leq (2M-1)|b|, \quad \text{whenever } \frac{1}{2} < M \leq 1,$$

and

$$\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)\delta(n,k)|a_{p+k}|\leq|b|,\quad\text{whenever}\quad M\geq 1,$$

then $f(z)\in R_{n,p,M}(b)$.

Corollary 2

Let the function $f(z)$ defined by (1.1) be analytic in U. If for some $\alpha,\lambda(0\leq\alpha<p,|\lambda|<\frac{\pi}{2})$,

$$\sum_{k=1}^{\infty}(p+k)\delta(n,k)|a_{p+k}|\leq(2M-1)(p-\alpha)\cos\lambda,\quad\text{whenever}\quad\frac{1}{2}<M\leq 1,$$

and

$$\sum_{k=1}^{\infty}(p+k)\delta(n,k)|a_{p+k}|\leq(p-\alpha)\cos\lambda,\quad\text{whenever}\quad M\geq 1,$$

then $f(z)\in R^{*\lambda}_{n,p,M}(\alpha)$.

Corollary 3

Let the function $f(z)$ defined by (1.1) be analytic in U. If for some $\alpha,\beta,\lambda(0\leq\alpha<p,0<\beta\leq 1,|\lambda|<\frac{\pi}{2})$,

$$\sum_{k=1}^{\infty}(p+k)\delta(n,k)|a_{p+k}|\leq\frac{2\beta(p-\alpha)\cos\lambda}{1+\beta}$$

then $f(z)$ belongs to $R^{\lambda}_{n,p}(\alpha,\beta)$.

Corollary 4

Let the function $f(z)$ defined by (1.1) be analytic in U. If for some $\alpha, \beta, \lambda (0 \leq \alpha < p, 0 < \beta \leq 1, |\lambda| < \frac{\pi}{2})$.

$$\sum_{k=1}^{\infty}(p+k)\delta(n,k)|a_{p+k}| \leq \frac{\beta(p-\alpha)\cos\lambda}{1-\beta}, \quad \text{whenever} \quad 0 < \beta \leq \frac{1}{2},$$

and

$$\sum_{k=1}^{\infty}(p+k)\delta(n,k)|a_{p+k}| \leq (p-\alpha)\cos\lambda, \quad \text{whenever} \quad \frac{1}{2} \leq \beta \leq 1,$$

then $f(z)$ belongs to $R^{\lambda}_{n,p,\beta}(\alpha)$.

Remark 1

1. Putting $n = 1-p, p \in N$ and $b = (1-\alpha)\cos\lambda e^{-i\lambda}, |\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1$, we get the corresponding sufficient condition obtained by Mogra [8].

2. Putting $n = 0,\ p = 1$ and $b = (1-\alpha)\cos\lambda e^{-i\lambda}, |\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1$, in Theorem 1, we get the corresponding sufficient condition by Ahuja [2].

3. COEFFICIENT ESTIMATES

Theorem 2

Let the function $f(z)$ defined (1.1) be in the class $R_{n,p}(b,\beta)(b \neq 0$, complex, $0 < \beta \leq 1, n > -p$ and $p \in N)$, then

$$|a_{p+k}| \leq \frac{2\beta p|b|}{(p+k)\delta(n,k)}(k = 1, 2, \cdots) \tag{3.1}$$

The result is sharp.

Proof

Since $f(z) \in R_{n,p}(b,\beta)$, we have

$$\frac{(D^{n+p-1}f(z))'}{pz^{p-1}} = \frac{1+[(2\beta-1)-2\beta b]w(z)}{1+(2\beta-1)w(z)} \tag{3.2}$$

where $w(z)=\sum_{m=1}^{\infty} t_m z^m \in \Omega$. From (3.2), we have

$$\left\{2\beta bpz^{p-1}+(2\beta-1)\sum_{m=1}^{\infty}(p+m)\delta(n,m)a_{p+m}z^{p+m-1}\right\}\left\{\sum_{m=1}^{\infty} b_m z^m\right\}$$
$$=-\sum_{m=1}^{k}(p+m)\delta(n,m)a_{p+m}z^{p+m-1} \tag{3.3}$$

Equating corresponding coefficients on both sides of (3.3) we observe that the coefficient a_{p+k} on the right of (3.3) depends only on $a_{p+1}, a_{p+2}, \cdots, a_{p+k-1}$ on the left of (3.3) for $k \geq 1$. Hence for $k \geq 1$, it follows from (3.3) that

$$\left\{2\beta pbz^{p-1}+(2\beta-1)\sum_{m=1}^{k-1}(p+m)\delta(n,m)a_{p+m}z^{p+m-1}\right\}w(z)$$
$$=-\sum_{m=1}^{\infty}(p+m)\delta(n,m)a_{p+m}z^{p+m-1}-\sum_{m=k+1}^{\infty}C_m z^{p+m-1}$$

where $C_m's$ being complex numbers. Then, since $|w(z)|<1$, we get

$$\left|2\beta pbz^{p-1}+(2\beta-1)\sum_{m=1}^{k=1}(p+m)\delta(n,m)a_{p+m}z^{p+m-1}\right|$$
$$\geq\left|\sum_{m=1}^{k}(p+m)\delta(n,m)a_{p+m}z^{p+m-1}+\sum_{m=k+1}^{\infty}C_m z^{p+m-1}\right| \tag{3.4}$$

Squaring both sides of (3.4) and integrating round $|z|=r, 0<r<1$, we obtain

$$\sum_{m=1}^{k}(p+m)^2(\delta(n,m))^2|a_{p+m}|^2r^{2(p+m-1)}+\sum_{m=k+1}^{\infty}|C_m|^2r^{2(p+m-1)}$$
$$\leq 4\beta^2p^2|b|^2r^{2(p-1)}+(2\beta-1)^2+\sum_{m=1}^{k-1}(p+m)^2(\delta(n,m))^2|a_{p+m}|^2r^{2(p+m-1)}$$

If we take limit as r approaches 1, then

$$\sum_{m=1}^{k}(p+m)^2(\delta(n,m))^2|a_{p+m}|^2 \leq 4\beta^2p^2|b|^2+(2\beta-1)^2\sum_{m=1}^{k}(p+m)^2(\delta(n,m))^2|a_{p+m}|^2$$

or

$$(p+k)^2(\delta(n,k))^2|a_{p+k}|^2 \le 4\beta^2p^2|b|^2 - 4\beta(1-\beta)\sum_{m=1}^{k-1}(p+m)^2(\delta(n,m))^2|a_{p+m}|^2$$

Since $0 < \beta \le 1$, we have

$$(p+k)^2(\delta(n,k))^2|a_{p+k}|^2 \le 4\beta^2p^2|b|^2$$

Whence follows that

$$|a_{p+k}| \le \frac{2\beta p|b|}{(p+k)\delta(n,k)}, k \ge 1.$$

Consider the function

$$D^{n+p-1}f(z) = \int_0^z pt^{p-1}\frac{1-[(2\beta-1)-2\beta b]t^k}{1-(2\beta-1)t^k}dt,\ z \in U,$$

where $b \neq 0$, complex, $0 < \beta \le 1, n > -p, p \in N$ and $k \ge 1$. Then it is easy to check that $f(z) \in R_{n,p}(b,\beta)$ and the function $D^{n+p-1}f(z)$ has the expansion

$$D^{n+p-1}f(z) = z^p + \frac{2\beta pb}{(p+k)}z^{p+k} + \cdots$$

for all $n > -p, p, p \in N, k \ge 1$ and $z \in U$ showing that the estimates are sharp.

Remark 2

Taking appropriate values of b and β in Theorem 2, we may get the corresponding coefficient estimates for functions in the classes $R_{n,p,m}(b)$, $R^{*\lambda}_{n,p,m}(\alpha)$, $R^{\lambda}_{n,p}(\alpha,\beta)$ and $R^{\lambda}_{n,p,\beta}(\alpha)$.

4. MAXIMIZATION OF $|a_3 - \mu a_2^2|$

We shall require the following lemma in our investigation:

Lemma 1

Let $w(z)=\sum_{k=1}^{\infty} c_k z^k \in \Omega$. Then

$$|c_2-\mu c_1^2| \leq \max\{1,|\mu|\}, \tag{4.1}$$

for any complex number μ. Equality in (4.1) may be attained with the functions $w(z)=z^2$ and $w(z)=z$ for $|\mu|<1$ and $|\mu|\geq 1$, respectively.

Theorem 3

Let the function $f(z)$ defined by (1.1) be in the class $R_{n,p}(b,\beta)$, then for complex number μ, we obtain

$$|a_{p+2}-\mu a_{p+1}^2| \leq \frac{2\beta p|b|}{(p+2)\delta(n,2)} \max\{1,|d|\}, \tag{4.2}$$

where

$$d=\frac{-(1-2\beta)(p+1)^2(\delta(n,1))^2+2\mu\beta pb(p+2)\delta(n,2)}{(p+1)^2(\delta(n,1))^2} \tag{4.3}$$

The result is sharp

Proof

Since $f(z)\in R_{n,p}(b,\beta)$, we have

$$\frac{(D^{n+p-1}f(z))'}{pz^{p-1}}=\frac{1+(2\beta-1-2\beta bw(z)}{1+(2\beta-1)w(z)}, \quad w(z)\in\Omega \tag{4.4}$$

From (4.4), we have

$$\begin{aligned} w(z) &= -\frac{\frac{(D^{n+p-1}f(z))'}{pz^{p-1}}-1}{2\beta b-(1-2\beta)\left(\left(\frac{(D^{n+p-1}f(z))'}{pz^{p-1}}\right)-1\right)} \\ &= -\frac{\sum_{k=1}^{\infty}(p+k)\delta(n,k)a_{p+k}z^k}{2\beta pb-(1-2\beta)\sum_{k=1}^{\infty}(p+k)\delta(n,k)a_{p+k}z^k} \\ &= -\frac{1}{2\beta pb}(p+1)\delta(n,1)a_{p+1}z+(p+2)\delta(n,k)a_{p+2}z^2 \\ &\quad +\frac{(1-2\beta)}{2\beta pb}(p+1)^2(\delta(n,1))^2a_{p+1}^2z^2+\cdots] \end{aligned} \tag{4.5}$$

and then comparing the coefficients of z and z^2 on both sides of (4.5), we have

$$c_1 = \frac{\delta(n,1)(p+1)}{2\beta pb} a_{p+1}$$

and

$$c_2 = -\frac{1}{2\beta pb}\left[(p+2)\delta(n,2)a_{p+2} + \frac{(1-2\beta)}{2\beta pb}(p+1)^2(\delta(n,1))^2 a_{p+1}^2\right]$$

Thus

$$a_{p+1} = \frac{-2\beta pb}{(p+1)\delta(n,1)} c_1$$

and

$$a_{p+2} = -\frac{2\beta pb}{(p+1)\delta(n,2)} c_2 - \frac{(1-2\beta)2\beta pb}{(p+2)\delta(n,2)} c_1^2.$$

Hence

$$a_{p+2} - \mu a_{p+1}^2 = \frac{-2\beta pb}{(p+2)\delta(n,2)} |c_2 - dc_1^2|,$$

where

$$d = -\frac{(1-2\beta)(p+1)^2(\delta(n,1))^2 + 2\mu\beta pb(p+2)\delta(n,2)}{(p+1)^2(\delta(n,1))^2}$$

Therefore

$$|a_{p+2} - \mu a_{p+1}^2| = \frac{2\beta p|b|}{(p+2)\delta(n,2)} |c_2 - dc_1^2| \tag{4.6}$$

Applying Lemma 1 in (4.6), we get

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{2\beta p|b|}{(p+2)\delta(n,2)} \quad \max\{1, |d|\},$$

which is (4.2). Since Lemma 1 is sharp, so that the equation (4.2) must also be sharp.

5. DISTORTION THEOREM

Theorem 4

If a function $f(z)$ defined by (1.1) is in the class $R_{n,p}(b,\beta)$, $\beta \neq \frac{1}{2}$, then for $|z| = r < 1$,

$$R_e\left\{\frac{(D^{n+p-1}f(z))'}{pz^{p-1}}\right\} \geq \frac{1-2\beta|b|r+(2\beta-1)[2\beta Re\{b\}-(2\beta-1)]r^2}{1-(2\beta-1)^2r^2} \tag{5.1}$$

and

$$R_e\left\{\frac{(D^{n+p-1}f(z))'}{pz^{p-1}}\right\} \leq \frac{1+2\beta|b|r+(2\beta-1)[2\beta Re\{b\}-(2\beta-1)]r^2}{1-(2\beta-1)^2r^2} \tag{5.2}$$

For $\beta = \frac{1}{2}$, the above estimates reduce to

$$Re\left\{\frac{(D^{n+p-1}f(z))'}{pz^{p-1}}\right\} \geq 1-|b|r \tag{5.3}$$

and

$$Re\left\{\frac{(D^{n+p-1}f(z))'}{pz^{p-1}}\right\} \leq 1+|b|r \tag{5.4}$$

The bounds are sharp.

Proof

Since $f(z) \in R_{n,p}(b,\beta)$, we observe that the condition (1.9) coupled with an application of Schwarz's Lemma [9] implies

$$\left|\frac{(D^{n+p-1}f(z))'}{pz^{p-1}} - \eta\right| < R \tag{5.5}$$

where

$$\eta = \frac{1-(2\beta-1)(2\beta-1-2\beta b)r^2}{1-(2\beta-1)^2r^2} \tag{5.6}$$

and

$$R = \frac{2\beta|b|r}{1-(2\beta-1)^2r^2}, |z| = r \tag{5.7}$$

Hence we have

$$R_e\left\{\frac{(D^{n+p-1}f(z))'}{pz^{p-1}}\right\} \geq \frac{1-2\beta|b|r+(2\beta-1)[2\beta Re\{b\}-(2\beta-1)]r^2}{1-(2\beta-1)^2r^2}$$

and

$$R_e\left\{\frac{(D^{n+p-1}f(z))'}{pz^{p-1}}\right\} \leq \frac{1+2\beta|b|r+(2\beta-1)[2\beta Re\{b\}-(2\beta-1)]r^2}{1-(2\beta-1)^2r^2}$$

By considering the function $f(z)$ defined by

$$\frac{(D^{n+p-1}f(z))'}{pz^{p-1}} = \frac{2\beta-1-2\beta b}{2\beta-1} + \frac{2\beta b}{(2\beta-1)(1+(2\beta-1)ze^{ir})}, \quad \beta \neq \frac{1}{2} \tag{5.8}$$

where

$$e^{ir} = \frac{|b|-(2\beta-1)bz}{b-(2\beta-1)|b|z}, \quad \beta \neq \frac{1}{2} \tag{5.9}$$

we find that the bounds in (5.1) and (5.2) are sharp at $z = \pm r$, respectively.

Remark 3

The corresponding distortion theorems for functions belonging to the classes $R_{n,p,M}(b)$, $R^{*\lambda}_{n,p,M}(\alpha)$, $R^{\lambda}_{n,p}(\alpha,\beta)$ and $R^{\lambda}_{n,p,\beta}(\alpha)$ can be obtained from Theorem 4 by taking appropriate values of b and β.

Remark 4

Putting (i) $n = 0, p = 1$ and $\beta = 1$ and (ii) $\beta = \frac{2M-1}{2M}(M > \frac{1}{2})$ in Theorem 4, we get:

Corollary 5

If a function $f(z)$ defined by (1.1) is in the class $R(b)$, then for $|z| = r < 1$, we have

$$Re\{f'(z)\} \geq \frac{1 - 2|b|r + (2Re\{b\} - 1)r^2}{1 - r^2} \tag{5.10}$$

and

$$Re\{f'(z)\} \leq \frac{1 + 2|b|r + (2Re\{b\} - 1)r^2}{1 - r^2} \tag{5.11}$$

By considering the function $f(z)$ defined by

$$f(z) = (1 - 2b)z + \frac{2b}{e^{i\gamma}} \log(1 + ze^{i\gamma}) \tag{5.12}$$

where

$$e^{ir} = \frac{|b| - bz}{b - |b|z}, \tag{5.13}$$

we find that the bounds in (5.10) and (5.11) are sharp at $z = \pm r$, respectively.

Corollary 6

If a function $f(z)$ defined by (1.1) is in the class $R_{n,p,M}(b)$, then for $|z| = r < 1$, we have

$$R_e\left\{\frac{(D^{n+p-1}f(z))'}{pz^{p-1}}\right\} \geq \frac{1 - (1+m)|b|r + m[(1+m)Re\{b\} - m]r^2}{1 - m^2r^2} \tag{5.14}$$

and

$$R_e\left\{\frac{(D^{n+p-1}f(z))'}{pz^{p-1}}\right\} \leq \frac{1 + (1+m)|b|r + m[(1+m)Re\{b\} - m]r^2}{1 - m^2r^2} \tag{5.15}$$

where $m = 1 - \frac{1}{M} (M > \frac{1}{2})$. By considering the function $f(z)$ defined by

$$\frac{(D^{n+p-1}f(z))'}{pz^{p-1}} = \frac{m - (1+m)b}{m} + \frac{(1+m)b}{m(1 + mze^{i\gamma})'} \tag{5.16}$$

where

$$e^{i\gamma} = \frac{|b| = mz}{b - m|b|z}, \tag{5.17}$$

we find that the bounds in (5.14) and (5.15) are sharp at $z = \pm r$, respectively.

6. CONVEX SET OF FUNCTIONS

Theorem 5

If $f(z)$ and $g(z)$ belong to the class $R_{n,p,M}(b)$, then $vf(z) + (1-v)g(z)(0 \leq v \leq 1)$, belongs to the class $R_{n,p,M}(b)$.

Proof

Since $f(z)$ and $g(z)$ belong to the class $R_{n,p,M}(b)$, we have

$$\left| \frac{\frac{(D^{n+p-1}f(z))'}{pz^{p-1}} - 1 + b}{Mb} - 1 \right| < 1 \tag{6.1}$$

and

$$\left| \frac{\frac{(D^{n+p-1}g(z))'}{pz^{p-1}} - 1 + b}{Mb} - 1 \right| < 1 \tag{6.2}$$

for some M, b satisfying $M > \frac{1}{2}$ and $b \neq 0$, complex. Using (6.1) and (6.2), it follows that

$$\left| \frac{\frac{v(D^{n+p-1}f(z))' + (1-v)(D^{n+p-1}g(z))'}{pz^{p-1}} - 1 + b}{Mb} - 1 \right|$$
$$\leq \gamma \left| \frac{\frac{(D^{n+p-1}f(z))'}{pz^{p-1}} - 1 + b}{Mb} - 1 \right| + (1-v) \left| \frac{\frac{(D^{n+p-1}g(z))'}{pz^{p-1}} - 1 + b}{Mb} - 1 \right|$$
$$< v + (1-v) = 1$$

for all $z \in U$. This proves that $vf(z) + (1-v)g(z)$ belongs to $R_{n,p,M}(b)$.

7. CONVOLUTION OF FUNCTIONS

Theorem 6

If

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$$

and

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k}$$

belong to $R_{n,p,M}(b)$, then

$$F(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right) a_{p+k} b_{p+k} z^{p+k}$$

is also a member of $R_{n,p,M}(b), 0 < |b| < 1$.

Proof

Since $f(z)$ and $g(z)$ belong to $R_{n,p,M}(b)$, we have

$$\left| \frac{\frac{(D^{n+p-1}f(z))'}{pz^{p-1}} - 1 + b}{b} - M \right| < M, \ M > \frac{1}{2} f \in U$$

and

$$\left| \frac{\frac{(D^{n+p-1}g(z))'}{pz^{p-1}} - 1 + b}{b} - M \right| < M, \ M > \frac{1}{2} z \in U$$

It is well known [9] that if $h(z) = \sum_{n=0}^{\infty} C_n z^n$ is regular in U and $|h(z)| \leq D$, then

$$\sum_{n=0}^{\infty} |C_n|^2 \leq D^2 \tag{7.1}$$

Applying the estimate (7.1) to the function.

$$\left\{ \frac{\frac{(D^{n+p-1}g(z))'}{pz^{p-1}} - 1 + b}{b} - M \right\}, \quad \text{we get}$$

$$\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2(\delta(n,k))^2|a_{p+k}|^2 \le (2M-1)|b|,\ M>\frac{1}{2}$$

Similarly, applying the estimate (7.1) to the function

$$\left\{\frac{\frac{(D^{n+p-1}g(z))'}{pz^{p-1}-1+b}}{b}-M\right\},\quad \text{we obtain}$$

$$\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2(\delta(n,k))^2|b_{p+k}|^2 \le (2M-1)|b|,\ (M>\frac{1}{2})$$

Since

$$F(z)=z^p+\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2 a_{p+k}b_{p+k}z^{p+k}$$

we have

$$\left|\frac{\frac{(D^{n+p-1}F(z))'}{p^{z^{p-1}}-1+b}}{b}-M\right|^2$$

$$= \left|(1-M)+\frac{1}{b}\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2(\delta(n,k))^2 a_{p+k}b_{p+k}z^{p+k}\right|^2$$

$$\le (1-M)^2+\frac{2(1-M)}{|b|}\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2(\delta(n,k))^2|a_{p+k}|\ |b_{p+k}|r^k +$$

$$\frac{1}{|b|^2}\left(\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2(\delta(n,k))^2|a_{p+k}|\ |b_{p+k}|r^k\right)^2 \quad (|z|=r)$$

$$\le (1-M)^2+\frac{2(1-M)}{|b|}\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2(\delta(n,k))^2|a_{p+k}||b_{p+k}| +$$

$$\frac{1}{|b|^2}\left(\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2(\delta(n,k))^2|a_{p+k}||b_{p+k}|\right)^2$$

$$\le (1-M)^2+\frac{2(1-M)}{|b|}\left[\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2(\delta(n,k))^2|a_{p+k}|^2\right]^{\frac{1}{2}}$$

$$\left[\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2(\delta(n,k))^2|b_{p+k}|^2\right]^{1/2}+\frac{1}{|b|^2}\left[\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2(\delta(n,k))^2|a_{p+k}|^2\right]$$
$$\left[\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2(\delta(n,k))^2|b_{p+k}|^2\right]$$
$$\leq (1-M)^2+2(1-M)(2M-1)|b|+(2M-1)^2|b|^2$$

Consequently

$$\left|\frac{\frac{(D^{n+p-1}F(z))'}{pz^{p-1}}-1+b}{b}-M\right|^2 < M^2$$

if

$$(1-M)^2+2(1-M)(2M-1)|b|+(2M-1)^2|b|^2 < M^2$$

that is, if

$$(2M-1)(|b|-1)[(2M-1)|b|+1] < 0$$

which is true for M and $b \neq 0$, complex satisfying $M > \frac{1}{2}$ and $0 < |b| < 1$. Hence

$$F(z) \in R_{n,p,M}(b)$$

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ON OPTICAL CONTROL FOR $n \times n$ ELLIPTIC SYSTEMS INVOLVING OPERATORS WITH AN INFINITE NUMBER OF VARIABLES

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Abstract

The necessary and sufficient conditions of optimality for the following $n \times n$ elliptic system with Dirichlet conditions are given

$$\begin{cases} \Im Y = MY + F & \text{in } \mathrm{R}^\infty \\ Y = 0 & \text{on} \Gamma, \end{cases} \quad (D)$$

where $\Im$ is an $n \times n$ diagonal matrix of the following second order self-adjoint operator with an infinite number of variables:

$$Ay(x) = -\sum_{k=1}^{\infty} \frac{1}{\sqrt{(p_k)(x_k)}} \frac{\partial^2}{\partial x_k^2} \sqrt{(p_k)(x_k)} y(x) + q(x)y(x) \tag{1}$$

M is a given n-square matrix coefficients and $F = (f_1, f_2, ..., f_n)$ is a given vector function. Also, the problem with Neumann conditions is elaborate.

1. INTRODUCTION

The necessary and sufficient conditions of optimality for systems governed by different type of partial differential operators defined on spaces with finite number of variables are discussed for example in $[7-11]$. The optimal control problem of systems governed by different type of operators defined on spaces with an infinite number of variables are initiated and proved by Gali et al in $[3,4]$. These systems consists of one equation. The optimal control of 2×2 elliptic systems defined on R^N are established for example in $[5,6,8,12,13]$. Here, we extend the discussion to $n \times n$ elliptic systems involving second order self-adjoint elliptic operator with an infinite number of variables. In section one, we introduce some function spaces defined on R^∞. Section two is devoted to study the optimal control for $n \times n$ elliptic system with Dirichlet conditions. We first prove the existence of solution for system (D); then we give the necessary and sufficient conditions for the control to be an optimal for the system. We also study the problem with Neumann conditions in section three. In all our considered problems the control of distributed type.

(I)Some function spaces defined on R^∞.

In this section, we give the definition of some function spaces of infinitely many variables $[1,2]$. For this purpose, we introduce the infinite product $R^\infty = R^1 \times R^1 \times ...$, with elements $(R^\infty \ni x = (x_n)_{n=1}^\infty, x_n \in R^1)$, and we denoted by $d_\rho(x)$ the product of measures $d_\rho(x) = p_1(x)dx_1 \times p_2(x)dx_2 \times ...$, where $(p_k(t))_{k=1}^\infty$ is a fixed weight such that

$$0 < p_k(t) \in C^\infty(R^1), \int_{R^1} p_k(t)dt = 1.$$

With respect to this measure and on R^∞ with sufficiently smooth boundary Γ, we construct the space $L^2(R^\infty, d_\rho(x))$ of function $u(x)$ which are measurable and such that

$$\|u\|_{L^2(R^\infty,\ d\rho(x))} = (\int_{R^\infty} |u|^2 d_\rho(x))^{\frac{1}{2}} < \infty. \tag{2}$$

We shall set

$$L^2(R^\infty, d\rho(x)) = L^2(R^\infty)$$

It is a classified results that $L^2(R^\infty)$ is a Hilbert space for the scaler product

$$(u,v)_{L^2(R^\infty)} = \int_{R^\infty} u(x)v(x)d\rho(x)$$

associated to the above norm (2). For function which are continuously differentiable up to the boundary Γ of R^∞ and which vanish in a neighborhood of ∞, we introduce the scalar product

$$(u, v) = \sum_{|\alpha|\leq 1} (D^\alpha(u), (D^\alpha(v)))_{L^2(R^\infty)} \tag{3}$$

we recall that $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ is a multi-index for differentiation,

$$|\alpha| = \sum_{i=1}^{N} \alpha_i, D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_N^{\alpha_N}},$$

and after the completion, we obtain the Sobolev space $W^1(R^\infty)$. In short,Sobolev space $W^1(R^\infty)$ is defined by:

$$W^1(R^\infty) = \{u|u, \frac{\partial u}{\partial x_i} \in L^2(R^\infty)\}.$$

This space form a Hilbert space endowed with the scalar product (3)(see[1, 2]). The space $W^1(R^\infty)$ form a positive space . We also construct the negative space $W^{-1}(R^\infty)$ with respect to the zero space $L^2(R^\infty)$ and then we have the following imbedding [1]:

$$W^1(R^\infty) \subseteq L^2(R^\infty) \subseteq W^{-1}(R^\infty), \tag{4}$$

$$\|u\|_{W^1(R^\infty)} \geq \|u\|_{L^2(R^\infty)} \geq \|u\|_{W^{-1}(R^\infty)}$$

Analogous to the above chain, we have a chain of the form

$$W_0^1(R^\infty) \subseteq L^2(R^\infty) \subseteq W_0^{-1}(R^\infty), \tag{5}$$

where $W_0^1(R^\infty)$ is the set of all function in $W^1(R^\infty)$, which vanish on the boundary of R^∞, i.e

$$W_0^1(R^\infty) = \{u|u \in W^1(R^\infty), \frac{\partial u}{\partial n}|_\Gamma = 0\},$$

where $\frac{\partial u}{\partial n}|_\Gamma$ is the derivative on Γ, oriented towards the exterior of R^∞, and $W_0^{-1}(R^\infty)$ its dual.
Then it is easy to construct the following Sobolev space $(W^1(R^\infty))^n$by cartesian product as follow:

$$(W^1(R^\infty))^n = \prod_{i=1}^{n} (W^1(R^\infty))^i,$$

with norm defined by ,

$$\|u\|_{(W^1(R^\infty))^n} = \sum_{i=1}^{n} \|u_i\|_{(W^1(R^\infty))}$$

so from (4), (5), we have the following two chains:-

$$(W^1(R^\infty))^n \subseteq (L^2(R^\infty))^n \subseteq (W^{-1}(R^\infty))^n, \tag{6}$$

$$(W_0^1(R^\infty))^n \subseteq (L^2(R^\infty))^n \subseteq (W_0^{-1}(R^\infty))^n \tag{7}$$

where $(W^{-1}(R^\infty))^n$ and $(W_0^{-1}(R^\infty))^n$, are denoted the dual of the spaces $(W^1(R^\infty))^n$ and $(W_0^1(R^\infty))^n$ respectively.

II-Distributed Control for $n \times n$ System Involving Operator with an Infinite Number of Variables with Dirichlet Conditions.

In this section, we consider the optimal control of system (D). Our model is defined on $(W_0^1(R^\infty))^n$ by $A(\Phi = \{\phi_1, \phi_2, ..., \phi_n\}) \rightarrow A\{\phi_1, \phi_2, ..., \phi_n\}$ $= (-\sum_{k=1}^{\infty} D_k^2\phi_1 + q(x)\phi_1 + \sum_{j=1}^{n} a_{1j}\phi_j, -\sum_{k=1}^{\infty} D_k^2\phi_2 + q(x)\phi_2 + \sum_{j=1}^{n} a_{2j}\phi_j,$ $..., -\sum_{k=1}^{\infty} D_k^2\phi_n + q(x)\phi_n + \sum_{j=1}^{n} a_{nj}\phi_j)$
that is

$$A\phi_i = -\sum_{k=1}^{\infty} D_k^2\phi_i + q(x)\phi_i + \sum_{j=1}^{n} a_{ij}\phi_j, i = 1, 2, ..., n \tag{8}$$

where

$$D_k^2\phi(x) = \frac{1}{\sqrt{p_k}(x_k)} \frac{\partial^2}{\partial x_k^2}(\sqrt{p_k(x_k)}\phi(x)), q(x) \geq \nu > 0.$$

System (D) can be written as:

$$\begin{cases} -\sum_{k=1}^{\infty} D_k^2(I) + q(x)y_i + \sum_{j=1}^{n} a_{ij}y_j = f_i & inR^\infty \\ y_i|_\Gamma = 0, & forall 1 \leq i \leq n, \end{cases}$$

$$a_{ij} = \begin{cases} 1 & if i \geq j \\ -1 & if i < j \end{cases}$$

Now we define on $(\omega_0^1(R^\infty))^n$ a continuous bilinear form

$$\Pi : (W_0^1(R^\infty))^n \times (W_0^1(R^\infty))^n \rightarrow R$$

by:

$$\Pi(Y,\Phi)=\sum_{i=1}^{n}\sum_{k=1}^{\infty}\int_{R^\infty}D_k y_i(x)D_k\phi_i(x)d\rho+\sum_{k=1}^{n}\int_{R^\infty}$$

$$q(x)y_i(x)\phi_i(x)d\rho+\sum_{i,j=1}^{n}\int_{R^\infty}a_{ij}y_j\phi_i d\rho \tag{9}$$

Then we have

Lemma 1

The bilinear form (9) is coercive on$(W_0^1(R^\infty))^n$, that is

$$\Pi(Y,Y)\geq\delta\|y\|^2_{(W_0^1(R^\infty))^n},\delta>0 \tag{10}$$

Proof

The bilinear form (9) can be written as
$\Pi(Y,\Phi)=\sum_{i=1}^{n}\sum_{k=1}^{\infty}\int_{R^\infty}D_k y_i(x)D_k\phi_i(x)d\rho+\sum_{i=1}^{n}\int_{R^\infty}q(x)y_i(x)\phi_i(x)d\rho$
$+\sum_{i=j=1}^{n}\int_{R^\infty}y_j(x)\phi_i(x)d\rho+\sum_{i>j}^{n}\int_{R^\infty}y_j(x)\phi_i(x)d\rho-\sum_{i<j}^{n}\int_{R^\infty}y_j(x)\phi_i(x)d\rho,$

then

$$\begin{aligned}\Pi(Y,Y)&=\sum_{k=1}^{\infty}\sum_{i=1}^{n}\int_{R^\infty}|D_k y_i(x)|^2d\rho+\sum_{i=1}^{n}\int_{R^\infty}q(x)|y_i(x)|^2d\rho+\sum_{i=1}^{n}\int_{R^\infty}|y_i|^2d\rho\\&\geq\sum_{k=1}^{\infty}\sum_{i=1}^{n}\|D_k y_i(x)\|^2_{L^2(R^\infty)}+\sum_{i=1}^{n}\|y_i(x)\|^2_{L^2(R^\infty)}+\nu\sum_{i=1}^{n}\|y_i(x)\|^2_{L^2(R^\infty)}\\&\geq\sum_{i=1}^{n}\|y_i(x)\|^2_{W_0^1(R^\infty)}+\nu\sum_{i=1}^{n}\|y_i(x)\|^2_{L^2(R^\infty)},0<\nu\leq1\\&\geq\delta\sum_{i=1}^{n}\|y_i(x)\|^2_{W_0^1(R^\infty)},\\&=\delta\|Y\|^2_{(W_0^1(R^\infty))^n}\end{aligned}$$

which prove the coerciveness condition. By Lax-Milgramm theorem we have,

Lemma 2

If (10) holds. Then for $F = (f_1, f_2, ..., f_n) \in (L^2(R^\infty))^n$, there exit a unique solution $Y = (y_1, y_2, ..., y_n) \in (W_0^1(R^\infty))^n$ of system (D).

Proof

Let $\Phi\{\phi_1, \phi_2, ..., \phi_n\} \to L(\Phi)$ be a continuous linear form defined on $(W_0^1(R^\infty))^n$ by

$$L(\Phi) = \sum_{i=1}^{n} \int_{R^\infty} f_i \Phi_i d\rho.$$

Since (10) holds, there exits a unique element $Y = (y_1, y_2, ..., y_n) \in (W_0^1(R^\infty))^n$ satisfying

$$\Pi(Y, \Phi) = L(\Phi) forall \Phi \in (W_0^1(R^\infty))^n$$

which equivalent to; there exists a unique solution $Y = (y_1, y_2, ..., y_n) \in (W_0^1(R^\infty))^n$ for system (D).

The Control Problem for system (D).

The space $(L^2(R^\infty))^n$ being the space of controls. For a control $u = (u_1, u_2, ..., u_n) \in (L^2(R^\infty))^n$ the state $Y = (y_1, y_2, ..., y_n) \in (W_0^1(R^\infty))^n$ of the system is given by the solution of,

$$\begin{cases} (-\sum_{k=1}^{\infty} D_k^2(I) + q(x))y_i(u) + \sum_{j=1}^{n} a_{ij} y_j(u) = f_i + u_i & in R^\infty \\ y_i(u)|_\Gamma = 0, & forall 1 \leq i \leq n. \end{cases} \tag{11}$$

The observation equation is given by :

$$Z(u) = \{z_1(u), z_2(u), ..., z_n(u)\} = Y(u) = \{y_1(u), y_2(u), ..., y_n(u)\}$$

i.e $z_i(u) = y_i(u)$. $forall \quad 1 \leq i \leq n.$

For given $Z_d = \{z_{1d}, z_{2d}, ..., z_{nd}\}, \in (L^2(R^\infty))^n$, the cost function is given by

$$J(v) = \sum_{i=1}^{n} \int_{R^\infty} (y_i(v) - z_{id})^2 d\rho + \sum_{i=1}^{n} (N_i v_i, v_i)_{L^2(R^\infty)} \tag{12}$$

where

$$N = \{N_1, N_2, ..., N_n\} \in L((L^2(R^\infty))^n, (L^2(R^\infty))^n)$$

is a diagonal matrix of Hermitian positive definite operators such that

$$(Nu, u)^n_{(L^2(R^\infty))} \geq \zeta \|u\|^2_{(L^2(R^\infty))})^n, \zeta > 0 \tag{13}$$

The control problem then is to find

$$u \in U_{ad} \quad suchthat \quad J(u) = \inf_{v \in U_{ad}} J(v) \tag{14}$$

where U_{ad} is a closed convex subset of $(L^2(R^\infty))^n$. Since the cost function (12) can be written as:

$$J(v) = a(v, v) - 2L(v) + \|Y(v) - Z_d\|^2_{(L^2(R^\infty))^n}$$

where $a(v, v)$ is a continuous coercive bilinear form and $L(v)$ is a continuous linear form on $(L^2(R^\infty))^n$. Then using the general theory of J.L.Lions [8] there exit a unique optimal control satisfies (14). Moreover we have the following theorem which gives the characterization of this optimal control.

Theorem 1

Assume that (10), (13) holds, the cost function is given by (12). A necessary and sufficient conditions for $u = (u_1, u_2, ..., u_n)$to be an optimal control is that that the following equations and inequalities are satisfied

$$\begin{cases} (-\sum_{k=1}^{\infty} D_k^2(I) + q(x))p_i(u) + \sum_{j=1}^{n} a_{ij}p_j(u) = y_i(u) - z_{id} & inR^\infty \\ p_i(u)|_\Gamma = 0, & forall 1 \leq i \leq n, \end{cases}$$

$$\sum_{i=1}^{n} \int_{R^\infty} (p_i(u) + N_i u_i)(v_i - u_i) d\rho \geq 0 \quad forallv = (v_1, v_2, ..., v_n) \in U_{ad}$$

together with (11), where a_{ji} is the transpose of a_{ij},$P(u) = (p_1(u), p_2(u), ..., p_n(u))$ is the adjoint state.

Proof

Since $J(v)$ is differential and U_{ad} is bounded then the optimal control $u = (u_1, u_2, ..., u_n) \in (L^2(R^\infty))^n$ is characterized by

$$\sum_{i=1}^{n} J_i'(u)(v_i - u_i) \geq 0, \forall v = (v_1, v_2, ..., v_n) \in U_{ad}$$

which is equivalent to:

$$\sum_{i=1}^{n}(y_i(u) - z_{id}, y_i(v) - y_i(u))_{L^2(R^\infty)} + \sum_{i=1}^{n}(N_i u_i, v_i - u_i)_{L^2(R^\infty)} \geq 0 \qquad (15)$$

since

$$(P, AY(u))_{(L^2(R^\infty))^n} = \sum_{i=1}^{n}(p_i, -\sum_{k=1}^{\infty} D_k^2 y_i + q(x) y_i + \sum_{j=1}^{n} a_{ij} y_j)_{L^2(R^\infty)},$$

then using Green's formula, we have

$$\begin{aligned}(P, AY(u))_{(L^2(R^\infty))^n} &= \sum_{i=1}^{n}(-\sum_{k=1}^{\infty} D_k^2 p_i(u) + q(x) p_i + \sum_{j=1}^{n} a_{ji} p_j, y_i)_{L^2(R^\infty)} \\ &= (A^* P(u), Y(u))_{(L^2(R^\infty))^n},\end{aligned}$$

where

$$A^*(P(u) = \{p_1(u), p_2(u), ..., p_n(u)\} = A^*\{p_1(u), p_2(u), ..., p_n(u)\} \rightarrow$$

$$(-\sum_{k=1}^{\infty} D_k^2 p_1(u) + q(x) p_1(u) + \sum_{j=1}^{n} a_{j1} p_j(u), -\sum_{k=1}^{\infty} D_k^2 p_2(u) + q(x) p_2(u) + \sum_{j=1}^{n} a_{j2} p_j(u),$$
$$..., -\sum_{k=1}^{\infty} D_k^2 p_n(u) + q(x) p_n(u) + \sum_{j=1}^{n} a_{jn} p_j(u))$$

Then the equation $A^* P(u) = Y(u) - Z_d$ can be written as:

$$-\sum_{k=1}^{\infty} D_k^2 p_i(u) + q(x) p_i(u) + \sum_{j=1}^{n} a_{ji} p_j(u) = y_i(u) - z_{id} \quad forall \quad 1 \leq i \leq n.$$

Now equation (15) is equivalent to:

$$\sum_{j=1}^{n}(-\sum_{k=1}^{\infty} D_k^2 p_i(u) + q(x)p_i(u) + \sum_{j=1}^{n} a_{ji}p_j(u), y_i(v) - y_i(u))_{L^2(R^\infty)} +$$
$$\sum_{i=1}^{n}(N_i u_i, v_i - u_i)_{L^2(R^\infty)} \geq 0.$$

From Green's formula and (11) we get

$$\sum_{i=1}^{n}(p_i(u) + N_i u_i, v_i - u_i)_{L^2(R^\infty)} \geq 0$$

So, $u = (u_1, u_2, ..., u_n) \in U_{ad}$ such that:

$$\sum_{i=1}^{n} \int_{R^\infty} (p_i(u) + N_i u_i, v_i - u_i) d\rho \geq 0 \quad forall \quad v = (v_1, v_2, ..., v_n) \in U_{ad}$$

which complete the proof.

Remark 1

When $n = 2$, the optimality system is given by:-

$$\begin{aligned}
-\sum_{k=1}^{\infty} D_k^2 y_1(u) + q(x)y_1(u) + y_1(u) - y_2(u) &= f_1 + u_1 \quad inR^\infty, \\
-\sum_{k=1}^{\infty} D_k^2 y_2(u) + q(x)y_2(u) + y_1(u) + y_2(u) &= f_2 + u_2 \quad inR^\infty \quad , \\
y_1(u)|_\Gamma = 0, \quad y_2(u)|_\Gamma &= 0, \\
-\sum_{k=1}^{\infty} D_k^2 p_1(u) + q(x)p_1(u) + p_1(u) + p_2(u) &= y_1(u) - z_{1d} \quad inR^\infty, \\
-\sum_{k=1}^{\infty} D_k^2 p_2(u) + q(x)p_2(u) - p_1(u) + p_2(u) &= y_2(u) - z_{2d} \quad inR^\infty, \\
p_1(u)|_\Gamma = 0, \quad p_2(u)|_\Gamma &= 0,
\end{aligned}$$

$$u = (u_1, u_2) \in U_{ad}, \int_{R^\infty} (p_1(u) + N_1 u_1)(v_1 - u_1) + (p_2(u) + N_2 u_2)(v_2 - u_2) d\rho \geq 0$$

$\forall v=(v_1,v_2)\in U_{ad}, where P(u)=(p_1(u),p_2(u))$ *is the adjoint state.*

(III) The System With Neumann Conditions.

We discuss here the following elliptic system with Neumann conditions

$$\begin{cases} (-\sum_{k=1}^{\infty} D_{k=1}^2(I)+q(x))y_i+\sum_{j=1}^{n} a_{ij}y_j=f_i \quad \text{in } \mathrm{R}^{\infty} \\ \frac{\partial}{\partial\nu_A}y_i|_\Gamma=h_i, \end{cases} \qquad (N)$$

where $Y=(y_1,y_2,...,y_n)\in(W^1(R^\infty))^n, F=(f_1,f_2,...,f_n)\in(L^2(R^\infty))^n$ and $h_i\in H^{\frac{-1}{2}}(\Gamma), forall 1\le i\le n.$

We prove the following lemma which gives the existence and uniqueness of solution for this system.

Lemma 3

For given $F=(f_1,f_2,...,f_n)\in(L^2(R^\infty))^n$ there exit a unique solution$Y=(y_1,y_2,...,y_n)\in(W^1(R^\infty))^n$ of the system (N).

Proof

Since the bilinear form (9) is also continuous and coercive on $(W^1(R^\infty))^n$, then there exit a unique element $Y=(y_1,y_2,...,y_n)\in(W^1(R^\infty))^n$, such that

$$\Pi(Y(u),\Phi)=L(\Phi)\quad\forall\Phi\in(W^1(R^\infty))^n \tag{16}$$

where $\Phi\to L(\Phi)$ is a continuous linear form defined on $(W^1(R^\infty))^n$ by

$$L(\Phi)=\sum_{i=1}^{n}\int_{R^\infty} f_i\phi_i d\rho+\sum_{i=1}^{n}\int_\Gamma h_i\phi_i d\Gamma \tag{17}$$

for all $\Phi=(\phi_1,\phi_2,...,\phi_n)\in(W^1(R^\infty))^n, (h_1,h_2,...,h_n)=h\in(H^{\frac{-1}{2}}(\Gamma))^n$, hence

$$(-\sum_{k=1}^{\infty} D_{k=1}^2(I)+q(x))y_i+\sum_{j=1}^{n} a_{ij}y_j=f_i \quad inR^\infty \quad \forall 1\le i\le n.$$

Multiplying both sides by $\Phi=(\phi_1,\phi_2,...,\phi_n)\in(W^1(R^\infty))^n$ and integrating over R^∞, we have

$$-\sum_{i=1}^{n}(\sum_{k=1}^{\infty}\int_{R^\infty} D_k^2(I)+q(x))y_i\phi_i d\rho+\sum_{j=i=1}^{n}\int_{R^\infty} a_{ij}y_j\phi_j d\rho=\sum_{i=1}^{n}\int_{R^\infty} f_i\phi_i d\rho$$

using Green's formula, we obtain

$$\sum_{i=1}^{n}\sum_{k=1}^{\infty}\int_{R^\infty} D_k y_i D_k \phi_i d\rho - \sum_{i=1}^{n}\int_\Gamma \frac{\partial y_i}{\partial \nu_A}\phi_i d\Gamma + \sum_{j=i=1}^{n}\int_{R^\infty} a_{ij} y_j \phi_i d\rho + $$
$$+\sum_{i=1}^{n}\int_{R^\infty} q(x) y_i \phi_i d\rho = \sum_{i=1}^{n}\int_{R^\infty} f_i \phi_i d\rho$$

using (9), we obtain

$$\Pi(Y(u),\Phi) - \sum_{i=1}^{n}\int_\Gamma \frac{\partial y_i}{\partial \nu_A}\phi_i d\Gamma = \sum_{i=1}^{n}\int_{R^\infty} f_i \phi_i d\rho$$

from (16), we get

$$\sum_{i=1}^{n}\int_\Gamma h_i \phi_i d\Gamma - \sum_{i=1}^{n}\int_\Gamma \frac{\partial y_i}{\partial \nu_A}\phi_i d\Gamma = 0$$

hence

$$\int_\Gamma (h_i - \frac{\partial y_i}{\partial \nu_A})\phi_i d\Gamma = 0 \rightarrow h_i|_\Gamma = \frac{\partial y_i}{\partial \nu_A}|_\Gamma \forall i = 1, 2, ..., n.$$

The control problem for system (N)

The space $(L^2(R^\infty))^n$ being the space of controls, for a control $u = (u_1(u), u_2(u), ..., u_n(u))$ $(L^2(R^\infty))^n$ the state $y(u) = (y_1, y_2, ..., y_n) \in (W^1(R^\infty))^n$ of the system is given by the solution of

$$\begin{cases} (-\sum_{k=1}^{\infty} D_{k=1}^2(I) + q(x))y_i(u) + \sum_{j=1}^{n} a_{ij} y_j(u) = f_i + u_i & \text{in } R^\infty \\ \frac{\partial}{\partial \nu_A} y_i|_\Gamma = h_i & \text{on}\Gamma, \text{for} 1 \leq i \leq n. \end{cases} \tag{18}$$

The cost function is again given by (12), then there exits a unique optimal control $u = (u_1, u_2, ..., u_n) \in (L^2(R^\infty))^n$ such that :$J(u) \leq J(v)$; Moreover it is characterized by:

Theorem 2

The optimal control $u = (u_1, u_2, ..., u_n) \in (L^2(R^\infty))^n$ in characterized by (18)together with:

$$\begin{cases} (-\sum_{k=1}^{\infty} D_{k=1}^2(I) + q(x))p_i(u) + \sum_{j=1}^{n} a_{ij} p_j(u) = y_i(u) + z_{id} & \text{in } R^\infty \\ \frac{\partial}{\partial \nu_{A^*}} p_i(u)|_\Gamma = 0 & for 1 \leq i \leq n. \end{cases} \tag{19}$$

$$u = (u_1, u_2, ..., u_n) \in U_{ad}, \sum_{i=1}^{n} \int_{R^\infty} (p_i(u) + N_i u_i)(v_i - u_i) d\rho \geq 0$$
$$\forall v = (v_1, v_2, ..., v_n) \in U_{ad}$$

Outline of the proof

Since $J(u)$ is differentiable and U_{ad} is bounded then the optimal control $u = (u_1, u_2, ..., u_n) \in U_{ad}$ is characterized by

$$\sum_{i=1}^{n} J_i'(u)(v_i - u_i) \geq 0, \quad \forall v = (v_1, v_2, ..., v_n) \in U_{ad}$$

therefore

$$\{\sum_{i=1}^{n}(y_i(u) - z_{id}, y_i(v) - y_i(u))_{L^2(R^\infty)} + \sum_{i=1}^{n}(N_i u_i, v_i - u_i)_{L^2(R^\infty)} \geq 0 \qquad (20)$$

since the adjoint state is given as in theorem (1) by (19), then (20) is equivalent to

$$\sum_{i=1}^{n}((-\sum_{k=1}^{\infty} D_k^2(I) + q(x))p_i(u) + \sum_{j=1}^{n} a_{ji} p_j(u), y_i(v) - y_i(u))_{L^2(R^\infty)} +$$
$$\sum_{i=1}^{n}(N_i u_i, v_i - u_i)_{L^2(R^\infty)} \geq 0,$$

using Green formula, we obtain :

$$\sum_{i=1}^{n}(p_i(u), (-\sum_{k=1}^{\infty} D_{k=1}^2(I) + q(x))(y_i(v) - y_i(u)) + \sum_{j=1}^{n} a_{ji}(y_i(v) - y_i(u))_{L^2(R^\infty)} +$$
$$+\sum_{i=1}^{n}(p_i(u), \frac{\partial}{\partial \nu_A}(y_i(v) - y_i(u))_{L^2(\Gamma)} - \sum_{i=1}^{n}(\frac{\partial}{\partial \nu_{A^*}} p_i(u), y_i(v) - y_i(u))_{L^2(\Gamma)} +$$
$$\sum_{i=1}^{n}(N_i u_i, v_i - u_i)_{L^2(R^\infty)} \geq 0,$$

Then from (18), (19), we have

$$u = (u_1, u_2, ..., u_n) \in U_{ad}, \sum_{i=1}^{n} \int_{R^\infty} (p_i(u) + N_i u_i)(v_i - u_i) d\rho \geq 0$$
$$\forall v = (v_1, v_2, ..., v_n) \in U_{ad}$$

which complete the proof.

Remark 2

Let $n = 2$, the optimality system is given by

$$\begin{aligned}
-\sum_{k=1}^{\infty} D_k^2 y_1(u) + q(x)y_1(u) + y_1(u) - y_2(u) &= f_1 + u_1 \quad inR^\infty, \\
-\sum_{k=1}^{\infty} D_k^2 y_2(u) + q(x)y_2(u) + y_1(u) + y_2(u) &= f_2 + u_2 \quad inR^\infty \quad , \\
\frac{\partial}{\partial \nu_A} y_1(u)|_\Gamma = h_1, \quad \frac{\partial}{\partial \nu_A} y_2(u)|_\Gamma &= h_2, \\
-\sum_{k=1}^{\infty} D_k^2 p_1(u) + q(x)p_1(u) + p_1(u) + p_2(u) &= y_1(u) - z_{1d} \quad inR^\infty, \\
-\sum_{k=1}^{\infty} D_k^2 p_2(u) + q(x)p_2(u) - p_1(u) + p_2(u) &= y_2(u) - z_{2d} \quad inR^\infty, \\
\frac{\partial}{\partial \nu_{A^*}} p_1(u)|_\Gamma = 0, \frac{\partial}{\partial \nu_{A^*}} \quad p_2(u)|_\Gamma &= 0,
\end{aligned}$$

$$u = (u_1, u_2) \in U_{ad}, \int_{R^\infty} (p_1(u) + N_1 u_1)(v_1 - u_1) + (p_2(u) + N_2 u_2)(v_2 - u_2) d\rho \geq 0$$

$$\forall v = (v_1, v_2) \in U_{ad}, where P(u) = (p_1(u), p_2(u)) \quad is \quad the \quad adjoint \quad state.$$

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FIXED COEFFICIENTS FOR CERTAIN SUBCLASS OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract

In this paper we consider the class $T_c(n, \lambda, \alpha)$ consisting of analytic and univalent functions with negative coefficients and fixed second coefficient.The object of the present paper is to show coefficient estimates, convex linear combinations, some distortion theorems and radii of starlikeness and convexity for $f(z)$ in the class $T_c(n, \lambda, \alpha)$. The results are generalized to families with finitely many fixed coefficients.

1. INTRODUCTION

Let S denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and univalent in the unit disc $U = \{z : |z| < 1\}$. Given two functions $f, g \in S$, where $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.2}$$

Their Hadamard product (or convolution) $f * g(z)$, is defined by

$$f * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \ (z \in U) \tag{1.3}$$

By using Hadamard product, Ruscheweyh [4] defined

$$D^{\beta} f(z) = \frac{z}{(1-z)^{\beta+1}} * f(z)(\beta \geq -1) \tag{1.4}$$

and observed that

$$D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!} \tag{1.5}$$

where $\beta = n \in N_0 = NU\{o\}$; and $N = \{1, 2, \cdots\}$. This symbol $D^n f(z)(n \in N_0)$ was called teh n-th order Ruscheweyh derivative of $f(z)$ by Al-Amiri [1]. We note that $D^0 f(z) = f(z)$ and $D^1 f(z) = zf'(z)$. It is easy to see that

$$D^n f(z) = z + \sum_{k=2}^{\infty} \delta(n, k) a_k z^k, \tag{1.6}$$

where

$$\delta(n, k) = \begin{pmatrix} n+k-1 \\ n \end{pmatrix} \tag{1.7}$$

Note that

$$z(D^n f(z))' = (n+1) D^{n+1} f(z) - n D^n f(z) \tag{1.8}$$

Let T denote the subclass of S consisting of functions of the form:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k (a_k \geq 0) \tag{1.9}$$

Further, we say that a function $f(z)$ belonging to T is in the class $T(n, \lambda, \alpha)$ if and only if

$$\operatorname{Re}\left\{\frac{\frac{D^{n+1}f(z)}{D^n f(z)}}{\lambda\frac{D^{n+1}f(z)}{D^n f(z)}+(1-\lambda)}\right\}>\alpha, \quad (n \in N_0) \tag{1.10}$$

for some $\alpha(0 \leq \alpha < 1), \lambda(0 \leq \lambda < 1)$, and for all $z \in U$. The class $T(n, \lambda, \alpha)$ was introduced by Aouf and Chen [3]. We note that by specializing the parameters n, λ and α, we obtain the following subclasses studied by various authors:

(1) $T(0, \lambda, \alpha) = T(\lambda, \alpha)$ (Altintas and Owa [2]);

(2) $T(0, 0, \alpha) = T^*(\alpha)$ (Silverman [5]);

(3) $T(n, 0, \alpha)$ represents the class of functions $f(z) \in T$ satisfying the condition

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\}>\alpha(0 \leq \alpha < 1; \quad n \in N_0). \tag{1.11}$$

For the class $T(n, \lambda, \alpha)$ Aouf and Chen [3] showed the following Lemma:

Lemma 1

Let the function $f(z)$ be defined by (1.9). Then $f(z) \in T(n, \lambda, \alpha)$ if and only if

$$\sum_{k=2}^{\infty} C_k(n, \lambda, \alpha)\delta(n, k)a_k \leq (1-\alpha)(n+1) \tag{1.12}$$

where

$$C_k(n, \lambda, \alpha) = n + k - \alpha[\lambda(k-1) + n + 1] \tag{1.13}$$

The result is sharp.

In view of Lemma 1, we can see that the coefficient a_2 of the function $f(z)$ defined by (1.9) and belonging to the class $T(n, \lambda, \alpha)$ satisfies the inequality:

$$a_2 \leq \frac{(1-\alpha)}{C_2(n,\lambda,\alpha)} \quad (n \in N_0, 0 \leq \alpha < 1, \quad 0 \leq \lambda < 1) \tag{1.14}$$

Thus we let $T_c(n, \lambda, \alpha)$ denote the class of functions $f(z)$ in $T(n, \lambda, \alpha)$ which are of the form:

$$f(z) = z - \frac{c(1-\alpha)}{C_2(n,\lambda,\alpha)} z^2 - \sum_{k=3}^{\infty} a_k z^k \quad (a_k \geq 0; \quad 0 \leq c \leq 1) \tag{1.15}$$

For the class $T_c(n, \lambda, \alpha)$ of analytic functions with negative coefficients and fixed second coefficients, defined above, we shall derive a number of interesting properties and characteristics (including, for example, coefficient estimates, closure properties involving convex linear combinations, growth and distortion theorems and radii of starlikeness and convexity). We also extend many of these results to hold true for analogous classes of functions with finitely many fixed coefficients.

2. COEFFICIENT ESTIMATES FOR THE CLASS $T_c(n, \lambda, \alpha)$

Theorem 1

Let the function $f(z)$ be defined by (1.15). Then $f(z) \in T_c(n, \lambda, \alpha)$ if and only if

$$\sum_{k=3}^{\infty} C_k(n,\lambda,\alpha)\delta(n,k)a_k \leq (1-c)(1-\alpha)(n+1) \tag{2.1}$$

where $C_k(n, \lambda, \alpha)$ is defined by (1.13) and $\delta(n, k)$ is defined by (1.7). The result is sharp.

Proof

Putting

$$a_2 = \frac{(1-\alpha)c}{C_2(n,\lambda,\alpha)} \quad (0 \leq c \leq 1) \tag{2.2}$$

in Lemma 1 and simplifying the resulting inequality, we readily arrive at the assertion (2.1) of Theorem 1. The result is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{c(1-\alpha)}{C_2(n,\lambda,\alpha)}z^2 - \frac{(1-c)(1-\alpha)(n+1)}{C_k(n,\lambda,\alpha)\delta(n,k)}z^k \quad (k = 3,4,5,\cdots) \quad (2.3)$$

Corollary 1

Let the function $f(z)$ defined by (1.15) be in the class $T_c(n,\lambda,\alpha)$. Then

$$a_k \leq \frac{(1-c)(1-\alpha)(n+1)}{C_k(n,\lambda,\alpha)\delta(n,k)} \quad (k = 3,4,\cdots) \quad (2.4)$$

The result is sharp for the function $f(z)$ given by (2.3)

3. CLOSURE THEOREMS FOR THE CLASS $T_c(n,\lambda,\alpha)$

Theorem 2

The class $T_c(n,\lambda,\alpha)$ is closed under convex linear combination.

Proof

Let the function $f(z)$ be defined by (1.15). Define the function $g(z)$ by

$$g(z) = z - \frac{c(1-\alpha)}{C_2(n,\lambda,\alpha)}z^2 - \sum_{k=3}^{\infty} b_k z^k \quad (b_k \geq 0) \quad (3.1)$$

Assuming that $f(z)$ and $g(z)$ are in the class $T_c(n,\lambda,\alpha)$, it is sufficient to prove that the function $h(z)$ defined by

$$h(z) = \mu f(z) + (1-\mu)g(z) \quad (0 \leq \mu \leq 1) \quad (3.2)$$

is also in the class $T_c(n,\lambda,\alpha)$. Since

$$h(z) = z - \frac{c(1-\alpha)}{C_2(n,\lambda,\alpha)}z^2 - \sum_{k=3}^{\infty}\{\mu a_k + (1-\mu)b_k\}z^k, \quad (3.3)$$

we observe that

$$\sum_{k=3}^{\infty} C_k(n,\lambda,\alpha)\delta(n,k)\{\mu a_k + (1-\mu)b_k\} \leq (1-c)(1-\alpha)(n+1) \tag{3.4}$$

with the aid of Theorem 1. Hence $h(z) \in T_c(n,\lambda,\alpha)$. This completes the proof of Theorem 2.

Theorem 3

Let the functions

$$f_j(z) = z - \frac{c(1-\alpha)}{C_2(n,\lambda,\alpha)} z^2 - \sum_{k=3}^{\infty} a_{k,j}\ z^k \quad (a_{k,j}, \geq 0) \tag{3.5}$$

be in the class $T_c(n,\lambda,\alpha)$ for every $j = 1,\cdots,m$. Then the function $F(z)$ defined by

$$F(z) = \sum_{j=1}^{\infty} \mu_j f_j(z) \quad (u_j \geq 0) \tag{3.6}$$

is also in the class $T_c(n,\lambda,\alpha)$, where

$$\sum_{j=1}^{m} \mu_j = 1 \tag{3.7}$$

Proof

Combining the definitions (3.5) and (3.6), we have

$$F(z) = z - \frac{c(1-\alpha)}{c_2(n,\lambda,\alpha)} z^2 - \sum_{k=3}^{\infty} \left(\sum_{j=1}^{m} \mu_j a_{k,j} \right) z^k, \tag{3.8}$$

where we have also used the relationship (3.7). Since $f_j(z) \in T_c(n,\lambda,\alpha)$ for every $j = 1,2,\cdots,m$, Theorem 1 yields.

$$\sum_{k=3}^{\infty} C_k(n,\lambda,\alpha)\delta(n,k)a_{k,j} \leq (1-c)(1-\alpha)(n+1) \quad (j = 1,2,\cdots,m) \tag{3.9}$$

Thus we obtain

$$\sum_{k=3}^{\infty} C_k(n,\lambda,\alpha)\delta(n,k)\left(\sum_{j=1}^{m}\mu_j a_{k,j}\right) = \sum_{j=1}^{m}\mu_j\left[\sum_{k=3}^{\infty} C_k(n,\lambda,\alpha)\delta(n,k)a_{k,j}\right]$$
$$\leq (1-c)(1-\alpha)(n+1)$$

which (in view of Theorem 1) implies that $F(z) \in T_c(n,\lambda,\alpha)$.

Theorem 4

Let

$$f_2(z) = z - \frac{c(1-\alpha)}{C_2(n,\lambda,\alpha)}z^2 \tag{3.10}$$

and

$$f_k(z) = z - \frac{c(1-\alpha)}{C_2(n,\lambda,\alpha)}z^2 - \frac{(1-c)(1-\alpha)(n+1)}{C_k(n,\lambda,\alpha)\delta(n,k)}z^k \quad (k = 3,4,5,\cdots), \tag{3.11}$$

Then $f(z)$ is in the class $T_c(n,\lambda,\alpha)$ if and only if it can be expressed in the form:

$$f(z) = \sum_{k=2}^{\infty}\mu_k f_k(z) \tag{3.12}$$

where

$$\mu_k \geq 0 \quad \text{and} \quad \sum_{k=2}^{\infty}\mu_k = 1 \tag{3.13}$$

Proof

We suppose that $f(z)$ can be expressed in the form (3.12). Then we have

$$f(z) = z - \frac{c(1-\alpha)}{C_2(n,\lambda,\alpha)}z^2 - \frac{(1-c)(1-\alpha)(n+1)\mu_k}{C_k(n,\lambda,\alpha)\delta(n,k)}z^k, \tag{3.14}$$

Since

$$\sum_{k=3}^{\infty}\frac{(1-c)(1-\alpha)(n+1)\mu_k}{C_k(n,\lambda,\alpha)\delta(n,k)}\cdot\frac{C_k(n,\lambda,\alpha)\delta(n,k)}{(1-\alpha)(n+1)} = (1-c)(1-\lambda_2)$$
$$\leq (1-c), \qquad (3.15)$$

it follows from (2.1) that $f(z)$ is in the class $T_c(n, \lambda, \alpha)$.

Conversely, we suppose that $f(z)$ defined by (1.15) is in the class $T_c(n, \lambda, \alpha)$. Then, by making use of (2.4), we get

$$a_k \leq \frac{(1-c)(1-\alpha)(n+1)}{C_k(n,\lambda,\alpha)\delta(n,k)} \quad (k \geq 3) \tag{3.16}$$

Setting

$$\mu_k = \frac{C_k(n,\lambda,\alpha)\delta(n,k)}{(1-c)(1-\alpha)(n+1)} \; a_k \quad (k \geq 3) \tag{3.17}$$

and

$$\mu_2 = 1 - \sum_{k=3}^{\infty} \mu_k, \tag{3.18}$$

we have (3.12). This completes the proof of Theorem 4.

Corollary 2

The extreme points of the class $T_c(n, \lambda, \alpha)$ are the functions $f_k(z)(k \in N/\{1\})$ given by Theorem 4.

4. GROWTH AND DISTORTION THEOREMS FOR THE CLASS $T_c(n, \lambda, \alpha)$

Lemma 2, 3, and 4 below will be required in our investigation of the growth and distortion properties of the general class $T_c(n, \lambda, \alpha)$.

Lemma 2

Let the function $f_3(z)$ be defined by

$$\begin{aligned} f_3(z) &= z - \frac{c(1-\alpha)}{[n+2-\alpha(\lambda+n+1)]} z^2 \\ &\quad - \frac{2(1-c)(1-\alpha)}{[n+3-\alpha(2\lambda+n+1)](n+2)} z^3 \end{aligned} \tag{4.1}$$

Then, for $0 \le r < 1$ and $0 \le c \le 1$,

$$|f_3(re^{i\theta})| \ge r - \frac{c(1-\alpha)}{[n+2-\alpha(\lambda+n+1)]}r^2 - \frac{2(1-c)(1-\alpha)}{[n+3-\alpha(2\lambda+n+1)](n+2)}r^3 \qquad (4.2)$$

with equality for $\theta = 0$. For either $0 \le c \le c_0$ and $0 \le r \le r_0$, or $c_0 \le c \le 1$,

$$|f_3(re^{i\theta})| \ge r + \frac{c(1-\alpha)}{[n+2-\alpha(\lambda+n+1)]}r^2 - \frac{2(1-c)(1-\alpha)}{[n+3-\alpha(2\lambda+n+1)](n+2)}r^3 \qquad (4.3)$$

with quality for $\theta = \pi$. Furthermore, for $0 \le c \le c_0$ and $r_0 \le r < 1$.

$$|f_3(re^{i\theta})| \le r\left\{\left[1 + \frac{c^2(1-\alpha)[n+3-\alpha(2\lambda+n+1)](n+2)}{8(1-c)[n+2-\alpha(\lambda+n+1)]^2}\right] + \left[\frac{c^2(1-\alpha)^2}{2[n+2-\alpha(\lambda+n+1)]^2} + \frac{4(1-c)(1-\alpha)}{[n+3-\alpha(2\lambda+n+1)](n+2)}\right]r^2 + \left[\frac{4(1-c)^2(1-\alpha)^2}{[n+3-\alpha(2\lambda+n+1)]^2(n+1)^2} + \frac{c^2(1-c)(1-\alpha)^3}{2[n+2-\alpha(\lambda+n+1)]^2[n+3-\alpha(2\lambda+n+1)](n+2)}\right]r^4\right\}^{1/2} \qquad (4.4)$$

with equality for

$$\theta = \cos^{-1}\left(\frac{2c(1-c)(1-\alpha)r^2 - c(n+2)[n+3-\alpha(2\lambda+n+1)]}{8(1-c)[n+2-\alpha(\lambda+n+1)]r}\right), \qquad (4.5)$$

where

$$c_0 = \frac{1}{4(1-\alpha)}[-\{8[n+2-\alpha(\lambda+n+1)] +$$

$$(n+2)[n+3-\alpha(2\lambda+n+1)]-2(1-\alpha)\}+$$
$$\{\{8[n+2-\alpha(\lambda+n+1)]+(n+2)[n+3-\alpha(2\lambda+n+1)]$$
$$-2(1-\alpha)\}^2+64(1-\alpha)[n+2-\alpha(\lambda+n+1)\}^{1/2}] \tag{4.6}$$

and

$$\begin{aligned} r_0 &= \frac{1}{2c(1-c)(1-\alpha)}\{-4(1-c)[n+2-\alpha(\lambda+n+1)] \\ &+\{16(1-c)^2[n+2-\alpha(\lambda+n+1)]^2 \\ &+2c^2(1-c)(1-\alpha)(n+2)[n+3-\alpha(2\lambda+n+1)]\}^{1/2}\}. \end{aligned} \tag{4.7}$$

Proof

We employ the same technique as used by Silverman and Silvia [6]. since

$$\begin{aligned} \frac{\partial|f_3(re^{i\theta})|^2}{\partial\theta} &= \frac{2(1-\alpha)}{[n+2-\alpha(\lambda+n+1)]}r^3\sin\theta[c+ \\ &\frac{8(1-c)[n+2-\alpha(\lambda+n+1)]}{[n+3-\alpha(2\lambda+n+1)](n+2)}r\cos\theta- \\ &\frac{2c(1-c)(1-\alpha)}{[n+3-\alpha(2\lambda+n+1)](n+2)}r^2], \end{aligned} \tag{4.8}$$

we can see that

$$\frac{\partial|f_3(re^{i\theta}|^2}{\partial\theta}=0 \tag{4.9}$$

for $\theta_1=0,\quad \theta_2=\pi$ and

$$\theta_3=\cos^{-1}\left(\frac{2c(1-c)(1-\alpha)r^2-c(n+2)[n+3-\alpha(2\lambda+n+1)]}{8(1-c)[n+2-\alpha(\lambda+n+1)]r}\right). \tag{4.10}$$

Since θ_3 is a valid root only when $-1\le\cos\theta_3\le 1$, we have a third root if and only if $r_0\le r<1$ and $0\le c<c_0$. Thus the results of Lemma 2 follow

upon comparing the extremal values $f_3(re^{i\theta_k})|(k = 1, 2, 3)$ on the appropriate intervals.

Lemma 3

Let the function $f_k(z)$ be defined by (3.11) and $k \geq 4$. Then

$$|f_k(re^{i\theta})| \leq |f_4(-r)| \qquad (k \geq 4). \tag{4.11}$$

Proof

$$\begin{aligned} f_k(z) &= z - \frac{c(1-\alpha)}{[n+2-\alpha(\lambda+n+1)]}z^2 \\ &\quad - \frac{(1-c)(1-\alpha)(n+1)}{\{n+k-\alpha[\lambda(k-1)+n+1]\}\delta(n,k)}z^k \end{aligned}$$

and

$$\frac{(1-c)(1-\alpha)(n+1)r^k}{\{n+k-\alpha[\lambda(k-1)+n+1]\}\delta(n,k)}$$

is a decreasing function of k, we have

$$\begin{aligned} |f_k(re^{i\theta})| &\leq r + \frac{c(1-\alpha)}{[n+2-\alpha(\lambda+n+1)]}r^2 \\ &\quad + \frac{6(1-c)(1-\alpha)}{(n+2)(n+3)\{n+4-\alpha[3\lambda+n+1]\}}r^4 \\ &= -f_4(-r), \end{aligned}$$

which proves (4.11).

Theorem 5

Let the function $f(z)$ defined by (1.15) belong to class $T_c(n, \lambda, \alpha)$. Then, for $0 \leq r < 1$,

$$|f(re^{i\theta})| \geq r - \frac{c(1-\alpha)}{[n+2-\alpha(\lambda+n+1)]}r^2 -$$

$$\frac{2(1-c)(1-\alpha)}{(n+2)[n+3-\alpha(2\lambda+n+1)]}r^3 \tag{4.12}$$

will equality for $f_3(z)$ at $z = r$, and

$$|f(re^{i\theta})| \leq \max\{\max_{\theta} |f_3(re^{i\theta})|, -f_4(-r)\}, \tag{4.13}$$

where

$$\max_{\theta} |f_3(re^{i\theta})|$$

is given by Lemma 2. The proof of Theorem 5 is obtained by comparing the bounds given by Lemma 2 and Lemma 3.

Remark 1

Putting $c = 1$ and $n = 0$ in Theorem 5 we obtain the following result obtained by Altintas and Owa [2].

Corollary 3

Let the function $f(z)$ defined by (1.9) be in the class $T_1(0, \lambda, \alpha) = T(\lambda, \alpha)$.

Then for $|z| = r < 1$, we have

$$r - \frac{(1-\alpha)}{[2-\alpha(1+\lambda)]}r^2 \leq |f(z)| \leq r + \frac{(1-\alpha)}{[2-\alpha(1+\lambda)]}r^2. \tag{4.14}$$

The result is sharp.

Lemma 4

Let the function $f_3(z)$ be defined by (4.1). Then, for $0 \leq r < 1$ and $0 \leq c \leq 1$,

$$|f_3'(re^{i\theta})| \geq 1 - \frac{2c(1-\alpha)}{[n+2-\alpha(\lambda+n+1)]}r - \frac{6(1-c)(1-\alpha)}{[n+3-\alpha(2\lambda+n+1)](n+2)}r^2 \tag{4.15}$$

with equality for $\theta = 0$. For either $0 \le c \le c_1$ and $0 \le r \le r_1$ or $c_1 \le c \le 1$,

$$\begin{aligned}|f_3'(re^{i\theta})| &\le 1 + \frac{2c(1-\alpha)}{[n+2-\alpha(\lambda+n+1)]}r - \\ &\frac{6(1-c)(1-\alpha)}{[n+3-\alpha(2\lambda+n+1)](n+2)}r^2\end{aligned} \tag{4.16}$$

with equality for $\theta = \pi$. Furthermore, for $0 \le c < c_1$ and $r_1 \le r < 1$,

$$\begin{aligned}|f_3'(re^{i\theta}) &\le \{[1 + \frac{c^2(1-\alpha)[n+3-\alpha(2\lambda+n+1)](n+2)}{6(1-c)[n+2-\alpha(\lambda+n+1)]^2}] \\ &+[\frac{2c^2(1-\alpha)^2}{[n+2-\alpha(\lambda+n+1)]^2} + \frac{12(1-c)(1-\alpha)}{(n+2)[n+3-\alpha(2\lambda+n+1)]}]r^2 \\ &+[\frac{6c^2(1-c)(1-\alpha)^3}{[n+2-\alpha(\lambda+n+1)]^2(n+2)[n+3-\alpha(2\lambda+n+1)]}] \\ &+\frac{36(1-c)^2(1-\alpha)^2}{(n+2)^2[n+3-\alpha(2\lambda+n+1)]^2}]r^4\}^{1/2}\end{aligned} \tag{4.17}$$

with equality for

$$\theta = \cos^{-1}\left(\frac{6c(1-c)(1-\alpha)r^2 - c(n+2)[n+3-\alpha(2\lambda+n+1)}{12[n+2-\alpha(\lambda+n+1)](1-c)r}\right), \tag{4.18}$$

where

$$\begin{aligned}c_1 &= \frac{1}{12(1-\alpha)}\{-12[n+2-\alpha(\lambda+n+1)] - \\ &(n+2)[n+3-\alpha(2\lambda+n+1)] + 6(1-\alpha) + \\ &[\{12[n+2-\alpha(\lambda+n+1)] + (n+2)[n+3-\alpha(2\lambda+n+1)] \\ &-6(1-\alpha)\}^2 + 288(1-\alpha)[n+2-\alpha(\lambda+n+1)]]^{1/2}\}\end{aligned} \tag{4.19}$$

and

$$r_1 = \frac{1}{6c(1-c)(1-\alpha)}\{-6(1-c)[n+2-\alpha(\lambda+n+1)] +$$

$$[36(1-c)^2[n+2-\alpha(\lambda+n+1)]^2 + 6c^2(1-c)(1-\alpha)(n+2)[n+3-\alpha(2\lambda+n+1)]]^{1/2}\}. \tag{4.20}$$

The proof of Lemma 4 is given in much the same way as Lemma 2.

Theorem 6

Let the function $f(z)$ defined by (1.15) be in the class $T_c(n,\lambda,\alpha)$. Then, for $0 \leq r < 1$,

$$|f'(re^{i\theta})| \geq 1 - \frac{2c(1-\alpha)}{[n+2-\alpha(\lambda+n+1)]} r - \frac{6(1-c)(1-\alpha)}{(n+2)[n+3-\alpha(2\lambda+n+1)]} r^2 \tag{4.21}$$

with equality for $f_3'(z)$ at $z = r$, and

$$|f'(re^{i\theta})| \leq \max\{\max_\theta |f_3'(re^{i\theta})|, \quad f_4'(-r)\}, \tag{4.22}$$

where

$$\max_\theta |f_3'(re^{i\theta})|$$

is given by Lemma 4.

Remark 2

Putting $c = 1$ and $n = 0$ in Theorem 6 we obtain the following result obtained by Altintas and Owa [2].

Corollary 4

Let the function $f(z)$ defined by (1.9) be in the class $T_1(0,\lambda,\alpha) = T(\lambda,\alpha)$. Then, for $|z| = r < 1$,

$$1 - \frac{2(1-\alpha)}{[2-\alpha(\lambda+1)]} r \leq |f'(z)| \leq 1 + \frac{2(1-\alpha)}{[2-\alpha(\lambda+1)]} r. \tag{4.23}$$

The result is sharp.

5. RADII OF STARLIKENESS AND CONVEXITY

Theorem 7

Let the function $f(z)$ defined by (1.15) be in the class $T_c(n,\lambda,\alpha)$. Then $f(z)$ is starlike of order $\rho(0 \leq \rho < 1)$ in the disc $|z| < r_1(n,\lambda,\alpha,c,\rho)$, where $r_1(n,\lambda,\alpha,c,\rho)$ is the largest value for which the following inequality holds true.

$$\begin{aligned} &\frac{c(1-\alpha)(2-\rho)r}{[n+2-\alpha(\lambda+n+1)]} + \\ &\frac{(1-c)(1-\alpha)(n+1)(k-\rho)}{\{n+k-\alpha[\lambda(k-1)+n+1]\}\delta(n,k)} r^{k-1} \\ \leq\ & 1-\rho (k=3,4,\cdots). \end{aligned} \tag{5.1}$$

The result is sharp, the external function being given by

$$\begin{aligned} f_k(z) = z - & \frac{c(1-\alpha)}{[n+2-\alpha(n+\lambda+1)]} z^2 - \\ & \frac{(1-c)(1-\alpha)(n+1)}{\{n+k-\alpha[\lambda(k-1)+n+1]\}\delta(n,k)} z^k \end{aligned} \tag{5.2}$$

for some k.

Proof

It suffices to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1-\rho \quad (0 \leq \rho < 1)$$

$|z| < r_1(n,\lambda,\alpha,c,\rho)$. We note that

$$\begin{aligned} \left|\frac{zf'(z)}{f(z)} - 1\right| &\leq \frac{\frac{c(1-\alpha)}{[n+2-\alpha(\lambda+n+1)]} r + \sum_{k=3}^{\infty}(k-1)a_k r^{k-1}}{1 - \frac{c(1-\alpha)}{[n+2-\alpha(\lambda+n+1)]} r - \sum_{k=3}^{\infty} a_k r^{k-1}} \\ &\leq 1-\rho (|z| \leq r) \end{aligned} \tag{5.3}$$

if and only if

$$\frac{c(1-\alpha)(2-\rho)}{[n+2-\alpha(\lambda+n+1)]}r+\sum_{k=3}^{\infty}(k-\rho)a_k r^{k-1}\leq 1-\rho. \tag{5.4}$$

Since $f(z)$ is in $T_c(n,\lambda,\alpha)$, from (2.1) we may take

$$a_k=\frac{(1-c)(1-\alpha)(n+1)\mu_k}{\{n+k-\alpha[\lambda(k-1)+n+1]\}\delta(n,k)}\quad (k=3,4,5,\cdots), \tag{5.5}$$

where

$$\mu_k\geq 0 (k=3,4,\cdots)\quad \text{and} \quad \sum_{k=3}^{\infty}\mu_k\leq 1. \tag{5.6}$$

For each fixed r, we choose the positive integer $k_0=k_0(r)$ for which

$$\frac{(k_0-\rho)}{\{n+k_0-\alpha[\lambda(k_0-1)+n+1]\}\delta(n,k_0)}r^{k_0-1}$$

is maximal. Then it follows that

$$\sum_{k=3}^{\infty}(k-\rho)a_k r^{k-1}\leq\frac{(1-c)(1-\alpha)(n+1)(k_0-\rho)}{\{n+k_0-\alpha[\lambda(k_0-1)+n+1]\}\delta(n,k_0)}r. \tag{5.7}$$

Hence $f(z)$ is starlike of order ρ in $|z|<r_1(n,\lambda,\alpha,c,\rho)$ provided that

$$\begin{aligned}&\frac{c(1-\alpha)(2-\rho)r}{[n+2-\alpha(\lambda+n+1)]}+\\&\frac{(1-c)(1-\alpha)(n+1)(k_0-\rho)}{\{n+k_0-\alpha[\lambda(k_0-1)+n+1]\}\delta(n,k_0)}r^{k_0-1}\\\leq\ &1-\rho.\end{aligned} \tag{5.8}$$

We find the value $r_0=r_0(n,\lambda,\alpha,c,\rho)$ and the corresponding integer $k_0(r_0)$ so that

$$\frac{c(1-\alpha)(2-\rho)r_0}{[n+2-\alpha(\lambda+n+1)]}+$$

$$\frac{(1-c)(1-\alpha)(n+1)(k_0-\rho)}{\{n+k_0-\alpha[\lambda(k_0-1)+n+1]\}\delta(n,k_0)} r_0^{k_0-1}$$
$$= \quad 1-\rho \tag{5.9}$$

Then this value r_0 is the radius of starlikeness of order ρ for functions $f(z)$ belonging to the class $T_c(n, \lambda, \alpha)$.

In a similar manner, we can prove the following theorem concerning the radius of convexity of order ρ for functions in the class $T_c(n, \lambda, \alpha)$.

Theorem 8

Let the function $f(z)$ defined by (1.15) be in the class $T_c(n, \lambda, \alpha)$.

Then $f(z)$ is convex of $\rho(0 \leq \rho < 1)$ in the disc $|z| < r_2(n, \lambda, \alpha, c, \rho)$, where $r_2(n, \lambda, \alpha, c, \rho)$ is the largest value for which the following inequality holds true:

$$\frac{2c(1-\alpha)(2-\rho)r}{[n+2-\alpha(\lambda+n+1)]} +$$
$$\frac{(1-c)(1-\alpha)(n+1)k(k-\rho)}{\{n+k-\alpha[\lambda(k-1)+n+1]\}\delta(n,k)} r^{k-1}$$
$$\leq \quad 1-\rho \quad (k=3,4,\ldots). \tag{5.10}$$

The result is sharp for the function $f(z)$ given by (5.2).

6. THE GENERAL CLASS $T_{cK,N}(n, \lambda, \alpha)$

Instead of fixing only the second coefficient, we can fix finitely many coefficients. Let $T_{c_k,N}(n, \lambda, \alpha)$ denote the class of functions $f(z)$ in $T_c(n, \lambda, \alpha)$ of the form:

$$f(z) \quad = \quad z - \sum_{k=2}^{N} \frac{c_k(1-\alpha)}{[n+2-\alpha(\lambda+n+1)]} z^k -$$
$$\sum_{k=N+1}^{\infty} a_k z^k \left(0 \leq \sum_{k=2}^{\infty} c_k = c \leq 1\right). \tag{6.1}$$

Observe that $T_{ck,2}(n,\lambda,\alpha) = T_c(n,\lambda,\alpha)$.

Theorem 9

The extreme points of the class $T_{cK,N}(n,\lambda,\alpha)$ are

$$z - \sum_{k=2}^{N} \frac{c_k(1-\alpha)(n+1)}{\{n+k-\alpha[\lambda(k-1)+n+1]\}\delta(n,k)} z^k$$

and

$$z - \sum_{k=2}^{N} \frac{c_k(1-\alpha)(n+1)}{\{n+k-\alpha[\lambda(k-1)+n+1]\}\delta(n,k)} z^k - \frac{(1-c)(1+\alpha)(n+1)}{\{n+k-\alpha[\lambda(k-1)+n+1]\}\delta(n,k)} z^k$$
$$(k = N+1,\ N+2,\ N+3,\ldots).$$

The details of the proof of Theorem 9 ae omitted.

Remark 3

The characterization of the extreme points for the general class $T_{ck,N}(n,\lambda,\alpha)$ enables us to solve the standard extremal problems in the same manner as was done for the special class $T_c(n,\lambda,\alpha)$. The details involved may be left as an exercise for the interested reader.

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A CALENDAR FOR 19,999 YEARS

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The calendar proposed herein is an improved version of a previous proposal of this author published in ASTROMATHICS (Vol III, No:1, Autumn 1959), Journal of Dyal Singh College Mathematics society, Lahore.

The Gregorian rule followed all over the world is: all years divisible by 4 are leap years but all centuries are ordinary years except 400 and its multiples which are considered to be leap years.

Now the mean length of the tropical year which is the basis of the Gregorian calendar is 365.24219878 days. Also according to recent researches this length is shrinking at a very low rate - so low that it is irrelevant for the proposed calendar. So we ignore this shrinkage. This ignoration is in consonance with a remark in the EXPLANATORY SUPPLEMENT TO THE ASTRONOMICAL EPHEMERIS (1961 edn. 1977 impression). The remark is this:

> 'The length of the synodic month is 29.530589 days and of the tropical year is 365.242199 days, for the epoch 1900. The very small, and some what uncertain, secular variations in the lengths of these periods are unimportant for chronological purposes.'
>
> (The supplement, p.407)

According to the Gregorian rule described above there are 97 intercalations (on account of leap years) in 400 years so that

$$400 \text{ years} = 400\text{x}365 + 97 = 146097 \text{ days}.$$

Dividing this number by 400 we obtain 365.2425 as the mean length of a Gregorian year. This value exceeds the value adopted by us by 0.00030122 d. The error will accumulate to a whole day in about 3320 years.

It is, therefore, proposed that the Gregorian rule may be modified by making the year 4000 and its its multiples ordinary years in place of regarding them as leap years. As a result of this modification the accumulated error will not amount to more than a day in 20,000 years.

The attached calendar has been framed according to this formula.

The error in the calender will become far less if the year 3200 and its multiples are treated as ordinary years in place of leap years. If this proposal is accepted on the international, the calendar will serve for 88000 years.

The calender given herein will remain usable by making a slight change in Table III. Only the phases shown on the top of the table will have to be changed from 1 to 3999 to from 1 to 3199A.D. etc.

It is for the international authorities to choose either of the cycles - 3200 years or 4000 years.

HOW TO USE THE CALENDAR

1. Divide the year, for which calender is desired, by 400 and take the remainder.

2. Look up the remainder in Table I to find the category of the year (A, B etc.). The figure in the hundred's place of the remainder to be looked up in the first column on the left of Table I and the remaining part from the other columns. The letter occurring at the intersection of the row through the hundred's figure and the column through the remaining part is the category letter. If the remainder is a complete century look up its category letter at the bottom of Table I.

3. In Table II, below every category letter and against each month there is a number, the number for the month calendar.

4. Locate the number obtained from Table II in the upper part of Table III in the row appropriate for the year in question. Below this number there is a permutation of the week days. This column of week days will govern the dates of the month shown on the right.

EXAMPLES

Example 1: Find the calendar for January 2003.

Sol: Dividing 2003 by 400 we get 03 as the remainder.

From Table I: Below 03 and against hundred 0 we find C, the category of the year.

From Table II: Below C and against Jan. (Ord.) we find 3, the number of the month calendar.

From Table III: Below 3 (first row) permutation of week days is W, Th etc. which read with dates on the right gives the desired calendar.

Example 2: Find the calendar for February 1960.

Sol: Dividing 1960 by 400 the remainder is 360.

From Table I: Below 60 and against 3 (hundred) we find F, the category of the given year.

From Table II: Below F and against February (Leap) we find 1, the number of the month calendar.

From Table III: Below 1 (in the first row) the permutation of week days is M, T etc. This permutation read with the dates on the right is the required calendar. The month starts and ends with Monday.

Example 3: Find the calendar for January, 2100.

Sol: On dividing 2100 by 400 the remainder is 100.

From Table I: Category of the year is E.

From Table II: Below E and against January (Ord.) we find 5 as the number of the month cal.

From Table III: Below 5 (in the first row) the permutation of week days is F, Sa etc. which read with dates on the right gives desired calendar.

THE CALENDAR

TABLE I

Categories of years

01 to 99

	01	02	03		04	05	06
	07		08	09	10	11	
	12	13	14	15		16	17
	18	19		20	21	22	23
		24	25	26	27		28
	29	30	31		32	33	34
	35		36	37	38	39	
H	40	41	42	43		44	45
U	46	47		48	49	50	51
N		52	53	54	55		56
D	57	58	59		60	61	62
R	63		64	65	66	67	
E	68	69	70	71		72	73
D	74	75		76	77	78	79
S		80	81	82	83		84
↓	85	86	87		88	89	90
	91		92	93	94	95	
	96	97	98	99			
0	A	B	C	D	E	F	G
1	F	G	A	B	C	D	E
2	D	E	F	G	A	B	C
3	B	C	D	E	F	G	A

CATEGORIES OF CENTURIES

Hereunder R_e denotes the remainder when given year is divided by 400 and C_y denotes its category.

R_e	100	200	300	nill
C_y	E	C	A	G

TABLE II

Numbers for month calendars

Categories of years / Months	A	B	C	D	E	F.	G
Jan. (Ordinary, Oct. May)	1 2	2 3	3 4	4 5	5 6	6 7	7 1
Feb. (Leap yr) August	3	4	5	6	7	1	2
Feb. (Ordinary) March, Nov. June	4 5	5 6	6 7	7 1	1 2	2 3	3 4
Sep. Dec.	6	7	1	2	3	4	5
Jan. (Leap) Apr. July	7	1	2	3	4	5	6

TABLE III

Months Calendars

1	2	3	4	5	6	7	1 to 3999 A.D.					
2	3	4	5	6	7	1	4000 to 7999 A.D.					
3	4	5	6	7	1	2	8000 to 11999 A.D.					
4	5	6	7	1	2	3	12000 to 15999 A.D.					
5	6	7	1	2	3	4	16000 to 19999 A.D.					
M	T	W	Th	F	Sa	S	1	8	15	22	29	
T	W	Th	F	Sa	S	M	2	9	16	23	30	D
W	Th	F	Sa	S	M	T	3	10	17	24	31	A
Th	F	Sa	S	M	T	W	4	11	18	25	x	T
F	Sa	S	M	T	W	Th	5	12	19	26	x	E
Sa	S	M	T	W	Ts	F	6	13	20	27	x	S
S	M	T	W	Th	F	Sa	7	14	21	28	x	

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