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An Improved Error Analysis for the Secant Method Under the Gamma Condition

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Abstract. We provide sufficient convergence conditions for the Secant method for approximating a locally unique solution of an operator equation in a Banach space. The main hypothesis is a type of gamma condition first introduced in [9] for the study of Newton's method. Our sufficient convergence condition reduces to the one obtained in [12] for Newton's method although in general it can be weaker. A numerical example is also provided.

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1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of the equation

$$F(x) = 0, \tag{1. 1}$$

where F is a Fréchet-differentiable operator mapping a convex subset D of a Banach space X into a Banach space Y.

The most popular methods for generating sequences approximating x^* are undoubtedly Newton's method

$$y_{n+1} = y_n - F'(y_n)^{-1}F(y_n) \quad (n \geq 0), \quad (y_0 \in D), \tag{1. 2}$$

and the Secant method

$$x_{n+1} = x_n - [x_{n-1}, x_n]^{-1}F(x_n) \quad (n \geq 0), \quad (x_{-1}, x_0 \in D). \tag{1. 3}$$

The advantages and disadvantages of using the Secant method over Newton's method are well known [1]–[14].
Here, $F'(x)$, $[x,y] \in L(X,Y)$ the space of bounded linear operators, by $[x,y]$ we mean $[x,y;F]$, and the divided difference of order one at (x,y) satisfying

$$[x,y](x-y) = F(x) - F(y) \tag{1. 4}$$

for all $x, y \in D$ with $x \neq y$ [4], [6], [9].

There is an extensive literature on methods (1.2) and (1.3). A survey of such results can be found in [1]–[9], [14], and the references there.

It turns out that so far there are two ways of studying method (1.2): Newton–Kantorovich-type local and semilocal convergence results depending on a domain containing the initial guess x_0 and Lipschitz conditions on $F'(x)$ [4], [6], [9]; Smale-type theorems that require information only at x_0 and the analyticity of F [11]–[14].

Moreover, Wang [12] introduced the weaker than Smale's gamma γ-condition and successfully applied it to Newton and Newton-type methods. Yakoubson [14] extended Smale's work for the Secant method using a strong analyticity assumption on operator F.

The results mentioned above are based on the assumption that the sequence

$$\left\| \frac{F'(x_0)^{-1}F^{(n)}(x_0)}{n!} \right\| \quad (n \geq 2), \tag{1. 5}$$

is bounded above by

$$\gamma(F, x_0) = \sup_{k\geq 2} \left\| \frac{F'(x_0)^{-1}F^{(n)}(x_0)}{n!} \right\|^{\frac{1}{n-1}}. \tag{1. 6}$$

However, this kind of assumption may not be reasonable. Particularly, for some concrete and special operators appearing in connection with the Durand–Kerner method, it is really so [8].

Here we provide a convergence analysis for the Secant method using an even weaker version of Wang's gamma condition (see (2.1)). It turns out that even in the special case when method (1.3) reduces to (1.2) our error bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$ are finer than the ones in [12] and the information on the location of the solution x^* at least as precise. Note also that these advantages are obtained under the same computational cost. Numerical examples are also provided.

2. Semilocal Convergence Analysis of Method (1.3)

Let $x_0 \in X$ and $r > 0$. We denote by $U(x_0, r) = \{x \in X\colon \|x - x_0\| < r\}$.

We introduce the (γ_0, γ) condition:

Definition 2.1. Suppose:

$$0 < \gamma_0 \leq \gamma. \tag{2. 1}$$

We say F satisfies the gamma (γ_0, γ) condition at $x_0 \in D$ in $\overline{U}(x_0, r) \subseteq D$ if operator F is Fréchet-differentiable at $x = x_0$, $F'(x_0)^{-1} \in L(Y, X)$ such that for all $r < \left(1 - \frac{\sqrt{2}}{2}\right)\frac{1}{\gamma_0}$, $x, y, w \in \overline{U}(x_0, r)$

$$\begin{aligned} &\|F'(x_0)^{-1}([x,y] - [y,w])\| \\ &\leq \int_0^1 \int_0^1 \frac{2\gamma[t\|x-y\| + (1-t)\|y-w\|]dsdt}{[1 - \gamma\|s(tx + (1-t)y) + (1-s)(ty + (1-t)w) - x_0\|]^3}, \end{aligned} \tag{2. 2}$$

and

$$\begin{aligned}
&\|F'(x_0)^{-1}([x,y]-F'(x_0))\| \\
&\leq \int_0^1\int_0^1 \frac{2\gamma_0\|x_0-tx-(1-t)y\|dsdt}{[1-s\gamma_0\|x_0-tx-(1-t)y\|]^3} \\
&= \int_0^1 [1-\gamma_0\|x_0-tx_{k-1}-(1-t)x_k\|]^{-2}-1. \qquad (2.\ 3)
\end{aligned}$$

Example 2.2. Let us provide a class of operators that satisfies both (2.2) and (2.3). For simplicity, we set $\gamma_0=\gamma$, and assume F is twice Fréchet-differentiable on $\overline{U}(x_0,r)$ satisfying:

$$\|F'(x_0)^{-1}F''(x)\| \leq \frac{2\gamma}{(1-\gamma\|x-x_0\|)^3}. \qquad (2.\ 4)$$

Note that condition (2.4) used in [12] requires the existence of the second Fréchet-derivative, whereas we only require the existence of the first derivative. It is known that $\gamma(F,x_0)\leq\gamma$ [11], [13], [14], which is the γ-motivation for our study. Moreover, assume divided difference $[x,y]$ is given by

$$[x,y]=\int_0^1 F'[y+t(x-y)]dt \qquad (2.\ 5)$$

for all $x,y\in\overline{U}(x_0,r)\subseteq D$, which holds in many interesting cases [7], [8]. Then using (2.4), we can have in turn:

$$\begin{aligned}
&\|F'(x_0)^{-1}([x,y]-[y,w])(y-w)\| \\
&= \Big\|\int_0^1\int_0^1 F''[s(tx+(1-t)y)+(1-s)(ty+(1-t)w]ds \\
&\qquad \cdot [t(x-y)+(1-t)(y-w)]dt(y-w)\Big\| \\
&\leq \int_0^1\int_0^1 \frac{2\gamma(t\|x-y\|+(1-t)\|y-w\|)\|y-w\|dsdt}{[1-\gamma\|s(tx+(1-t)y)+(1-s)(ty+(1-t)w)-x_0\|]^3}, \qquad (2.\ 6)
\end{aligned}$$

which justifies condition (2.2). Moreover using again (2.4) we can obtain

$$\begin{aligned}
&\|F'(x_0)^{-1}([x,y]-F'(x_0)) = \Big\|F'(x_0)^{-1}\int_0^1 [F'(tx+(1-t)y)-F'(x_0)]dt\Big\| \\
&= \Big\|\int_0^1\int_0^1 F'(x_0)^{-1}F''[((1-s)x_0 \\
&\quad + s(tx+(1-t)y))][x_0-ty-(1-t)x]dsdt\Big\| \\
&\leq \int_0^1\int_0^1 \frac{2\gamma\|x_0-tx-(1-t)y\|dsdt}{[1-s\gamma\|x_0-tx-(1-t)y\|]^3}, \qquad (2.\ 7)
\end{aligned}$$

which justifies condition (2.3).

It is convenient for us to define scalar function f, and scalar sequences $\{r_n\}$, $\{s_n\}$,

$\{t_n\}$, for some $\alpha \geq 0$, $a \geq 0$, $b \geq 0$ by

$$
\begin{aligned}
f(t) &= \frac{\alpha}{\gamma} - t + \frac{\gamma t^2}{1-\gamma t}, \quad t \neq \frac{1}{\gamma}, && (2.\ 8)\\
r_{-1} &= -a, \quad r_0 = 0, \quad r_1 = b,\\
r_{n+1} &= r_n - \\
& \int_0^1 \int_0^1 \frac{2\gamma[t(r_{n-1}-r_{n-2})+(1-t)(r_n-r_{n-1})]dsdt}{[1-\gamma s(tr_{n-2}+(1-t)r_{n-1})-\gamma(1-s)(tr_{n-1}+(1-t))r_n]^3}\\
& \quad \times (1-t)r_n]^{-2}\}g(r_{n-1},r_n)(r_n-r_{n-1})dt \ (n \geq 1), && (2.\ 9)\\
s_{-1} &= -a, \quad s_0 = 0, \quad s_1 = b,\\
s_{n+1} &= s_n - f(s_n)g(s_{n-1},s_n) \ \ (n \geq 0), && (2.\ 10)
\end{aligned}
$$

and

$$
\begin{aligned}
t_{-1} &= -a, \quad t_0 = 0,\\
t_{n+1} &= t_n - \frac{t_n - t_{n-1}}{f(t_n)-f(t_{n-1})} f(t_n), && (2.\ 11)
\end{aligned}
$$

where, function g is given by:

$$
g(r,s) = \frac{(1-\gamma_0 r)(1-\gamma_0 s)}{2(1-\gamma_0 r)(1-\gamma_0 s)-1} \quad \text{for all } r,s \in \left[0, \left(1-\frac{\sqrt{2}}{2}\right)\frac{1}{\gamma_0}\right). \qquad (2.\ 12)
$$

We need the following lemma on majorizing sequence $\{t_n\}$.

Lemma 2.3. *Assume:*

$$
\alpha = b\gamma \frac{1+2a\gamma}{1+a\gamma} \leq 3-2\sqrt{2}. \qquad (2.\ 13)
$$

Then sequence $\{t_n\}$ generated by (2.11) is monotonically increasing and converges to the smallest root

$$
t^* = \frac{1+\alpha-\sqrt{(1+\alpha)^2-8\alpha}}{4\gamma} \qquad (2.\ 14)
$$

of equation $f(t)=0$, with the largest root being

$$
t^{**} = \frac{1+\alpha+\sqrt{(1+\alpha)^2-8\alpha}}{4\gamma}. \qquad (2.\ 15)
$$

Moreover, the following estimate holds for

$$
q = \frac{1-\gamma t^{**}}{1-\gamma t^*}, \quad q_0 = \lambda \frac{t^*}{t^{**}}, \quad q_1 = q\frac{t^*-b}{t^{**}-b},
$$

and p_n be the Fibonacci sequence:

$$
t^* - t_n = \begin{cases} e_n(t^{**}-t^*), & \alpha < 3-2\sqrt{2} \\ h_n^{-1}, & \alpha = 3-2\sqrt{2}, \end{cases} \quad (n \geq 0), \qquad (2.\ 16)
$$

where,

$$
e_n = \frac{q_0^{p_{n-2}} q_1^{p_{n-1}}}{q - q_0^{p_{n-2}} q_1^{p_{n-1}}}, \qquad (2.\ 17)
$$

and

$$
h_n = \frac{\gamma p_{n-1}}{1-\gamma t^*} + \frac{p_{n-1}}{t^*-b} + \frac{p_{n-2}}{t^*}. \qquad (2.\ 18)
$$

Proof. We shall show estimates:

$$t_k < t_{k+1}, \tag{2. 19}$$

and

$$t_k < t^* \tag{2. 20}$$

hold true for all $k \geq 0$. Estimates (2.19) and (2.20) hold true by the initial conditions for $k = 0$. Let us assume that they hold true for $k = 0, 1, \ldots, n-1$ for $n \geq 1$ a fixed natural number.

In view of the induction hypotheses and (2.11), we can obtain in turn for $[s,t] = [s,t;f]$:

$$\begin{aligned} t^* - t_n &= t^* - t_{n-1} + [t_{n-2}, t_{n-1}, f]^{-1}(f(t_{n-1}) - f(t^*)) \\ &= [t_{n-2}, t_{n-1}]^{-1}([t_{n-2}, t_{n-1}] - [t_{n-1}, t^*])(t^* - t_{n-1}) \\ &= -[t_{n-2}, t_{n-1}]^{-1}(t^* - t_{n-1})(t^* - t_{n-2})[t_{n-1}, t_{n-2}, t^*], \end{aligned} \tag{2. 21}$$

where by $[s,t,u]$ we mean $[s,t,u;f]$ the divided difference of order two of scalar function f at the points s, t and u.

It follows that there exist $\beta_0 \in (t_{n-2}, t_{n-1})$, and $\beta \in (t_{n-2}, t^*)$

$$[t_{n-2}, t_{n-1}] = f'(\beta_0) < 0 \tag{2. 22}$$

and

$$[t_{n-1}, t_{n-2}, t^*] = \frac{f''(\beta)}{2} > 0, \tag{2. 23}$$

since

$$-1 < f'(t) < 0, \tag{2. 24}$$

and

$$f''(t) = \frac{2\gamma}{(1-\gamma t)^3} > 0, \tag{2. 25}$$

for $t \in \left[0, \left(1 - \frac{\sqrt{2}}{2}\right)\frac{1}{\gamma}\right)$, which together with (2.21) imply (2.20) for $n = k$.

Using (2.11) we can write

$$t_{n+1} - t_n = (t^* - t_n)[t_{n-1}, t_n]^{-1}[t^*, t_n] > 0, \tag{2. 26}$$

which implies (2.19) for $n = k$. That completes the induction for estimates (2.19) and (2.20). It follows that sequence $\{t_n\}$ converges to t^*.

In view of (2.14), (2.15) and (2.21), we can easily see that

$$\begin{aligned} q\frac{t^* - t_{n+1}}{t^{**} - t_{n+1}} &= q\frac{t^* - t_n}{t^{**} - t_n} q\frac{t^* - t_{n-1}}{t^{**} - t_{n-1}} \\ &= q_0^{p_{n-1}} q_1^{p_n} \quad (n \geq 0). \end{aligned} \tag{2. 27}$$

Clearly, if $\alpha < 3 - 2\sqrt{2}$, then $t^* \neq t^{**}$. It then follows from (2.28) that the first part of estimate (2.16) holds true. Otherwise, set $\lambda_n = \gamma(t^* - t_n)$ and $\mu_n = \sqrt{2}\lambda_n$. It then follows from (2.21) that

$$t^* - t_{n+1} = \frac{\gamma(t^* - t_n)(t^* - t_{n-1})}{[1 - 2(1 - \gamma t_{n-1})(1 - \gamma t_n)](1 - \gamma t^*)} \quad (n \geq 0), \tag{2. 28}$$

from which it follows that

$$\lambda_{n+1} = \frac{\lambda_n \lambda_{n-1}}{\lambda_{n-1} + \lambda_n + \sqrt{2}\lambda_{n-1}\lambda_n} \quad (n \geq 0), \tag{2. 29}$$

and

$$\mu_{n+1} = \frac{\mu_n \mu_{n-1}}{\mu_{n-1} + \mu_n + \mu_{n-1}\mu_n} \quad (n \geq 0), \tag{2. 30}$$

or

$$\frac{1}{\mu_n} = \frac{p_{n-2}}{\mu_0} + \frac{p_{n-1}}{\mu_1} + p_n - 1 \quad (n \geq 0), \tag{2. 31}$$

by the definition of the Fibonacci sequence ($p_{-2} = 1$, $p_{-1} = 0$, $p_{n+1} = p_n + p_{n-1}$ $(n \geq -1)$). It then follows by the definition of λ_n that the second part of estimate (2.16) also holds true.

That completes the proof of Lemma 2.3.

Corollary 2.4. *If:*

(a) $a < 3 - 2\sqrt{2}$, *then for all* $n \geq 0$

$$0 \leq t^* - t_n \leq \frac{q_0^{p_n}}{q - q_0^{p_n}}(t^{**} - t^*) \leq \frac{t^{**} - t^*}{q - q_0}\left(q_0^{\frac{1}{\sqrt{5}}}\right)^{\left(\frac{1+\sqrt{5}}{2}\right)^n}. \tag{2. 32}$$

(b) $a = 3 - 2\sqrt{2}$, *then for all* $n \geq 1$

$$0 \leq t^* - t_n \leq \frac{t^* - b}{p_{n-1}} \leq \sqrt{5}(t^* - b)\left(\frac{2}{1+\sqrt{5}}\right)^{n-1}. \tag{2. 33}$$

Proof. The result follows immediately from estimate (2.16) and the fact that

$$p_n \geq \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n \quad (n \geq 0). \tag{2. 34}$$

Remark 2.5. (a) For $F = f$, $D = \left(-\infty, \frac{1}{\gamma}\right)$, $\gamma = \gamma_0$, and $X = Y = \mathbf{R}$, x_n becomes t_n and x^* is t^*. That is estimate (2.16) is sharp. Note also that f satisfies (2.5).

(b) In the special case when $x_{-1} = x_0$ condition (2.13) reduces to Wang's [12] sufficient convergence condition for Newton's method

$$\alpha = b\gamma \leq 3 - 2\sqrt{2}. \tag{2. 35}$$

(c) If we set $X = Y = \mathbf{R}$, then it can easily be seen that condition (2.5) is satisfied. Other examples which satisfy (2.5) can be found in [7], [8].

Using induction on n it follows immediately from the definitions of sequences $\{r_n\}$, $\{s_n\}$, $\{t_n\}$ that the following relationship holds between them:

Lemma 2.6. *If* $\gamma_0 < \gamma$, *and (2.13) holds true, then*

$$r_n < s_n < t_n \quad (n > 1), \tag{2. 36}$$

$$0 < r_{n+1} - r_n < s_{n+1} - s_n < t_{n+1} - t_n \quad (n > 1), \tag{2. 37}$$

$$0 \leq r^* - r_n \leq s^* - s_n \leq t^* - t_n \quad (n \geq -1), \tag{2. 38}$$

and

$$r^* \leq s^* \leq t^*, \tag{2. 39}$$

where, $r^* = \lim_{n\to\infty} r_n$, *and* $s^* = \lim_{n\to\infty} s_n$.

Note that if $\gamma_0 = \gamma$ (2.37)–(2.40) hold true as equalities.

Remark 2.6. In view of (2.37)–(2.40), one hopes that sequences $\{r_n\}$ and $\{s_n\}$ may converge under conditions weaker than (2.14). Such conditions already exist in the literature. We refer the reader to [5, 4, 6] where we provided sufficient convergence conditions for sequences more general than $\{r_n\}$ and $\{s_n\}$.

However, we do not pursue this here. Instead we provide the main semilocal convergence theorem for the Secant method (1.3), under the (γ_0, γ) condition:

Theorem 2.7. *Let operator F satisfy the (γ_0, γ) condition at $x_0 \in D$ in*

$$\overline{U\left(x_0, \left(1 - \frac{\sqrt{2}}{2}\right)\frac{1}{\gamma_0}\right)} \subseteq D,$$

let $x_{-1}, x_0 \in D$ with $\|x_0 - x_{-1}\| \le a$, and

$$\|[x_{-1}, x_0]^{-1} F(x_0)\| \le b. \tag{2. 40}$$

Furthere, assume condition (2.13) holds true.

Then, sequence $\{x_n\}$ generated by Secant method (1.3) is well defined, remains in $\overline{U}(x_0, r^*)$ for all $n \ge 0$, and converges to a unique solution x^* of equation $F(x) = 0$ in $\overline{U}(x_0, r^*)$.

Moreover, the following estimates hold for all $n \ge -1$

$$\|x_{n+1} - x_n\| \le r_{n+1} - r_n, \tag{2. 41}$$

and

$$\|x_n - x^*\| \le r^* - r_n. \tag{2. 42}$$

Furthermore, if there exists $R \in \left(r^*, \left(1 - \frac{\sqrt{2}}{2}\right)\frac{1}{\gamma_0}\right]$ satisfying

$$\int_0^1 [1 - \gamma(tR + (1-t)r^*)]^{-2} dt = 2, \tag{2. 43}$$

then the solution x^* is unique in $U(x_0, R)$.

Proof. We shall show:

$$\|x_{k+1} - x_k\| \le r_{k+1} - r_k, \tag{2. 44}$$

and

$$\overline{U}(x_{k+1}, r^* - r_{k+1}) \subseteq \overline{U}(x_k, r^* - r_k) \tag{2. 45}$$

hold for all $k \ge -1$.

For every $z \in \overline{U}(x_1, r^* - r_1)$

$$\|z - x_0\| \le \|x - x_1\| + \|x_1 - x_0\| \le r^* - r_1 + r_1 = r^* - r_0$$

implies $z \in \overline{U}(x_0, r^* - r_0)$. We also have that (2.41) holds, and

$$\|x_1 - x_0\| = \|[x_{-1}, x_0]^{-1} F(x_0)\| = b.$$

Therefore (2.45) and (2.46) hold for $k = -1, 0$. Let us assume $x_1, x_2, \ldots, x_k$ are well defined and (2.45), (2.46) hold true for $n = 0, 1, \ldots, k-1$, where $k \ge 1$ is a fixed natural number.

We shall establish the existence of $[x_{k-1}, x_k]^{-1}$ which will also imply that x_{k+1} is well defined. Using condition (2.3) for $x = x_{k-1}$ and $y = x_k$, and the induction hypotheses we obtain

$$\begin{aligned}
&\|F'(x_0)^{-1}(F'(x_0)-[x_{k-1},x_k])\| \\
&\le \int_0^1\int_0^1 \frac{2\gamma_0\|x_0-tx_{n-1}-(1-t)x_k\|dsdt}{[1-s\gamma_0\|x_0-tx_{k-1}-(1-t)x_k\|]^3} \\
&= \int_0^1 [1-\gamma_0\|x_0-tx_{k-1}-(1-t)x_k\|]^{-2}dt-1 \\
&\le \int_0^1 [1-\gamma_0(r_k+t(r_{k-1}-r_k))]^{-2}dt-1 \\
&= \frac{1}{(1-\gamma_0 r_{k-1})(1-\gamma_0 r_k)}-1<\frac{1}{(1-\gamma_0 r^*)^2}-1\le 1. \qquad (2.\ 46)
\end{aligned}$$

It follows from (2.46) and the Banach Lemma on invertible operators [5], [9] that $[x_{k-1},x_k]^{-1}$ exists, and

$$\begin{aligned}
\|[x_{k-1},x_k]^{-1}F'(x_0)\| &\le \left[1-\left(\frac{1}{(1-\gamma_0 r_{k-1})(1-\gamma_0 r_k)}-1\right)\right]^{-1} \\
&= \frac{(1-\gamma_0 r_{k-1})(1-\gamma_0 r_k)}{2(1-\gamma_0 r_{k-1})(1-\gamma_0 r_k)-1}=g(r_{k-1},r_k).
\end{aligned} \qquad (2.\ 47)$$

In view of (1.3), condition (2.2) for $x=x_{k-2}$, $y=x_{k-1}$, and $w=x_k$ gives:

$$\begin{aligned}
&\|F'(x_0)^{-1}F(x_k)\|=\|F'(x_0)^{-1}([x_{k-2},x_{k-1}]-[x_{k-1},x_k])(x_{k-1}-x_k)\| \\
&\le \int_0^1\int_0^1 \frac{2\gamma[t\|x_{k-2}-x_{k-1}\|+(1-t)\|x_{k-1}-x_k\|]\|x_k-x_{k-1}\|dsdt}{[1-\gamma\|s(tx_{k-2}+(1-t)x_{k-1})+(1-s)(tx_{k-1}+(1-t)x_k)-x_0\|]^3} \\
&\le \int_0^1\int_0^1 \frac{2\gamma[t(r_{k-1}-t_{k-2})+(1-t)(r_k-r_{k-1})](r_k-r_{k-1})dsdt}{[1-\gamma s(tr_{k-2}+(1-t)r_{k-1})-\gamma(1-s)(tr_{k-1}+(1-t)r_k]^3} \\
&= h(r_{k-1},r_k). \qquad (2.\ 48)
\end{aligned}$$

By (1.3), (2.10), (2.47) and (2.48) we get:

$$\begin{aligned}
\|x_{k+1}-x_k\| &\le \|[x_{k-1},x_k]^{-1}F'(x_0)\|\,\|F'(x_0)^{-1}F(x_k)\| \\
&\le g(r_{k-1},r_k)h(r_{k-1},r_k)=r_{k+1}-r_k, \qquad (2.\ 49)
\end{aligned}$$

which shows (2.45) for all $k\ge -1$.

We also have that for every $z\in\overline{U}(x_{k+1},x^*-r_{k+1})$ we get

$$\|z-x_k\|\le\|z-x_{k+1}\|+\|x_{k+1}-x_k\|\le r^*-r_{k+1}+r_{k+1}-r_k=r^*-r_k.$$

That is,

$$z\in\overline{U}(x_k,r^*-r_k), \qquad (2.\ 50)$$

which implies (2.46). The induction for (2.45) and (2.46) is now complete.

Lemma 2.6 imply that sequence $\{x_n\}$ is Cauchy (since $\{r_n\}$ is Cauchy (since $\{r_n\}$ is a Cauchy sequence) in a Banach space X and as such it converges to some $x^*\in\overline{U}(x_0,r^*)$ (since $\overline{U}(x_0,r^*)$ is a closed set). By letting $n\to\infty$ in (1.3) (or $k\to\infty$ in (2.48)) we obtain $F(x^*)=0$.

We shall show uniqueness of the solution x^* first in $\overline{U}(x_0,r^*)$. Let y^* be a solution of equation $F(x)=0$ in $\overline{U}(x_0,r^*)$. Set $L=[x^*,y^*]$. In view of (2.3), we get

$$\begin{aligned}
&\|F'(x_0)^{-1}(F'(x_0)-L)\| \\
&\leq \int_0^1 [1-\gamma_0(t\|x_0-y^*\|+(1-t)\|x^*-x_0\|)]^{-2}dt-1 \qquad (2.\ 51)\\
&\leq \int_0^1 [1-\gamma_0(tr^*+(1-t)r^*)]^{-2}dt-1 \\
&= (1-\gamma_0 r^*)^{-2}-1<1. \qquad (2.\ 52)
\end{aligned}$$

It follows from (2.52) and the Banach Lemma on invertible operators that L^{-1} exists. Thus from the identity

$$F(x^*)-F(y^*)=[x^*,y^*](x^*-y^*), \qquad (2.\ 53)$$

we deduce $x^*=y^*$.

If $R\in\left(r^*,\left(1-\frac{\sqrt{2}}{2}\right)\frac{1}{\gamma_0}\right]$ satisfies (2.44) and y^* is a solution of equation $F(x)=0$ in $U(x_0,R)$, then as in (2.51) we get

$$\|F'(x_0)^{-1}[F'(x_0)-L]\|<\int_0^1 [1-\gamma_0(tR+(1-t)r^*]^{-2}dt-1=1.. \qquad (2.\ 54)$$

Hence, again we deduce $x^*=y^*$.

That completes the proof of the theorem.

Remark 2.8. In view of Lemma 2.6 $\{s_n\}$, s^* or $\{t_n\}$, t^* can replace $\{r_n\}$, r^* respectively in Theorem 2.7. Note that we could have used easier $\{t_n\}$, t^* in Theorem 2.7 but we wanted to leave the results as uncluttered as possible using the finer possible majorizing sequence $\{r_n\}$.

We now complete this study with numerical examples.

Example 2.9. Let $X=Y=\mathbf{R}$, $\gamma_0=\gamma=\alpha>0$, $D=\left[0,\frac{1}{\gamma}\right)$, and define function f on D by

$$f(t)=1-t+\frac{\gamma t^2}{1-\gamma t}. \qquad (2.\ 55)$$

We shall use the Secant method (1.3) to find the smallest positive zero of equation $f(t)=0$. Let $t_{-1}=-.000001$, and $t_0=0$. Using (2.17) we can have for $\alpha=\frac{1}{2}(3-2\sqrt{2})=.0857864=\alpha_0$ and $\alpha=\frac{3}{4}(3-2\sqrt{2})=.1286797=\alpha_1$ the following table:

Table 1: Numerical Values for t^*-t_n

n	α_0	α_1
0	1.119	1.232
1	1.188×10^{-1}	2.322×10^{-1}
2	1.522×10^{-2}	5.891×10^{-2}
3	2.618×10^{-4}	4.362×10^{-3}
4	5.937×10^{-7}	9.145×10^{-5}
5	2.324×10^{-11}	1.463×10^{-7}

Example 2.10. Let $X=C[0,1]$, the space of all functions v, continuous on the interval $[0,1]$, with norm

$$\|v\|=\max_{0\leq s\leq 1}|v(s)|, \quad D=\overline{U}(0,1), \quad \lambda\in\mathbf{R}, \quad K(s,t)$$

a continuous function of two variables $s, t \in [0,1]$, and $h(s)$ a continuous function on $[0,1]$. Consider nonlinear integral equation

$$v(s) = \lambda v(s) \int_0^1 K(s,t)v(t)dt + h(s). \tag{2. 56}$$

Equations like (2.56) appear in connection with radiative transfer, neutron transport, and in the kinetic theory of gasses [2], [3], [10].

In order for us to solve equation (2.56), we define operator T on D by

$$T(v(s)) = \lambda v(s) \int_0^1 K(s,t)v(t)dt + h(s) - x(s). \tag{2. 57}$$

Let us consider some special cases of interest:

Case 1 (Chandrasekhar's equation [2], [3], [10]). Set $\lambda = \frac{1}{4}$, $K(s,t) = \frac{s}{s+t}$, $s + t \neq 0$, and $h(s) = 1$. Choose $v_0(s) = 1$, and $v_{-1}(s) = 1.0000001$. Let us also denote by δ an upper bound on $\|T'(v_0(s))^{-1}\|$. That is,

$$\|T'(v_0(s))^{-1}\| \leq \delta. \tag{2. 58}$$

We can have:

$$\|T'(v_0(s))^{-1}T(v_0(s))\| \leq \delta\|T(v_0(s))\| \leq \delta|\lambda| \ln 2 = b, \tag{2. 59}$$

and

$$\|T''(v(s))\| \leq 2|\lambda| \max_{0\leq s\leq 1}\left|\int_0^1 \frac{s}{s+t}dt\right| \leq 2|\lambda| \ln 2, \tag{2. 60}$$

since,

$$\max_{0\leq s\leq 1}\left|\int_0^1 \frac{s}{s+t}dt\right| = \ln 2. \tag{2. 61}$$

Condition (2.4) certainly holds if

$$2\delta|\lambda| \ln 2 \leq 2\gamma.$$

Hence, we can set $\gamma = \delta|\lambda| \ln 2$. Using the choices above we get

$$b = \gamma = .265197108.$$

Hypothesis (2.13) is satisfied, since

$$\alpha = .070329508 < 3.2\sqrt{2} = .17157287.$$

Hence, the conclusions of Theorem 2.7 can apply, since any solution $v^*(s)$ of equation $F(v(s)) = 0$, satisfies (2.56).

Case 2. Let $D = U(0, 1-c)$ for some $c \in [0,1)$, set $h(s) = v^3(s) - c + 1$, and $v_0(s) = 1$. As above it can easily be seen that we can set for

$$d = \max_{0\leq s\leq 1}\left|\int_0^1 K(s,t)dt\right| < \infty:$$

$$b = [1 - c + d|\lambda|]\delta, \quad \gamma = [2 - c + d|\lambda|]\delta, \tag{2. 62}$$

and

$$\gamma_0 = \frac{1}{2}[3 - c + 2d|\lambda|]\delta. \tag{2. 63}$$

In view of (2.62), and (2.63) we have:

$$\gamma_0 < \gamma \text{ for all } d, \delta, \lambda \in R \text{ and } c \in [0,1). \tag{2. 64}$$

It also follows from the above choices of b and γ that for v_{-1}, c close enough to v_0, 1 respectively, and λ sufficiently small condition (2.13) holds true. That is as in Case 1, the conclusions of Theorem 2.7 apply. Note however that in this case finer sequence $\{s_n\}$ than $\{t_n\}$ can be used as a majorizing sequence for Secant method (1.3) (see also Lemma 2.6).

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On θ-Euclidean Fuzzy k-ideals of Semirings

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Abstract. The concept of fuzzy ideals is extended by introducing θ-Euclidean fuzzy k-ideals in semirings. In this paper, factorization theorem of homomorphism on θ-Euclidean fuzzy k-ideal of semiring R is proved.

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1. Introduction

The concept of fuzzy subset was introduced by Zadeh [15]. Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. The concept of fuzzy subgroup was introduced by Rosenfeld [11] and was further studied by several authors [9, 6]. Liu [8] studied fuzzy ideals in rings. Subsequently, many authors [10, 12, 14] fuzzified certain concepts on rings and ideals. The theory of semirings has been studied by many authors [1, 3]. The fuzzy ideal of semiring is a good tool for us to study the fuzzy algebraic structure. Dutta and Biswas [4] studied different kinds of fuzzy ideals such as fuzzy k-ideals and fuzzy prime k-ideals of semirings and characterized fuzzy prime k-ideals of semirings of non-negative integers and determined all its prime k-ideals. Baik *et al.* [2] studied further results on fuzzy k-ideals of semirings. Further, Zhan *et al.* [19] fuzzified the concept of left k-ideals of semirings over t-norm and studied their related results. Then, Jun *et al.* [13] extended the concept of L-fuzzy ideal of a ring to a semiring. Moreover, Zhan *et al.* [17] fuzzified the concept h-ideals of hemirings and studied several results. Zhan [16] extended the concept of fuzzy left h-ideals of hemirings to fuzzy left h-ideals of hemirings over t-norm and studied many properties. Also, Zhan [18] introduced intuitionistic M-fuzzy h-ideals in M-hemirings. Koç and Balkanay [5] introduced the concept of θ-Euclidean L-fuzzy ideals of rings. Latha and Williams [7] introduced a notion of θ-Euclidean fuzzy k-ideals of semirings. In this paper, we study the factorization theorem on homomorphism for a θ-Euclidean fuzzy k-ideals in semirings.

2. Preliminaries

In this section, we cite the fundamental definitions that are used in the sequel:
Definition 1. [1] An algebraic system $(R;+,.)$ is said to be a *semiring* if it satisfies the following conditions:

(1): $(R;+)$ and $(R;.)$ are semigroups,
(2): $a.(b+c) = a.b + a.c$ and $(b+c).a = b.a + c.a$ for all $a, b, c \in R$.

A semiring R may have an identity 1, defined by $1.a = a = a.1$ and a zero 0, defined by $0 + a = a = a + 0$ and $a.0 = 0 = 0.a$ for all $a \in R$.
In what follows,we denote $x.y = xy$, for all $x, y \in R$.

Definition 2. [9] A non-empty subset I of R is said to be a *left ideal*(resp., *right ideal*) if $x, y \in I$ and $r \in R$ imply that $x + y \in I$ and $rx \in I$ $(resp., xr \in I)$.

If I is both left and right ideal of R, we say I is a *two-sided ideal, or simply ideal,* of R.
Definition 3. [4] An ideal I of a semiring R is said to be a *k-ideal* if $a \in I$ and $x \in R$, and if $x + a \in I$ or $a + x \in I$ then $x \in I$.
Definition 4. [10] A fuzzy subset $\mu : R \longrightarrow [0,1]$ of a semiring R is said to be a fuzzy left (resp., *right*) ideal of R if

(F1): $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$
(F2): $\mu(xy) \geq \mu(y)$ $(resp. right, \mu(xy) \geq \mu(x))$ for all $x, y \in R$.

Definition 5. [4] A fuzzy ideal of a semiring R is said to be a fuzzy k-ideal of R if

$$\mu(x) \geq \min\{\max\{\mu(x+y), \mu(y+x)\}, \mu(y)\},$$

for all $x, y \in R$.

If R is an additively commutative semiring then the condition reduces to

$$\mu(x) \geq \min\{\mu(x+y), \mu(y)\}$$

for all $x, y \in R$.
Definition 6. [7] Let R be a semiring and let $\theta : R \to [0,1]$ be a non-constant fuzzy subset of R. A fuzzy ideal $\mu : R \to [0,1]$ is called a θ-Euclidean k-fuzzy ideal if μ satisfies the following axioms:

(F3): $\mu(x) \geq \min\{\max\{\mu(x+y), \mu(y+x)\}, \mu(y)\}$, for all $x, y \in R$;
(F4): For any $x, y \in R$ with $y \neq 0$, there exist elements $q, r \in R$ such that $x = yq + r$, where either $r = 0$ or else $\max\{\mu(r), \theta(r)\} \geq \max\{\mu(y), \theta(y)\}$ for all $x, y \in R$.

Example
Let R be the set of Natural Numbers including zero and $\phi : R \to [0,1]$ be a fuzzy subset defined by

$$\phi(a) = \begin{cases} 1 & if \quad a = 0, \\ \frac{1}{3} & if \quad a \text{ is non-zero even}, \\ 0 & if \quad a \text{ is odd}. \end{cases}$$

Let $\theta : R \rightarrow [0,1]$ be a fuzzy subset defined by

$$\theta(a) = \begin{cases} 0 & if \quad a = 0, \\ \frac{1}{3} & if \quad a = 3, 5, 7, ..., \\ \frac{1}{|a|} & otherwise. \end{cases}$$

Clearly ϕ is a fuzzy k-ideal of R, also ϕ is a θ-Euclidean fuzzy k-ideal of R.

Example

Let R be the set of Natural Numbers including zero and $\phi : R \rightarrow [0,1]$ be a fuzzy subset defined by

$$\phi(a) = \begin{cases} 1 & if \quad a = 0, \\ \frac{1}{3} & if \quad a \text{ is non} - \text{zero even}, \\ 0 & if \quad a \text{ is odd}. \end{cases}$$

Let $\theta_1 : R \rightarrow [0,1]$ be a fuzzy subset defined by

$$\theta_1(a) = \begin{cases} 0 & if \quad a = 0 \\ \frac{1}{|a|} & otherwise. \end{cases}$$

So ϕ is a fuzzy k-ideal but θ_1 is not a θ-Euclidean fuzzy k-ideal of R.

3. Fuzzy Quotient Semiring

In this section, we introduce the notion of *fuzzy quotient semiring* and study the factorization theorem of homomorphism on θ-Euclidean fuzzy k-ideal of semiring R.

Definition 7. Let μ be a fuzzy ideal of R. For all $x \in R$, let $x + \mu$ be the fuzzy subset of R defined by

$$(x + \mu)(y) = \mu(y - x)$$

for all $y \in R$. The fuzzy subset $x + \mu$ is called a fuzzy coset of the fuzzy ideal R.

The set of all such fuzzy cosets will be denoted by $\frac{R}{\mu}$. Two binary operations (denoted by + and .) on $\frac{R}{\mu}$ are defined as follows:

$$(x + \mu) + (y + \mu) = (x + y) + \mu$$

and

$$(x + \mu) . (y + \mu) = (x.y) + \mu$$

for all $x, y \in R$. The above two operations are well defined and makes $\frac{R}{\mu}$ into a semiring, called the *fuzzy quotient semiring* of R by μ [5].

Theorem 8. *Let R be a semiring and $\mu : R \rightarrow [0,1]$ be a $\theta-$Euclidean k-fuzzy ideal, $n : R \rightarrow \frac{R}{R_\mu}$ be the natural homomorphism. Also suppose that $\theta(a) = \theta(b)$ when $a - b \in \ker n$, for all $a, b \in R$. Let $\phi : \frac{R}{R_\mu} \rightarrow [0,1]$ be defined by*

$$\phi(a + R_\mu) = \mu(a).$$

Then there exists a unique $\theta^* (= n(\theta)) -$Euclidean k-fuzzy ideal $\phi : \frac{R}{R_\mu} \rightarrow [0,1]$ with the property that $\mu = \phi \circ n$.

Proof: First we shall show that the function ϕ is well defined. For all $a + R_\mu, b + R_\mu \in \frac{R}{R_\mu}$ and $a + R_\mu = b + R_\mu$, then there exists $x \in R_\mu$ such that $a - b = x$. Using the definition of R_μ we obtain

$$\begin{aligned}\mu(x) &= \mu(0)\\ \mu(0) &= \mu(x)\\ \mu(0) &= \mu(a-b)\\ 0 &= \mu(a)-\mu(b)\\ \mu(a) &= \mu(b)\\ \phi(a+R_\mu) &= \phi(b+R_\mu)\end{aligned}$$

Hence ϕ is well defined.
(i) For any $a+R_\mu, b+R_\mu \in \frac{R}{R_\mu}$,for all $a,b \in R$,then

$$\begin{aligned}\phi[(a+R_\mu)+(b+R_\mu)] &= \phi[(a+b)+R_\mu]\\ &= \mu(a+b)\\ &\geq \min[\mu(a),\mu(b)], by (F1)\end{aligned}$$

Thus

$$\phi[(a+R_\mu)+(b+R_\mu)] \geq \min[\phi(a+R_\mu)+\phi(b+R_\mu)].$$

Also,we have

$$\phi[(a+R_\mu).(b+R_\mu)] = \phi[(ab)+R_\mu] = \mu(ab) \geq \mu(b), by (F2)$$

Thus

$$\phi[(a+R_\mu).(b+R_\mu)] \geq \phi(b+R_\mu).$$

(ii) For any $a+R_\mu, b+R_\mu \in \frac{R}{R_\mu}$, for all $a,b \in R$, then

$$\begin{aligned}\phi[(a+R_\mu)] &= \mu(a)\\ &\geq \min\{\max\{\mu(a+b),\mu(b+a)\},\mu(b)\},\end{aligned}$$

since μ is a $\theta-$ Euclidean k-fuzzy ideal

$$\begin{aligned}&= \min\{\max\{\phi(a+b+R_\mu),\phi(b+a+R_\mu)\},\phi(b+R_\mu)\}\\ &= \min\{\max\{\phi[(a+R_\mu)+(b+R_\mu)],\phi[(b+R_\mu)+(a+R_\mu)]\},\phi(b+R_\mu)\}\end{aligned}$$

Thus, ϕ is a k-fuzzy ideal.
(iii) For any $a+R_\mu, b+R_\mu \in \frac{R}{R_\mu}$ such that $R_\mu \neq b+R_\mu$, for all $a,b \in R$, then

$$\begin{aligned}&\Rightarrow b \notin R_\mu \Rightarrow \mu(b) \neq \mu(0)\\ &\Rightarrow b \neq 0.\end{aligned}$$

So $0 \neq b \in R$. Since μ is a $\theta-$Euclidean fuzzy k-ideal of R, then there exist elements $q,r \in R$ such that $a = bq+r$ where $r=0$ or else $\max\{\mu(r),\theta(r)\} \geq \max\{\mu(b),\theta(b)\}$.

$$\begin{aligned}a &= bq+r\\ a+R_\mu &= bq+r+R_\mu\\ a+R_\mu &= (bq+R_\mu)+(r+R_\mu)\\ a+R_\mu &= (b+R_\mu)(q+R_\mu)+(r+R_\mu)\end{aligned}$$

If $r=0$ then $r+R_\mu = 0+R_\mu$,thus $r+R_\mu = R_\mu$.For all $r,q \in R$, then $r+R_\mu, q+R_\mu \in \frac{R}{R_\mu}$.
Let $r+R_\mu = r'$.If $n(z) = r'$ and $n(r) = r'$ then $n(z-r) = 0$. This means $z-r \in \ker n$. we get $\theta(z) = \theta(r)$. Thus

$$n(\theta)\left(r'\right) = \bigcup\left\{\theta(z) \,|\, z \in n^{-1}\left(r'\right)\right\} = \theta(r).$$

If $\max\{\mu(r), \theta(r)\} \geq \max\{\mu(b), \theta(b)\}$, then

$$\max\{\phi(r+R_\mu), n(\theta)(r')\} \geq \max\{\phi(b+R_\mu), n(\theta)(b').\}$$

Thus

$$\max\{\phi(r+R_\mu), \theta^*(r+R_\mu)\} \geq \max\{\phi(b+R_\mu), \theta^*(b+R_\mu)\}.$$

Hence, μ is θ^*–Euclidean fuzzy k-ideal from $\frac{R}{R_\mu}$ to $[0,1]$.

Also, for all $a \in \mu, \mu(a) = \phi(a+R_\mu) = \phi(n(a)) = (\phi \circ n)(a)$. It means that $\mu = \phi \circ n$. Now, we need to show that this factorization is unique. Suppose $\phi' \circ n = \mu$, for some other $\theta^*(= n(\theta))$ –Euclidean fuzzy k-ideal $\phi' : \frac{R}{R_\mu} \longrightarrow [0,1]$.

If $a + R_\mu \in \frac{R}{R_\mu}$, then

$$\phi(a+R_\mu) = \mu(a) = (\phi' \circ n)(a) = \phi'(a+R_\mu),$$

for all $a \in R$. Thus $\phi = \phi'$. Hence,ϕ is unique $\theta^*(= n(\theta))$– Euclidean fuzzy k-ideal from $\frac{R}{R_\mu}$ into [0,1] with the property that $\mu = \phi \circ n$.

Corollary 9. *Let μ be a θ–Euclidean fuzzy k-ideal,$n : R \to \frac{R}{R_\mu}$ be the natural homomorphism.. Suppose that $\theta(a) = \theta(b)$ when $a - b \in \ker n$. Then there exists a θ^*–Euclidean fuzzy k-ideal from $\frac{R}{\mu} \to [0,1]$.*

Proof: Since $\mu : R \to [0,1]$ is a θ–Euclidean fuzzy k-ideal from Theorem 8, $\phi : \frac{R}{R_\mu} \to [0,1]$ is a θ^*–Euclidean fuzzy k-ideal . Also, the semirings $\frac{R}{\mu}$ and $\frac{R}{R_\mu}$ are isomorphic. So there exists a θ^*–Euclidean fuzzy k-ideal from $\frac{R}{\mu}$ to [0,1].

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Almost Periodic Functions Defined on $\mathbb{R}^n$ With Values in Fuzzy Setting

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Abstract. In this paper we develop the theory of almost periodic functions defined on $\mathbb{R}^n$ with values in fuzzy setting.

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1. Introduction

The theory of almost periodic functions was developed in its main features by Bohr [5] in three rather long papers in the Acta Mathematica (Volumes 45, 46 and 47) under the common title "Zur theorie der Fast Periodische Funktionen" in 1923; the first of these deals with the almost periodic functions of a real variable, while the third takes up the case of a complex variable. Afterwards theory was continuously getting established by several mathematicians like Besicovitch [3], Bochner [4], Amerio and Prouse [1], Levitan [8], Levitan and Zhikov [9], Corduneaue [6], Fink [7] and Zaidman [11] etc. In 1933, Bochner defined and studied the almost periodic functions with values in Banach spaces. He showed that these functions include certain earlier generalizations of the notion of almost periodic functions. The theory of almost periodic functions was further developed by replacing Banach spaces by complete Hausdroff locally convex spaces and Fréchet spaces by N'Guérékata [10]. The theory of almost periodicity as known in Banach spaces, is studied in fuzzy setting that is based on the work of Bede and Gal [2]. The theory of almost periodic functions defined on $\mathbb{R}^n$ with values in Banach spaces is given in monograph of Zaidman [11]. However the theory of almost periodic functions defined on $\mathbb{R}^n$ with values in fuzzy-number-type spaces was not yet developed. It is the main goal of this present paper to develop this theory in section 3.

To this end we first recall the following:

2. Preliminaries

Definition 1. Let us denote by $\mathbb{R}_F$ the class of fuzzy subsets of real axis $\mathbb{R}$ (i.e. $u : \mathbb{R} \longrightarrow [0,1]$), satisfying the following properties:

(i) $\forall u \in \mathbb{R}_F$, u is normal i.e.with $u(x) = 1$.

(ii) $\forall u \in \mathbb{R}_F$, u is convex fuzzy set i.e.
$$u(tx + (1-t)y) \geq \min\{u(x), u(y)\}, \forall t \in [0,1].$$

(iii) $\forall\ u \in \mathbb{R}_F$, u is upper semi-continuous on $\mathbb{R}$.

(iv) $\overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact.

The set $\mathbb{R}_F$ is called the space of fuzzy real numbers.

Remark 2. It is clear that $\mathbb{R} \subset \mathbb{R}_F$, because any real number $x_0 \in \mathbb{R}$, can be described as the fuzzy number whose value is 1 for $x = x_0$ and zero otherwise.

We will collect some other definitions and notations needed in the sequel. For $0 < r \leq 1$ and $u \in \mathbb{R}_F$, we define
$$[u]^r = \{x \in \mathbb{R} : u(x) \geq r\}$$
$$[u]^0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$$
Now it is well known that for each $r \in [0,1]$, $[u]^r$, is bounded closed interval. For $u, v \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$, we have the sum $u \oplus v$ and the product $\lambda \odot u$ are defined by $[u \oplus v]^r = [u]^r + [v]^r$, $[\lambda \odot u]^r = \lambda [u]^r$, $\forall r \in [0,1]$, where $[u]^r + [v]^r$ means the usual addition of two intervals as subsets of $\mathbb{R}$ and $\lambda [u]^r$means the usual product between a scalar and a subset of $\mathbb{R}$.

Now we define $D : \mathbb{R}_F \times \mathbb{R}_F \longrightarrow \mathbb{R} \cup \{0\}$ by
$$D(u,v) = \sup_{r \in [0,1]} \left(\max\left\{\left|u_-^r - v_-^r\right|, \left|u_+^r + u_+^r\right|\right\}\right)$$
where $[u]^r = [u_-^r, u_+^r]$, $[v]^r = [v_-^r, v_+^r]$ then $(D, \mathbb{R}_F)$ is a metric space and it possesses the following properties:

(i) $D(u \oplus w, v \oplus w) = D(u,v)$, $\forall u, v, w \in \mathbb{R}_F$.

(ii) $D(\lambda \odot u, \lambda \odot v) = \lambda D(u,v), \forall u, v \in \mathbb{R}_F, \forall \lambda \in \mathbb{R}$.

(iii) $D(u \oplus v, w \oplus e) \leq D(u,w) + D(v,e)$, $\forall u, v, w, e \in \mathbb{R}_F$ and $(\mathbb{R}_F, D)$ is a complete metric space.

Also we have the following theorem.

Theorem 3. (i) If we denote $\widetilde{0} = \mathcal{X}_{\{0\}}$ *then* $\widetilde{0} \in \mathbb{R}_F$ is neutral element with respect to $\oplus$, *i.e.* $u \oplus \widetilde{0} = \widetilde{0} \oplus u$, *for all* $u \in \mathbb{R}_F$.

(ii) *With respect to* $\widetilde{0}$ *none of* $u \in \mathbb{R}_F \backslash \mathbb{R}$ *has opposite in* $\mathbb{R}_F$ *with respect to* $\oplus$.

(iii) *For any* $a, b \in \mathbb{R}$ *with* $a, b \geq 0$ *or* $a, b \leq 0$, *any* $u \in \mathbb{R}_F$, *we have* $(a+b) \odot u = a \odot u \oplus b \odot u. \forall a, b \in \mathbb{R}$ *the above property does not hold.*

(iv) *For any* $\lambda \in \mathbb{R}$*and any* $u, v \in \mathbb{R}_F$, *we have* $\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot v$.

(v) *For any* $\lambda, \mu \in \mathbb{R}$ *and any* $u \in \mathbb{R}$, *we have* $\lambda \odot (\lambda \odot v) = (\lambda.\mu) \odot v$.

(vi) *If we denote*$\|u\|_F = D(u, \widetilde{0})$, $\forall u \in \mathbb{R}_F$ *then* $\|.\|_F$ *has the properties of a usual norm on* $\mathbb{R}_F$, *i.e.* $\|u\|_F = 0$ *if and only if* $u = \widetilde{0}$,
$\|\lambda \odot u\|_F = |\lambda| . \|u\|_F$ *and* $\|u \oplus v\|_F \leq \|u\|_F + \|v\|_F, |\|u\|_F + \|v\|_F| \leq D(u,v)$.

Remark 4. The propositions (ii) and (iii) in theorem show us that $(\mathbb{R}_F, \oplus, \odot)$ is not a linear space over $\mathbb{R}$ and consequently $(\mathbb{R}_F, \|.\|_F)$ cannot be a normed space. However, the properties of D and those in theorem (iv)-(vi), have as an effect that most of the metric properties of a functions defined on $\mathbb{R}$ with values in a Banach space, can be extended to functions $f : \mathbb{R} \longrightarrow \mathbb{R}_F$, called fuzzy functions.

We now recall the following definitions and theorems

Definition 5. A function $f : \mathbb{R} \longrightarrow \mathbb{R}_F$ is said to be continuous at $x_0 \in \mathbb{R}$ if for every $\varepsilon > 0$ we can find $\delta > 0$ such that $D(f(x), f\ (x_0)) < \varepsilon$, whenever $|x - x_0| < \delta$. f is said to be continuous on $\mathbb{R}$ if it is continuous at every $x \in \mathbb{R}$.

Definition 6. Let $f : \mathbb{R} \longrightarrow \mathbb{R}_F$ be continuous on $\mathbb{R}$. We say that f is B-almost periodic if $\forall\varepsilon > 0, \exists\ l > 0$ such that any interval $[a, a + l]$ of length l contains at least one point τ with

$D(f(t + \tau), f\ (t)) < \varepsilon, \forall t \in \mathbb{R}$.

Definition 7. We say that f is normal if for any sequence $F_n : \mathbb{R} \longrightarrow \mathbb{R}_F$ of the form

$F_n(x) = f\ (x + h_n), n \in \mathbb{N}$, where $(h_n)_n$ is a sequence of real numbers, one can extract a subsequence of $(F_n)_n$, converging uniformly on $\mathbb{R}$ i.e. for every sequence $(h_n)_n$ of real numbers there exists a subsequence $(h_{n_k})_{n_k}$, and $F : \mathbb{R} \longrightarrow \mathbb{R}_F$ which may depend on $(h_n)_n$, such that

$D(F_{n_k}(x), F(x)) \longrightarrow 0$ as $k \longrightarrow \infty$, uniformly with respect to x .

Theorem 8. *If $f : \mathbb{R} \longrightarrow \mathbb{R}_F$ is B-almost periodic then f is bounded i.e.$\exists$, $M > 0$ with*

$$D(f(x), f\ (y)) < M, \forall x, y \in \mathbb{R}.$$

Theorem 9. *If $f : \mathbb{R} \longrightarrow \mathbb{R}_F$ is B-almost-periodic then f is uniformly continuous on $\mathbb{R}$.*

Theorem 10. *If $f_n : \mathbb{R} \longrightarrow \mathbb{R}_F$, $n \in \mathbb{N}$ are B-almost periodic and $f_n \longrightarrow f$ as $n \longrightarrow \infty$ uniformly on $\mathbb{R}$, then f is B-almost periodic.*

Theorem 11. *If $f : \mathbb{R} \longrightarrow \mathbb{R}_F$ is B-almost periodic, then the set of values of f is relatively compact in the complete metric space $(\mathbb{R}_F, D)$.*

Theorem 12. *If $f : \mathbb{R} \longrightarrow \mathbb{R}_F$ is B-almost periodic, then $\lambda \odot f, \lambda \in \mathbb{R}$, $F_h(x) = f\ (x+h)$, and $G(x) = \|f(x)\|_F$, $x \in \mathbb{R}$ are* B-*almost periodic functions.*

Theorem 13. *The sum $\oplus$ of two B-almost periodic functions is B-almost periodic.*

Remark 14. Let us denote $AP(\mathbb{R}_F) = \{f : \mathbb{R} \longrightarrow \mathbb{R}_F : f \text{ is B-almost periodic}\}$, and for $f \in AP(\mathbb{R}_F)$, let us define$\|f\| = \sup\ \{\|f(x)\|_F : x \in \mathbb{R}\}$. By theorem 8 we get $\|f\| < +\infty$. Also by theorems 3, 12 and 13 $AP(\mathbb{R}_F, \oplus, \odot)$, is not a linear space, and consequently $AP(\mathbb{R}_F, \|.\|_F)$ is not a normed space. However, endowed with the metric

$D^* : AP(\mathbb{R}_F) \times AP(\mathbb{R}_F) \longrightarrow \mathbb{R}_+ \cup \{0\}$ defined by

$D^*(f, g) = \sup_{x\in\mathbb{R}} D(f(x), g(x)), f, g \in AP(\mathbb{R}_F)$

becomes a complete metric space. Indeed if we denote

$C_b(\mathbb{R}_F) = \{f : \mathbb{R} \longrightarrow \mathbb{R}_F : f$ is continuous and bounded on $\mathbb{R}\}$, then because $(\mathbb{R}_F, D)$ is a complete metric space, it follows that $(C_b(\mathbb{R}_F), D^*)$ is a complete metric space. Then theorems 8 and 11 show that $AP(\mathbb{R}_F)$ is a closed subset of $C_b(\mathbb{R}_F)$, i.e. $(AP(\mathbb{R}_F), D^*)$ is a complete metric space. For all of the above C.f [5].

3. Almost Periodic Functions Defined On $\mathbb{R}^n$ With Values In Fuzzy-Number-Type Space

Now we recall the following facts about the Euclidean n-dimensional space $\mathbb{R}^n$

Let $\mathbb{R}^n$ the usual Euclidean n-dimensional space. The elements x of $\mathbb{R}^n$ are the n-tuples $x = (x_1, x_2, ..., x_n)$ and a norm of $x \in \mathbb{R}^n$ is given by $\|x\| = (x_1 + x_2 + ... + x_n)^{\frac{1}{2}}$. A closed ball $\overline{B}(x_0; r)$ in $\mathbb{R}^n$ with center x and radius $r > 0$ is defined by the set $\overline{B}(x_0; r) = \{x \in \mathbb{R}^n : \|x - x_0\| \leq r\}$. A set P is said to be relatively dense in $\mathbb{R}^n$ if there exists a number $r > 0$ such that $P \cap \overline{B}(x_0; r) \neq \varnothing$, for all $x \in \mathbb{R}^n$. We also have the following two important theorems for our further discussion.

Theorem 15. *A subset P of $\mathbb{R}^n$ is relatively dense in $\mathbb{R}^n$ if and only if, for some $r > 0$, we have the relation $\mathbb{R}^n = \underset{p \in P}{\cup} \overline{B}(p; r)$.*

Theorem 16. *A subset P of $\mathbb{R}^n$ is relatively dense if and only if there exists a compact set K in such that $K + P = \mathbb{R}^n$ (vector sum of K and P). Now we define almost periodic functions defined on $\mathbb{R}^n$ and taking values in $\mathbb{R}_F$ but before that we define continuity and uniform continuity of functions defined on $\mathbb{R}^n$ and taking values in $\mathbb{R}_F$ then we define the almost periodicity of functions defined on $\mathbb{R}^n$ with values in the fuzzy-number-type spaces.*

Definition 17. A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ is said to be continuous at $x_0 \in \mathbb{R}^n$ if for every $\varepsilon > 0$ we can find $\delta > 0$ such that $D(f(x), f\ (x_0)) < \varepsilon$, whenever $\|x - x_0\| < \delta$. f is said to be continuous on $\mathbb{R}^n$ if it is continuous at every $x \in \mathbb{R}^n$.

Definition 18. A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ is said to be uniformly continuous on $\mathbb{R}^n$ if such that

$D(f(x_1), f\ (x_2)) < \varepsilon$, whenever $\|x_1 - x_2\| < \delta, \forall x_1, x_2 \in \mathbb{R}^n$.

Definition 19. A continuous function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$, is said to be B-almost periodic, if for every $\varepsilon > 0$, we can find a relatively dense set which we denote by $T(f; \varepsilon)$ in $\mathbb{R}^n$ such that

$D(f(t+\tau), f\ (t)) < \varepsilon, \forall t \in \mathbb{R}^n, \tau \in T(f; \varepsilon)$. Hence to any $\varepsilon > 0$, we may associate a number, $r = r(\varepsilon) > 0$, in such manner that in any closed ball $\overline{B}(x; r)$ there exists at least one element of the set $T(f; \varepsilon)$.

Theorem 20. *If $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ is B-almost periodic function then given any any $\varepsilon > 0$, there are two positive numbers $r_1 = r_1(\varepsilon)$ and $r_2 = r_2(\varepsilon)$ such that any ball $\overline{B}(x; r_1)$ in $\mathbb{R}^n$ contains a ball of radius r_2 which is contained in $T(f; \varepsilon)$.*

Proof. Consider the set $T(f; \frac{\varepsilon}{2})$ which is relatively dense in $\mathbb{R}^n$ ($f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ is assumed to be almost periodic) and the associated number $R = R(\frac{\varepsilon}{2})$ such that $\overline{B}(a; R) \cap T(f; \frac{\varepsilon}{2}) \neq \varnothing$, $\forall a \in \mathbb{R}^n$. Using now the uniform continuity of f over $\mathbb{R}^n$ we find a number $\delta_1 = \delta_1(\frac{\varepsilon}{2})$ such that if $y \in \mathbb{R}^n$ and $\|y\| \leq \delta_1$ it follows that $D(f(x + y), f\ (x)) < \frac{\varepsilon}{2}, \forall\ x \in \mathbb{R}^n$ we say that $r_1 = R + 2\delta$ and $r_1 = \delta_1$ form required numbers. In fact, given $x \in \mathbb{R}^n$, take $z \in \mathbb{R}^n$ such that $\|z\| = r_1$. Then $\exists$ $y \in \overline{B}(x_0 + z; R) \cap T(f; \frac{\varepsilon}{2})$. Hence $\|y - x_0\| \leq R + \delta_1 < r_1$ so that $y \in \overline{B}(x_0; r_1)$. Furthermore $\forall y \in \mathbb{R}^n$, $\|y\| \leq \delta_1$,

$\left\|x' + y - x_0\right\| \leq R + \delta_1 + \delta_1 = R + 2\delta_1 = r_1$. Hence $x' + y \in \overline{B}(x_0; r_1)$. Therefore the whole ball $\in \overline{B}(x'; \delta_1)$ is contained in $\in \overline{B}(x_0; r_1)$. Finally any vector in this ball belong to$T(f; \varepsilon)$, this is because $x' + y$ with $\|y\| \leq \delta_1$ is such a vector, we have $\forall\ y \in \mathbb{R}^n$

$$
\begin{aligned}
& D(f(x + y + x'), f\ (x)) \\
&= D(f(x + y) \oplus f(x + y + x'), f(x) \oplus f(x + y)) \\
&= D(f(x + y) \oplus f(x + y + x'), f(x + y) \oplus f(x))
\end{aligned}
$$

$\leq D(f(x+y), f(x+y+x')) + D(f(x+y), f(x)) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$.

Here we have used the fact that $x' \in T(f; \frac{\varepsilon}{2})$, $\|y\| \leq \delta_1$ and uniform continuity of f over $\mathbb{R}^n$ is proved. The following result shows the boundedness of B-almost periodic. □

Theorem 21. *If a continuous function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$, is B-almost periodic, then f is bounded i.e. $\exists M > 0$ with $D(f(x), f\ (y)) < M, \forall x, y \in \mathbb{R}$.*

Proof. Because $D(f(x), f\ (y)) \leq D(f(x), \widetilde{0}) + D(\widetilde{0}, f\ (y)) = \|f(x)\|_F + \|f(y)\|_F$ it is sufficient to prove that M_1 with $\|f(x)\|_F \leq M_1$. Take any $\varepsilon > 0$ and associative relatively set $T(f; \varepsilon)$. Therefore, for some $r = r(\varepsilon) > 0$, $\mathbb{R}^n = \bigcup_{\tau \in T(f;\varepsilon)} \overline{B}(\tau; r)$ and consequently $\forall t' \in \mathbb{R}^n, \exists\ \tau \in T(f; \varepsilon)$ such that $\left\|t' - y\right\| \leq r$. Then, if t is $t' - y$, we have $t' = t - y$ where $\tau \in T(f; \varepsilon)$. Therefore

$$\left\|f(t')\right\|_F$$
$$= D(f(t'), \widetilde{0})$$
$$= D(f(t) \oplus f(t+\tau), f(t) \oplus \widetilde{0})$$
$$\leq D(f(t), f(t+\tau)) + D(f(t) \oplus \widetilde{0})$$
$$< \varepsilon + \|f(t)\|_F$$
$$< \varepsilon + \sup_{t \in \overline{B}(0;r)} (\|f(t)\|_F)$$

For instance, if we take $\varepsilon = 1$, $\left\|f(t')\right\|_F < \varepsilon + \sup_{\|t\| \leq r(1)} (\|f(t)\|_F)$ which gives us an upper bound for f over $\mathbb{R}^n$. □

Next theorem shows that the range of B-almost periodic functions is relatively compact.

Theorem 22. *If $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ is B-almost periodic, then the set of values of f is relatively compact in the complete metric space $(\mathbb{R}_F, D)$.*

Proof. In complete metric spaces, the relatively compact sets coincides with totally bounded sets, it is sufficient to show that the values of the functions can be embedded in a finite number of spheres of radius 2ε. Take any $\varepsilon > 0$ and the and the number $r = r(\varepsilon) > 0$. The range of f when t runs over the compact ball $\overline{B}(0; r) = \{t \in \mathbb{R}^n : \|t\| \leq r\}$ is compact in $\mathbb{R}_F$. Therefore, there are ν points $f(t_1), f(t_2), ..., f(t_\nu)$ where $\|t_i\| \leq r, 1 \leq i \leq \nu$ and $f(t) \in \bigcup_{i=1}^{\nu} \overline{B}(f(t_i); \varepsilon), \forall t, \|t\| \leq r$. Take now any arbitrary $t' \in \mathbb{R}^n$, it can be written as $t' = t + \tau$ where $\|t\| \leq r$ and $\tau \in T(f; \varepsilon)$, hence, there is an $i \in \{1, 2, 3, ..., \nu\}$ such that $D(f(t'), f\ (t_i)) \leq \varepsilon$. It follows that

$$D(f(t'), f\ (t_i))$$
$$= D(f(t+\tau), f\ (t_i))$$
$$= D(f(t+\tau) \oplus f\ (t_i), f(t_i) \oplus f(t))$$
$$\leq D(f(t+\tau), f\ (t_i)) + D(f(t_i) \oplus f(t)) \leq \varepsilon + \varepsilon = 2\varepsilon.$$

Thus $R_f \subset \bigcup_{i=1}^{\nu} \overline{B}(f(t_i); 2\varepsilon)$. □

Remark 23. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ is B-almost periodic, and let us consider the sequence $(f(t_n))_n$of values. Denote $A = \{f(t_n) : n \in \mathbb{N}\}$,and take the closure $\overline{A} \subset$

$\overline{f(\mathbb{R}^n)} \subset \mathbb{R}_F$. It follows that $\overline{A}$ is compact, so is $\overline{A}$ sequentially compact too, which by $A \subset \overline{A}$ implies that the sequence has a convergent subsequence in $\mathbb{R}_F$.

Theorem 24. *If $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ is B-almost periodic, then it is uniformly continuous over $\mathbb{R}^n$.*

Proof. Let $\varepsilon > 0$ be given, we can find $r = r(\frac{\varepsilon}{3}) > 0$ such that for any $t' \in \mathbb{R}^n$ can be represented as $t' = t + \tau$ where $\tau \in T(f; \frac{\varepsilon}{3})$. Next we note the uniform continuity of the function f on the closed ball $\overline{B}(0; 2r) = \{t \in \mathbb{R}^n : \|t\| \leq 2r\}$. Therefore, there is a $\delta > 0$, $(\delta < r(\frac{\varepsilon}{3}))$ such that, if $\|t_1\| \leq 2r, \|t_2\| \leq 2r$ and $\|t_1 - t_2\| < \delta$, then we have $D(f(t_1), f\ (t_2)) \leq \frac{\varepsilon}{3}$. Take now any pair $t_1', t_2' \in \mathbb{R}^n$, such that $\left\|t_1' - t_2'\right\| < \delta$. We can write, for some $\tau \in T(f; \frac{\varepsilon}{3})$, the decomposition $t_1' = t_1 + \tau$, where $\|t_1\| \leq r$. Then define t_2 as $t_2' - \tau$. It follows that $\|t_1 - t_2\| = \left\|t_1' - t_2'\right\| < \delta$. Also $\|t_2\| = \|t_1 - t_2\| + \|t_1\| < \delta + r \leq 2r$. From the above we derive that $D(f(t_1), f\ (t_2)) \leq \frac{\varepsilon}{3}$, and accordingly, as $\tau \in T(f; \frac{\varepsilon}{3})$, we find that

$$
\begin{aligned}
&D(f(t_2'), f\ (t_1')) \\
&= D(f(t_2 + \tau), f\ (t_1 + \tau)) \\
&= D(f(t_1) \oplus f(t_2 + \tau), f\ (t_1 + \tau) \oplus f(t_1)) \\
&= D(f(t_1) \oplus f(t_1 + \tau), f\ (t_1) \oplus f(t_2 + \tau)) \\
&\leq D(f(t_1), f(t_1 + \tau)) + D(f\ (t_1), f(t_2 + \tau)) \\
&= D(f(t_1), f(t_1 + \tau)) + D(f(t_1) \oplus f(t_2 + \tau), f\ (t_1) \oplus f(t_2 + \tau)) \\
&\leq D(f(t_1), f(t_1 + \tau)) + D(f(t_1), f(t_1 + \tau)) + D(f\ (t_1 + \tau), f(t_2 + \tau)) \\
&= \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon.
\end{aligned}
$$

□

The next result shows that the set $AP(\mathbb{R}_F)$ is closed with respect to uniform convergence.

Theorem 25. *If $f_k : \mathbb{R}^n \longrightarrow \mathbb{R}_F$, $n \in \mathbb{N}$ are B-almost periodic and $f\ _k \longrightarrow f$ as $k \longrightarrow \infty$ uniformly on $\mathbb{R}^n$, then is B-almost periodic.*

Proof. Let $\varepsilon > 0$. Since $f_k(t) \longrightarrow f(t)$ uniformly over $\mathbb{R}^n$ as $k \longrightarrow \infty$, so we can find a natural number k_0 such that$\forall k \geq k_0$, we have $D(f_k(t), f\ (t)) < \frac{\varepsilon}{3}$. Since $f\ _k : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ is almost periodic for $k = 1, 2, 3, \ldots$ so for already chosen $\varepsilon > 0$, we can find a relatively dense set $T(f_k; \frac{\varepsilon}{3})$ such that

$D(f_k(t + \tau), f\ _k(t)) < \frac{\varepsilon}{3}, \forall \tau \in T(f_k; \frac{\varepsilon}{3}), t \in \mathbb{R}^n,\ k = 1, 2, 3, \ldots.$ Now

$$
\begin{aligned}
&D(f_k(t + \tau), f\ _k(t)) \\
&= D(f(t + \tau) \oplus f_k(t + \tau), f_k(t + \tau) \oplus f\ (t)) \\
&\leq D(f(t + \tau), f_k(t + \tau)) + D(f_k(t + \tau), f\ (t)) \\
&= D(f(t + \tau), f_k(t + \tau)) + D(f_k(t + \tau) \oplus f_k(t), f_k\ (t) \oplus f(t)) \\
&\leq D(f(t + \tau), f_k(t + \tau)) + D(f_k(t + \tau), f_k(t)) + D(f_k\ (t), f(t)) \\
&= \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon, \forall \tau \in T(f_k; \tfrac{\varepsilon}{3}),\ t \in \mathbb{R}^n,\ k = 1, 2, 3, \ldots.
\end{aligned}
$$

which implies that $\tau \in T(f; \varepsilon)$, and hence $T(f; \varepsilon)$ is a relatively dense set in $\mathbb{R}^n$ so f is proved to be almost periodic function. □

We now give the following simple theorem.

Theorem 26. *If $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ is B-almost periodic, then $\lambda \odot f, \lambda \in \mathbb{R}$, $F_h(x) = f\ (x+h)$, and $G(x) = \|f(x)\|_F, x \in \mathbb{R}^n$ are B-almost periodic functions.*

Proof. Because $D(\lambda \odot f(t + \tau), \lambda \odot f(t)) = |\lambda|\, D(f(t + \tau), f(t))$, for all, it follows that λf is almost periodic whenever f is B-almost periodic. And since

$\left|\|f(t+\tau)\|_F - \|f(t)\|_F\right| \leq D(f(t+\tau), f(t))$, then it is immediate that $G(x) = \|f(x)\|_F$, $x \in \mathbb{R}^n$, is B-almost periodic. Now at the last let h be fixed and for every $\varepsilon > 0$, let $r = r(\varepsilon) > 0$ be attached to f in the definition of B-almost periodic.

By $D(f(t+\tau), f\ (t)) < \varepsilon$, $\forall \tau \in T(f;\varepsilon)$ and $\forall t \in \mathbb{R}^n$, we get (by taking $t = u+h$),

$D(f(u+h+\tau), f\ (u+h)) < \varepsilon$, $\forall \tau \in T(f;\varepsilon)$ and $\forall t \in \mathbb{R}^n$.

$D(F_h(u+\tau), F(u)) < \varepsilon$, $\forall \tau \in T(f;\varepsilon)$ and $\forall t \in \mathbb{R}^n$. This implies that $T(f;\varepsilon) \subset T(F_h;\varepsilon)$, therefore F is B-almost periodic. □

We now define normal functions and prove some results.

Definition 27. We say that f is normal if for any sequence $F_k : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ of the form

$F_k(x) = f\ (x+h_k), k \in \mathbb{N}$, where $(h_k)_k$ is a sequence of real numbers, one can extract a subsequence of $(F_k)_k$, converging uniformly on $\mathbb{R}^n$ i.e. for every sequence $(h_k)_k$ of real numbers there exists a subsequence $(h_{k_l})_{k_l}$, and $F : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ which may depend on $(h_k)_k$, such that

$D(F_{k_l}(x), F(x)) \longrightarrow 0$ as $l \longrightarrow \infty$, uniformly with respect to x.

We now apply this to prove the following theorem.

Theorem 28. *The sum of two B-almost periodic functions are B-almost periodic.*

Proof. Let f and g be two B-almost periodic functions, and let $(h_k)_k$ an arbitrary sequence in $\mathbb{R}^n$. From the sequence $(f_{h_k})_k$of translates, we can choose a uniformly convergent subsequence on $\mathbb{R}^n$ say $(f_{l_k})_k$. From the sequence $(g_{k_k})_k$,we choose a subsequence uniformly convergent on $\mathbb{R}^n$, say $(g_{l_k})_k$. Then the sequence $(f_{l_k} + g_{k_k})_k$, which is a subsequence of the sequence $(f_{h_k} + g_{h_k})_k$, is uniformly convergent on $\mathbb{R}^n$. □

To prove the equivalence between the normal functions and B-almost periodic functions we need the following lemma.

Lemma 29. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ be a B-almost periodic function and a sequence $(x_k)_k \subset \mathbb{R}^n$ be given then to any $\varepsilon > 0$ we may associate a subsequence $(x_{k_i})_{k_i}$ such that the inequality*

$\sup_{x\in\mathbb{R}^n} D(f(x+x_{k_p}), f(x+x_{k_q})) < \varepsilon, \forall p, q \in \mathbb{N}$, *is satisfied.*

Proof. We know that any vector $x_k \in \mathbb{R}^n$, can be written as $x_k = y_k + z_k$, where $z_k \in T(f;\frac{\varepsilon}{2})$ and $\|y_k\| \leq r(\frac{\varepsilon}{2}) = r > 0$ (r is independent of k). Let y be the limit point of the sequence $(y_k)_k$ then $\|y\| \leq r$. Since $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ be a B-almost periodic function so it is uniformly continuous over $\mathbb{R}^n$ so we can find $\delta > 0$ such that $\|x_1 - x_2\| < \delta \Longrightarrow \|f(x_1) - f(x_2)\| < \frac{\varepsilon}{2}$. Then in the ball $\{x \in \mathbb{R}^n : \|x\| \leq \delta\}$ we find an infinite sequence of y_k, s which we denote by $(x_{k_i})_{k_i}$. Take now two vectors x_{k_p} and x_{k_q} then $x_{k_p} = y_{k_p} + z_{k_p}$, $x_{k_q} = y_{k_q} + z_{k_q}$. We deduce that

$$\sup_{x\in\mathbb{R}^n} D(f(x+x_{k_p}), f(x+x_{k_q})) =$$
$$\sup_{x\in\mathbb{R}^n} D(f(x+y_{k_p}+z_{k_p}), f(x+y_{k_q}+z_{k_q}))$$
$$= \sup_{x\in\mathbb{R}^n} D(f(x+y_{k_p}+z_{k_p}-y_{k_q}-z_{k_q}), f(x))$$
$$= \sup_{x\in\mathbb{R}^n} D(f(x+y_{k_p}+z_{k_p}-y_{k_q}-z_{k_q}) \oplus f(x+y_{k_p}-y_{k_q}), f(x) \oplus f(x+y_{k_p}-y_{k_q}))$$
$$\leq \sup_{x\in\mathbb{R}^n} D(f(x+y_{k_p}+z_{k_p}-y_{k_q}-z_{k_q}), f(x+y_{k_p}-y_{k_q})) + D(f(x), f(x+y_{k_p}-y_{k_q}))$$

$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

The last inequality is a consequence of fact that

$y_{k_p} - y_{k_q} \in T(f; \frac{\varepsilon}{2})$ and $\|y_{k_p} - y_{k_q}\| \leq \|y_{k_p} - y\| + \|y - y_{k_q}\| \leq \delta + \delta = 2\delta.$ □

Theorem 30. *Any B-almost periodic $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$, is normal.*

Proof. Let $(x_k)_k \subset \mathbb{R}^n$ be a given sequence. Then by Lemma 29 we can find a subsequence $(x_{k_i,1})_i$ such that

$\sup_{x \in \mathbb{R}^n} D(f(x + x_{k_p,1}), f(x + x_{k_q,1})) < 1, \forall p, q \in \mathbb{N}.$

$D(f(x + x_{k_p,1}), f(x + x_{k_q,1})) < 1, \forall p, q \in \mathbb{N}, \forall x \in \mathbb{R}^n.$

Next we choose a subsequence $(x_{k_i,2})_i \subset (x_{k_i,1})_i$ with the property

$D(f(x + x_{k_p,2}), f(x + x_{k_q,2})) < \frac{1}{2}, \forall p, q \in \mathbb{N}, \forall x \in \mathbb{R}^n.$

We can choose a further subsequence $(x_{k_i,3})_i \subset (x_{k_i,2})_i$ with the property. $D(f(x + x_{k_p,3}), f(x + x_{k_q,3})) < \frac{1}{3}, \forall p, q \in \mathbb{N}, \forall x \in \mathbb{R}^n.$

And so on. Consider now a diagonal sequence, $(f(x + x_{k_i,i})_i$ of translated functions. Now if $p, q \in \mathbb{N}$, with $p \leq q$, then we have

$\sup_{x \in \mathbb{R}^n} D(f(x + x_{k_p,p}), f(x + x_{k_q,q})) < \frac{1}{3}, \forall p, q \in \mathbb{N}.$ This proves that the sequence $(f(x + x_{k_i,i}))_i$ is uniformly convergent over $\mathbb{R}^n$ this also proves the normality of f. □

The converse of this theorem also holds true as proved in the following theorem.

Theorem 31. *Any normal function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ is B-almost periodic.*

Proof. On contrary suppose that f is not B-almost periodic function then there exists an $\varepsilon > 0$ such that the set $T(f; \varepsilon)$ is not relatively dense in $\mathbb{R}^n$. This implies that for all $r(\varepsilon) = r > 0$ there is a ball $\overline{B}(a; r)$ which contains no element of the set $T(f; \varepsilon)$. Consider now an arbitrary element $x_1 \in \mathbb{R}^n$ and take $r_1 > \|x_1\|$, hence there exists a ball $\overline{B}(x_2; r_2)$ which is disjoint with $T(f; \varepsilon)$. Since $x_2 - x_1 \in \overline{B}(x_2; r_2) \Longrightarrow x_2 - x_1 \notin T(f; \varepsilon)$. Next take $r_1 > \|x_1\| + \|x_2\|$ and find a ball $\overline{B}(x_3; r_3)$ which is disjoint of $T(f; \varepsilon)$. Now both the vectors $x_2 - x_1$ and $x_2 - x_3$ belong to $\overline{B}(x_3; r_3)$ but $x_2 - x_1 \notin T(f; \varepsilon)$ and $x_2 - x_3 \notin T(f; \varepsilon)$. Continuing this procedure, we can find an infinite sequence $(x_k)_k \subset \mathbb{R}^n$ such that $\forall k, l \in \mathbb{N}, k \neq l \Longrightarrow x_k - x_l \notin T(f; \varepsilon)$. It follows that, by replacing x by $x - x_k$

$$D(f(x + x_k), f(x + x_l)) = D(f(x + x_k - x_l), f(x)) > \varepsilon, \forall x \in \mathbb{R}^n.$$

This shows that the sequence, $(f(x + x_k))_k$, $x \in \mathbb{R}^n$ contains no subsequence which converges uniformly over $\mathbb{R}^n$. A contradiction to the fact that f is normal function. So our assumption that f is not B-almost periodic function is wrong. Therefore f is proved to be almost periodic. □

Remark 32. Let us denote

$AP(\mathbb{R}_F) = \{f : \mathbb{R}^n \longrightarrow \mathbb{R}_F : f \text{ is B-almost periodic}\}$, and for $f \in AP(\mathbb{R}_F)$, let us define $\|f\| = \sup \{\|f(x)\|_F : x \in \mathbb{R}^n\}$. By theorem 20 we get $\|f\| < +\infty$. $AP(\mathbb{R}_F)$, is a complete metric space with respect to the metric

$D^* : AP(\mathbb{R}_F) \times AP(\mathbb{R}_F) \longrightarrow \mathbb{R}_+ \cup \{0\}$ defined by

$D^*(f, g) = \sup_{x \in \mathbb{R}^n} D(f(x), g(x)), \ f, g \in AP(\mathbb{R}_F)$

Let us denote

$C_b(\mathbb{R}_F) = \{f : \mathbb{R}^n \longrightarrow \mathbb{R}_F : f \text{ is continuous and bounded on } \mathbb{R}^n\}$ Then because

$(\mathbb{R}_F, D)$ is a complete metric space, it follows that $(C_b(\mathbb{R}_F), D^*)$ is a complete metric space.

Then theorems 20 and 23 show that $AP(\mathbb{R}_F)$ is a closed subset of $C_b(\mathbb{R}_F)$, i.e. $(AP(\mathbb{R}_F), D^*)$ is a complete metric space.

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Rotational Effects On Rayleigh Wave Speed In Orthotropic Medium

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Abstract. A rotational effect on rayleigh wave speed in orthotropic materials is studied. A formula for the wave speed is derived. Rayleigh wave speed for some rotating and non-rotating orthotropic materials is calculated.

1. Introduction

In 1885, Rayleigh[5] studied the surface waves (called the rayleigh after his name) which propagate along the plane surface of elastic solid. After that a number of researches [4, 3, 6, 10, 8, 11, 2] studied the Rayleigh wave speed by using different techniques in different kind of materials. Recently Pham and Ogden [9] discussed the Rayleigh wave speed in orthotropic elastic solids. In this article we have extended the work of Pham and Ogden [9] and derived the formula for Rayleigh wave speed in rotating orthotropic materials with and without rotational effect is studied.

2. Boundary Value Problem & Secular Equation

Consider the semi-infinite stress-free surface of orthotropic material. We choose the rectangular co-ordinate system in such a way that $x_3 - axis$ is normal to the boundary and the material occupies region $x_3 \leq 0$. By following Pham and Ogden [9] we consider the plane harmonic waves in $x_1 - direction$ in $x_1x_3 - plane$ with displacement components (u_1, u_2, u_3) such that
Generalized Hook's law gives

$$\left.\begin{aligned} \sigma_{11} &= c_{11}u_{1,1} + c_{13}u_{3,3} \\ \sigma_{33} &= c_{13}u_{1,1} + c_{33}u_{3,3} \\ \sigma_{13} &= c_{55}(u_{1,3} + u_{3,1}) \end{aligned}\right\} \qquad (2.\ 1)$$

where the elastic constants $c_{11}, c_{33}, c_{13}, c_{55}$ satisfy the inequalities

$$c_{ii} > 0, i = 1, 3, 4\ , c_{11}c_{33} - c_{13}^2 > 0 \tag{2. 2}$$

which are the necessary and sufficient conditions for the straign energy of the material to be positive definite. If a homogeneous elastic body is rotating about an axis, we may choose x_3-axis, with a constant angular velocity $\mathbf{\Omega}$ then equations of motion for infinitesimal deformation may be written as follows [7]

$$\sigma_{ij,j} = \rho\{\ddot{u}_i + \Omega_j u_j \Omega_i - \Omega^2 u_i + 2\varepsilon_{ijk}\Omega_j \dot{u}_k\} \tag{2. 3}$$

where $\mathbf{\Omega} = \Omega(0, 0, 1)$
The Eqs. (2.3), for the problem may be written as

$$\left.\begin{array}{l} \sigma_{11,1} + \sigma_{13,3} = \rho(\ddot{u}_1 - \Omega^2 u_1) \\ \sigma_{31,1} + \sigma_{33,3} = \rho\ddot{u}_3 \end{array}\right\} \tag{2. 4}$$

In view of (2.1), Eqs. (2.4) can be written as

$$\left.\begin{array}{l} c_{11}u_{1,11} + c_{13}u_{3,31} + c_{55}(u_{1,33} + u_{3,13}) = \rho(\ddot{u}_1 - \Omega^2 u_1) \\ c_{55}(u_{1,31} + u_{3,11}) + c_{13}u_{1,13} + c_{33}u_{3,33} = \rho\ddot{u}_3 \end{array}\right\} \tag{2. 5}$$

The boundary conditions of zero traction are

$$\sigma_{3i} = 0\ , i = 1, 3 \ \textit{on the plane}\ x_3 = 0 \tag{2. 6}$$

Usual requirements that the displacements and the stress components decay away from the boundary implies

$$u_i \to 0,\ \sigma_{ij} \to 0\ (i.j = 1, 3)\ as\ x_3 \to \infty \tag{2. 7}$$

Considering the harmonic waves propagating in x-direction, by following Pham and Ogden [9]we write;

$$u_j = \phi_j(kx_3)exp(ik(x_1 - ct))\ ;\ j = 1, 3 \tag{2. 8}$$

where k is the wave number and c is the wave speed and $\phi_j, j = 1, 3$ are the functions to be determined. Substituting (2.8) into (2.5) gives

$$\begin{aligned} c_{55}k^2\phi_1'' + ik(c_{55} + c_{13})\phi_3' + \{k^2(\rho c^2 - c_{11}) + \rho\Omega^2\}\phi_1 = 0, \\ c_{33}\phi_3'' + ik(c_{55} + c_{13})\phi_1' + (\rho c^2 - c_{55})\phi_3 = 0. \end{aligned} \tag{2. 9}$$

In terms of $\phi_j; j = 1, 3$ after taking into account (2.1) and (2.8) the boundary conditions (2.6) give

$$\begin{aligned} ic_{13}\phi_1 + c_{33}\phi_3' = 0 \\ \phi_1' + i\phi_3 = 0 \ \textit{on the plane}\ x_3 = 0. \end{aligned} \tag{2. 10}$$

while from (2.7) we have

$$\phi_j, \phi_j' \to 0\ as\ x_3 \to -\infty \tag{2. 11}$$

Laplace transform of (2.9) by using (2.10) we have

$$\begin{aligned} \{k^2(c_{55}s^2 + \rho c^2 - c_{11})_{+\rho\Omega^2}\}\overline{\phi_1}(s) + ik^2(c_{13} + c_{55})s\overline{\phi_3}(s) = \\ c_{55}k^2\{s\phi_1(0) + \phi_1'(0)\} + ik^2(c_{13} + c_{55})\phi_3(0) \end{aligned}$$

$$\begin{aligned} i(c_{13} + c_{55})s\overline{\phi_1}(s) + (c_{33}s^2 - c_{55} + \rho c^2)\overline{\phi_3}(s) \\ = i(c_{13} + c_{55})\phi_1(0) + c_{33}\{s\phi_3(0) + \phi_3'(0)\} \end{aligned} \tag{2. 12}$$

From (2.12) we have

$$\overline{\phi_1}(s) = \frac{\begin{vmatrix} c_{55}k^2\{s\phi_1(0)+\phi_1'(0)\}+ik^2(c_{13}+c_{55})\phi_3(0) & ik^2(c_{13}+c_{55})s \\ i(c_{13}+c_{55})\phi_1(0)+c_{33}\{s\phi_3(0)+\phi_3'(0)\} & (c_{33}s^2-c_{55}+\rho c^2) \end{vmatrix}}{k^2c_{33}c_{55}s^4+[k^2\{(c_{13}+c_{55})^2+c_{33}(\rho c^2-c_{11})+c_{55}(\rho c^2-c_{55})\}+c_{33}\rho\Omega^2]s^2+(\rho c^2-c_{55})\{k^2(\rho c^2-c_{11})+\rho\Omega^2\}} \quad (2.\ 13)$$

Let ${s_1}^2$, ${s_2}^2$ be the roots of quadratic equation in s^2 (where s_1, s_2 must have positive real parts) of the denominator,

$$k^2c_{33}c_{55}s^4+[k^2\{(c_{13}+c_{55})^2+c_{33}(\rho c^2-c_{11})+c_{55}(\rho c^2-c_{55})\}+c_{33}\rho\Omega^2]s^2 +(\rho c^2-c_{55})\{k^2(\rho c^2-c_{11})+\rho\Omega^2\}=0 \quad (2.\ 14)$$

By considering (2.11) the inverse Laplace transform of $\overline{\phi_1}(s)$ gives

$$\phi_1(y)=A_1exp[s_1y]+A_2exp[s_2y] \quad (2.\ 15)$$

where $y=kx_3$. By using (2.15), (2.9) and (2.11) we have .

$$\phi_3(y)=\alpha_1A_1exp[s_1y]+\alpha_2A_2exp[s_2y] \quad (2.\ 16)$$

where

$$\alpha_j=\frac{i[k^2\{c_{55}{s_j}^2+(\rho c^2-c_{11})\}+\rho\Omega^2]}{k^2(c_{13}+c_{55})s_j},\ j=1,2$$

As ${s_1}^2, {s_2}^2$ are the roots of (2.14), therefore, we must have

$${s_1}^2+{s_2}^2=-\frac{[k^2\{(c_{13}+c_{55})^2+c_{33}(\rho c^2-c_{11})+c_{55}(\rho c^2-c_{55})\}+c_{33}\rho\Omega^2]}{k^2c_{33}c_{55}}$$

$${s_1}^2{s_2}^2=\frac{(\rho c^2-c_{55})\{k^2(\rho c^2-c_{11})+\rho\Omega^2\}}{k^2c_{33}c_{55}} \quad (2.\ 17)$$

Substituting (2.15) and (2.16) into (2.10) we get

$$(ic_{13}+c_{33}\alpha_1s_1)A_1+(ic_{13}+c_{33}\alpha_2s_2)A_2=0$$
$$(s_1+i\alpha_1)A_1+(s_2+i\alpha_2)A_2=0 \quad (2.\ 18)$$

For non-trivial solution of (2.18), the determinant of the coefficients must vanish i.e

$$(ic_{13}+c_{33}\alpha_1s_1)(s_2+i\alpha_2)-(ic_{13}+c_{33}\alpha_2s_2)(s_1+i\alpha_1)=0 \quad (2.\ 19)$$

Substituting the value of α_1 and α_2 from (2.16) into (2.19) and simplifying we have the following equation

$$(\rho c^2-c_{55})[k^2{c_{13}}^2+c_{33}\{k^2(\rho c^2-c_{11})+\rho\Omega^2\}] -k\rho c^2\sqrt{c_{33}c_{55}}\sqrt{\{k^2(\rho c^2-c_{11})+\rho\Omega^2\}(\rho c^2-c_{55})}=0 \quad (2.\ 20)$$

The Eq. (2.20) may be written as

$$\sqrt{\frac{c_{33}}{c_{55}}\frac{\frac{\rho c^2}{c_{11}}-\frac{c_{55}}{c_{11}}}{\frac{\rho c^2}{c_{11}}-1+\frac{\rho\Omega^2}{k^2c_{11}}}}[\frac{{c_{13}}^2}{c_{11}c_{33}}+\frac{\rho c^2}{c_{11}}-1+\frac{\rho\Omega^2}{k^2c_{11}}]-\frac{\rho c^2}{c_{11}}=0 \quad (2.\ 21)$$

The Eq. (2.21) is the required Rayleigh wave speed formula for orthotropic materials.

3. Rayleigh wave speed in some rotating and non-rotating orthotropic materials

Value of Ω may be chosen at random for convenience we take

$$\frac{\Omega^2}{k^2} = \frac{c_{11}}{\rho}.$$

Therefore, the above equation becomes

$$\sqrt{\frac{c_{33}}{c_{55}} \frac{\frac{\rho c^2}{c_{11}} - \frac{c_{55}}{c_{11}}}{\frac{\rho c^2}{c_{11}}}} [\frac{c_{13}{}^2}{c_{11}c_{33}} + \frac{\rho c^2}{c_{11}}] - \frac{\rho c^2}{c_{11}} = 0 \qquad (3.\ 1)$$

and can be written as

$$\rho^3 c_{33}(c_{33} - c_{55})c^6 + \rho^2 c_{33}(2c_{13}{}^2 - c_{33}c_{55})c^4$$
$$+\rho c_{13}{}^2(c_{13}{}^2 - 2c_{33}c_{55})c^2 - c_{55}c_{13}{}^4 = 0 \qquad (3.\ 2)$$

Now using the computer software Mathematica and the following Table [1]
We have

Table 1

Materials	Stiffness$(10^{10} N/m^2)$				Density (Kg/m^3)
	c_{11}	c_{13}	c_{33}	c_{55}	ρ
Iodic acid HIO_3	3.01	1.11	4.29	2.06	4.64
Bariumsodium niobate $Ba_2NaNb_5O_{15}$	23.9	5.00	13.5	6.60	5.30

It is evident from (3.2) that there will be six values of c, but we have taken those

Table 2. For Rotating Materials

Materials	Speed(Km/s)
Iodic acid	82.41
Bariumsodium Niobate	120.97

values which satisfy Eq. (3.1)

Similarly for other values of Ω we can find Rayleigh wave speed in the given materials.

If $\Omega = 0$(stationary case), then the Eq. (2.21) becomes

$$\sqrt{\frac{c_{33}}{c_{55}} \frac{\frac{\rho c^2}{c_{11}} - \frac{c_{55}}{c_{11}}}{\frac{\rho c^2}{c_{11}} - 1}} [\frac{c_{13}{}^2}{c_{11}c_{33}} + \frac{\rho c^2}{c_{11}} - 1] - \frac{\rho c^2}{c_{11}} = 0 \qquad (3.\ 3)$$

which may be written as

$$\rho^3 c_{33}(c_{33} - c_{55})c^6 + \rho^2 c_{33}\{2{c_{13}}^2 - c_{33}c_{55} - c_{11}(2c_{33} - c_{55})\}c^4 + \rho({c_{13}}^2 - c_{11}c_{33})({c_{13}}^2 - c_{11}c_{33} - 2c_{33}c_{55})c^2 - c_{55}({c_{13}}^2 - c_{11}c_{33})^2 = 0 \quad (3.\ 4)$$

Again using the Table [1] and computer software Mathematica we have from (3.4) that there are three waves which propagate in the non-rotating material with distinct velocities as shown in the following

Thus tremendous rotational effects on the Rayleigh wave speed can be seen from

TABLE 3. For Non-Rotating Materials

Materials	Speed(Km/s)
Iodic acid	53.44, 80.94, 125.37
Bariumsodium Niobate	102.55, 213.59, 296.44

the last two Tables.

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Cubic Coefficients Estimates and Asymptotic Properties of the Estimates from the Two Parameters Rayleigh Distribution

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Abstract. A particular linear estimate with four terms approximation called Cubic Coefficient Estimate (CCE) using ordered observations is derived by using general theory of linear coefficients with polynomial coefficients Downton [3]. This estimate is applicable for complete samples and is shown to yield highly efficient estimators even in the case of small samples. The asymptotic properties of the Cubic Coefficients estimates are also described. The Cubic Coefficient Estimates also applicable to, and can even be simplified for, one parameter distributions of the type $F(x/\lambda)$.

1. Introduction

A common statistical problem is to estimate the unknown location and scale parameters from the distribution of the form $F\left(\frac{x-\mu}{\lambda}\right)$. In general, μ and λ may be mean and standard deviation and this is not always necessary and in certain circumstances μ may be the percentage point of the distribution and λ may be defined as the range of variation of the variate X.
Lloyd [9] using least squares method obtained the estimates of μ and λ using ordered observations Lloyd's method gives the exact solution of the minimum variance unbiased estimation of the location and scale parameters. This method becomes impracticable for certain distribution as the ordered moments of certain distributions are not available for a sample of size more than 10.
Moreover it involves the calculation of $\frac{1}{2}n(n-1)$ double integrals and 2n single integrals. Lloyds method can be used for singly or doubly censored data.
Various approximate methods have been devised to overcome this difficulty. Blom [1] and Weiss [11] devised approximate method using ordered observations. Blom's nearly unbiased method is quite efficient for small samples but its standard error may be poor.

The approximate methods suggest that the efficiency of the determination of linear estimates does not seem particularly sensitive to the changes in the coefficients and may be chosen for convenience. Hirai [4] considered the estimation of the parameters from the Rayleigh distribution by linear coefficient method and in [6]

by the Quadratic Coefficients method as a particular case of Downtion's General Theory to estimate μ and λ from the Rayleigh distribution.
In this paper, therefore we discuss the properties of the moments of ordered random variables and then derive the Cubic Coefficient Estimate (CCE) and apply Cubic Coefficient Estimate to estimate the two parameters of the Rayleigh distribution and also discuss its asympotic properties.
To this end we first recall the following:

2. Properties of the Moments of Ordered Random Variables

If $x_1^{(n)} \leq x_2^{(n)} \leq x_3^{(n)} \leq \cdots x_n^{(n)}$ are the available ordered observations from the distribution $F(\frac{x-\mu}{\lambda})$, where μ and λ are the unknown parameters to be estimated. We make the transformation $y_i^{(n)} = \frac{x_i^{(n)}-\mu}{\lambda}$ $(i = 1, 2, \cdots, n)$ such that $y_1^{(n)} \leq y_2^{(n)} \leq \cdots y_n^{(n)}$ are the realizations of the set of random variables $Y_1^{(n)} \leq Y_2^{(n)}, \cdots, Y_n^{(n)}$.
We denote

$$E(Y_i^{(n)}) = \alpha_i^{(n)} = \int_{-\infty}^{\infty} x \frac{n!}{(i-1)!(n-i)!}[F(x)]^{i-1}[1-F(x)]^{n-i} dF(x) \qquad (2.\ 1)$$

$$E(Y_i^{(n)})^2 = W_{ii}^{(n)} = \int_{-\infty}^{\infty} x^2 \frac{n!}{(i-1)!(n-i)!}[F(x_i)]^{i-1}[1-F(x_i)]^{n-i} dF(x) \qquad (2.\ 2)$$

and

$$\begin{aligned} E(Y_i^{(n)}, Y_j^{(n)}) i < j = W_{ij}^{(n)} &= \int_{-\infty}^{\infty}\int_{-\infty}^{y} xy \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \times \\ &\quad [F(x)]^{i-1}[F(Y) - F(x)]^{j-i-1} \times \\ &\quad [1 - F(y)^{n-j}] dF(x) dF(y) \end{aligned} \qquad (2.\ 3)$$

Also

$$Cov\ (Y_i^{(n)}, Y_i^{(n)}) = W_{ij}^{(n)} - \alpha_i^{(n)}\alpha_j^{(n)} \qquad (2.\ 4)$$

We denote $m^{(r)}$ with m and r integers, as the rth factorial power of m i.e. $m^{(r)} = \frac{m!}{(m-r)!} = m(m-1)\cdots(m-r+1)$ and introduce also the two identities

$$(a+b)^{(m)} = \sum_{r=0}^{m} \binom{m}{r} a^{(r)} b^{(m-r)} \qquad (2.\ 5)$$

$$(a-b)^{(m)} = \sum_{r=0}^{m} (-1)^r (a-r)^{(m-r)} b^{(r)} \qquad (2.\ 6)$$

We note that

$$\sum_{i=1}^{n} (i-1)^{(k)} = \frac{1}{k+1} n^{(k+1)} \qquad (2.\ 7)$$

Property 1

$$\begin{aligned}\sum_{i=1}^{n}(i-1)^{(k)}\alpha_i^{(n)} &= \sum_{i=k+1}^{n}(i-1)^{(k)}\alpha_i^{(n)}\\ &= \int_{-\infty}^{\infty} xf(x)\sum_{i=k+1}^{n}\frac{n!(i-1)^{(k)}}{(i-1)!(n-i)!}\times\\ &\quad [F(x)]^{i-1}[1-F(x)]^{n-1}dF(x)\\ &= n^{(k+1)}\int_{-\infty}^{\infty}[F(x)]^k xf(x)\sum_{i=k+1}^{n}\binom{n-k-1}{i-k-1}\\ &\quad [F(x)]^{i-k-1}[1-F(x)]^{n-1}dx\end{aligned}$$

Setting $s=i-k-1$ we get after simplification

$$\sum(i-1)^{(k)}\alpha_i^{(n)}=\frac{n^{(k+1)}}{(k+1)}\alpha_{k+1}^{(k+1)} \tag{2. 8}$$

Property 2

$$\sum_{i=1}^{n}(i-1)^{(k)}w_{ii}^{(n)}=\frac{n^{(k+1)}}{(k+1)}W_{k+1,k+1}^{(k+1)} \tag{2. 9}$$

The result follows immediately from property 1

Property 3

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}(i-1)^{(p)}(j-1)^{(q)}W_{ij}^{(n)} &= \sum_{i=1}^{n}(i-1)^{(p)}(i-1)^{(q)}w_{ii}^{(n)}\\ &\quad+\sum_{i<j}\{(i-1)^{(p)}(j-1)^{(q)}\\ &\quad+(j-1)^{(p)}(i-1)^{(q)}\}w_{ij}^{(n)}\end{aligned} \tag{2. 10}$$

Property 4

$$\sum_{i=1}^{n-1}(i-1)^{(k)}(n-i)^{(l)}W_{ij}^{(n)}=\frac{n!k!l!}{(k+l+2)!(n-k-l-1)!}W_{l+1,k+1}^{(k+l+2)} \tag{2. 11}$$

For the proofs of the last 3 and 4 properties we refer to Hirai [5], David [2] and Hirai [8].

3. Coraviance between Linear Function

$$\psi_k = \frac{k+1}{n^{(k+1)}}\sum_{i=1}^{n}(i-1)^{(k)}Y_i^{(n)} \tag{3. 12}$$

$$\psi_s = \frac{s+1}{n^{(s+1)}}\sum_{i=1}^{n}(i-1)^{(s)}Y_i^{(n)} \tag{3. 13}$$

By definition we have

$$Cov(\phi_i,\phi_j)=\Omega_{ij}=\Omega_{ji}=\frac{(i+1)(j+1)}{n^{(i+1)}n^{(j+1)}}\times$$

$$E\left\{\sum_{r=1}^{n}\sum_{s=1}^{n}(r-1)^{(i)}(s-1)^{(j)}Y_r^{(n)}Y_s^{(n)}\right\}-\alpha_{i+1}^{(i+1)}\alpha_{j+1}^{(j+1)} \tag{3. 14}$$

Now

$$\begin{aligned} E\sum_{r=1}^{n}\sum_{s=1}^{n}(r-1)^{(i)}(s-1)^{(j)}Y_r^{(n)}Y_s^{(n)} &= \sum_{s=r=1}^{n}(r-1)^{(i)}(r-1)^{(j)}w_{rr}^{(n)} \\ &+\sum_{r}\sum_{<s}\{(r-1)^{(i)}(s-1)^{(j)} \\ &+(s-1)^{(i)}(r-1)^{(j)}\}W_{rs}^{(n)} \quad (3.\ 15)\end{aligned}$$

Using the properties of ordered random variables we have

$$\begin{aligned} &= \sum_{t=0}^{i}\binom{i}{t} j^{(i-t)}(j+1)!\binom{n}{j+t+1}W_{j+t+1,j+t+1}^{(j+t+1)} \\ &+\sum_{t=0}^{i}(-1)^t\binom{i}{t}(n-t-1)^{(i-t)}j!t!\binom{n}{j+t+2}W_{j+1,j+2}^{(j+t+2)} \\ &+\sum_{t=0}^{i}(-1)^t\binom{j}{t}(n-t-1)^{(j-t)}j!t!\binom{n}{j+t+2}W_{i+1,i+2}^{(i+t+2)} \\ &= S_1+S_2+S_3 \quad (3.\ 16)\end{aligned}$$

In S_1 we substitute $v=j+t+1$ and obtain

$$S_1=\sum_{v=j+1}^{i+j+1}\frac{i!j!n^{(v)}}{(i+j+1-v)!(v-j-1)!}W_{u,v}^{(v)} \tag{3. 17}$$

Using the identity (2.5) we see that

$$\begin{aligned}(n-t-1)^{(i-t)} &= (n-j-t-2+j+1)^{(i-t)} \\ &= \sum_{r=0}^{i-t}\binom{i-t}{r}(n-j-t-2)^{(r)}(j+1)^{(i-t-r)} \quad (3.\ 18)\end{aligned}$$

and writing $t=s-r$ we obtain

$$S_2 = \sum_{t=0}^{i}\binom{i}{t}(-1)^t\frac{j!t!}{(j+t+2)}W_{j+1,j+2}^{(j+t+2)}\sum_{s=t}^{i}\binom{i-t}{s-t}n^{(j+s+2)}(j+1)^{(i-s)} \tag{3. 19}$$

Changing the order of summation and putting $v=j+s^1+2$ we have

$$S_2=\sum_{u=j+2}^{i+j+2}\sum_{t=0}^{u-j-2}\frac{(-1)^t!j!(j+1)^{(i+j+2-v)}n^{(v)}}{(v-j-2-t)!(i+j+2-v)!(j+t+2)!}W_{j+1,j+2}^{(j+t+2)} \tag{3. 20}$$

By symmetry we put $i=j$ in (3.20) then

$$S_3=\sum_{s=0}^{i}\sum_{t=0}^{s}(-1)^t\frac{i!j!(n)^{(i+s+2)}}{(i+t+2)!(s-t)!(j-2)!}(i+1)^{(j-s)}W_{i+1,i+2}^{(i+t+2)}$$

Writing the value of $W_{i+1,i+2}^{(i+t+2)}$ and after little simplification we get

$$S_3 = \sum_{s=0}^{j} \frac{j!n^{(i+s+2)}(i+1)!}{(j-s)!s!(i+1-j+s)!} \int_{-\infty}^{\infty} \int_{-\infty}^{y} xy[F(x)]^i [F(y)]^s dF(x)\, dF(y) \quad (3.\ 21)$$

Now we have

$$\begin{aligned}\int_{-\infty}^{\infty} \int_{-\infty}^{y} xy[F(x)]^i [F(y)]^s dF(x) dF(y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy[F(x)]^i [F(y)]^s dF(x) dF(y) \\ &\quad - \int_{-\infty}^{\infty} \int_{y}^{\infty} xy[F(y)]^s [1 - \overline{1 - F(x)}]^i \\ &\quad dF(x) dF(y) \\ &= \frac{i!\alpha_{i+1}^{(i+1)}}{(i+1)!} - \frac{s!\alpha_{s+1}^{(s+1)}}{(s+1)!} \\ &\quad - \int_{-\infty}^{\infty} \int_{-\infty}^{y} yx[F(x)]^t \sum_{r=0}^{i} (-1)^r \binom{i}{r} \\ &\quad [1 - F(y)]^r dF(x) dF(y) \qquad (3.\ 22)\end{aligned}$$

Putting (3.22) in (3.21) and writing $v = i + s + 2$.

$$S_3 = \sum_{v=i+2}^{i+j+2} \frac{j!n^{(v)}(i+1)^{(i+j+2-v)}}{(i+j+2-v)!} \left[\frac{\alpha_{i+1}^{(i+1)} \alpha_{(v-i-1)}^{(v-i-1)}}{(i+1)!(v-i-1)!} - \sum_{r=u}^{i} (-1)^r i! W_{v-1-i,v-1}^{(v-1+r)} \right]. \quad (3.\ 23)$$

This vanishes for $i = j,\ v = j + 1$
Adding (3.17), (3.20) and (3.23) we have factorial series of the form i.e.

$$\begin{aligned} S_1 + S_2 + S_3 &= \sum_{r=1}^{n} \sum_{s=1}^{n} (r-1)^i (s-1)^{(j)} W_{rs}^{(n)} \\ &= \sum_{v=j+1}^{i+j+2} a_{ij}^{(v)} n^{(v)} \\ &= \sum_{t=j+1}^{i+j+2} a_{ij}^{(t)} n^{(t)} \qquad (3.\ 24)\end{aligned}$$

Hence we get

$$\Omega_{ij} = \frac{(i+1)(j+1)\sum_{t=j+1}^{i+j+2} a_{ij}^{(t)} n^{(t)}}{n^{(i+1)} n^{(j+1)}} - \alpha_{i+1}^{(i+1)} \alpha_{j+1}^{(j+1)} \qquad (3.\ 25)$$

We can write

$$\begin{aligned} n^{(t)} &= n(n-1)\cdots(n-j)(n-j-1)\cdots(n-t+1) \\ &= n^{(j+1)}(n-j-1)^{(t-j-1)} \qquad (3.\ 26)\end{aligned}$$

Using again the identity (2.5)

$$n^{(t+1)} = (n-j-1+j+1)^{(i+1)} = \sum_{s=0}^{i+1} \binom{i+1}{s} (n-j-1)^{(s)} (j+1)^{(i+1-s)} \quad (3.\ 27)$$

Putting (3.26) and (3.27) in (3.25) we get after simplification

$$\Omega_{ij} = \Omega_{ji} = \frac{(i+1)(j+1)}{n^{(t+1)}} \left[\sum_{s=0}^{i+1} a_{ij}^{(s+j+1)} (n-j-1)^{(s)} \times \right.$$

$$\left. \frac{\alpha_{i+1}^{(i+1)} \alpha_{j+1}^{(j+1)}}{(i+1)(j+1)} \sum_{s=0}^{i+1} \binom{i+1}{s} (n-j-1)^{(s)} (j+1)^{(i+1-s)} \right], \; i \leq j$$

$$= \frac{(i+1)(j+1)}{n^{(i+1)}} \sum_{S=0}^{i} b_{ij}^{(S)} (n-j-1)^{(S)} \tag{3. 28}$$

Using Downton's notation we have

$$b_{ij}^{(s)} = i!j! W_{s+j+1,s+j+1}^{(s+j+1)} / (i-s)!(s+j+i)!(s+j-i)!(s+j+1)!s!$$

$$+ i!j!(j+1)^{(i+1-s)} \sum_{r=0}^{s-1} (-1)^r W_{j+1,j+2}^{(j+2+r)} / (i+1-s)!(j+2+r)!(s-1-r)!$$

$$+ i!j!(i+1)^{(i+1-s)} \sum_{r=0}^{i} (-1)^r W_{s+j-i,s+j-i+1}^{(s+j-i+1+r)} / (i+1-s)!(i-r)!$$

$$(s+j-i+1+r)! + i!j! \alpha_{i+1}^{(i+1)} [\alpha_{s+j-i}^{(s+j-i)} - \alpha_{j+1}^{(j+1)}] / (i+1-s)!(s+j-i)!S! \tag{3. 29}$$

When $s = 0$, some of these terms vanish in $b_{ij}^{(s)}$. These coefficients depend only upon diagonal and next diagonal terms of relatively small variance matrices of ordered observations and expected value of the largest observations. In the evaluation of $b_{ij}^{(s)}$ we note that

$$\begin{array}{ll} (i) & W_{12}^{(2)} = (\alpha_1^{(1)})^2 \\ (ii) & W_{23}^{(4)} = 4W_{23}^{(3)} - 3(\alpha_2^{(2)})^2 \\ (iii) & W_{23}^{(3)} = 3\alpha_1^{(1)}\alpha_2^{(2)} - 3(\alpha_1^{(1)})^2 + W_{12}^{(3)} \end{array} \tag{3. 30}$$

(i) By definition we have

$$W_{12}^{(2)} = 2 \int_{-\infty}^{\infty} \int_{-\infty}^{y} xy dF(x) dF(y)$$

put

$$\int_{-\infty}^{y} x f(x) dx = g(y)$$

then

$$g'(y) = y f(y)$$

thus

$$W_{12}^{(2)} = 2 \int_{-\infty}^{\infty} g(y) g'(y) dy = [g(y)^2]_{-\infty}^{\infty} = [\int_{-\infty}^{\infty} x f(x) dx]^2 = (\alpha_1^{(1)})^2 \tag{3. 31}$$

Similarly (i) and (ii) can be proved.
and hence $b_{ij}^{(S)}$ can further be simplified.

4. Cubic Coefficients Estimates (CCE)

In Downton's general theory if we take 4 terms approximation and call it Cubic Coefficient Estimate (CCE). Suppose we have an ordered sample $x_1^{(n)} \leq x_2^{(n)} \cdots \leq x_n^{(n)}$ from the distribution $F\left(\frac{x-\mu}{\lambda}\right)$ where μ and λ are unknown parameters. We want to estimate a parametric function $p = k_1\mu + k_2\lambda$ using the available ordered observation where k_1 and k_2 are known constants.
Let

$$y_1^{(n)} = \frac{x_i^{(n)} - \mu}{\lambda} \quad i = 1, 2, \cdots, n \tag{4. 32}$$

then $y_1^{(n)} \leq y_2^{(n)} \cdots \leq y_n^{(n)}$ are the realizations of the random variables $Y_1^{(n)} \leq Y_2^{(n)} \cdots \leq Y_n^{(n)}$.
We define an estimate function Cubic Coefficient Estimate (CCE) for p as

$$U = at + bg + cs + de \tag{4. 33}$$

where

$$t = \sum_{i=1}^{n} (i-1)^{(0)} x_i^{(n)}, \; g = \sum_{i=1}^{n} (i-1)^{(1)} x_i^{(n)}, \; s = \sum_{i=1}^{n} (i-1)^{(2)} x_i^{(n)}$$

$$e = \sum_{i=1}^{n} (i-1)^{(3)} x_i^{(n)} \tag{4. 34}$$

Now using (2.9) we see that

$$\begin{aligned} E(U) &= \left(\frac{an^{(1)}}{1} + \frac{bn^{(2)}}{2} + \frac{cn^{(3)}}{3} + \frac{dn^{(4)}}{4}\right)\mu \\ &+ \left(\frac{an^{(1)}}{1}\alpha_1^{(1)} + \frac{bn^{(2)}}{2}\alpha_2^{(2)} + \frac{cn^{(3)}}{3}\alpha_3^{(3)} + \frac{dn^{(4)}}{4}\alpha_4^{(4)}\right)\lambda \end{aligned} \tag{4. 35}$$

U will be unbiased estimator of p if the coefficients are so chosen such that

$$\vec{a}\vec{\psi} = \vec{k}$$

where

$$\vec{k} = [k_1 k_2] \tag{4. 36}$$

and

$$a = [a\ b\ c\ d] : \ \psi' = \begin{bmatrix} \frac{n^{(1)}}{1} & \frac{n^{(2)}}{2} & \frac{n^{(3)}}{3} & \frac{n^{(4)}}{4} \\ \frac{n^{(1)}}{1}\alpha_1^{(1)} & \frac{n^{(2)}}{2}\alpha_2^{(2)} & \frac{n^{(3)}}{3}\alpha_3^{(3)} & \frac{n^{(4)}}{4}\alpha_4^{(4)} \end{bmatrix} \tag{4. 37}$$

We define

$$\begin{aligned} T^* &= \sum_{i=1}^{n} (i-1)^{(0)} Y_i^{(n)}; \; G^* = \sum_{i=1}^{n} (i-1)^{(1)} Y_i^{(n)} \\ S^* &= \sum_{i=1}^{n} (i-1)^{(2)} Y_i^{(n)}; \; E = \sum_{i=1}^{n} (i-1)^{(3)} Y_i^{(n)} \end{aligned} \tag{4. 38}$$

as random variables.
Now we denote

$$Var\left(\overset{*}{T}\right) = \Omega_{00}; \; Var\left(\overset{*}{G}\right) = \Omega_{11}$$

$$\Omega_{01} = Cov\begin{pmatrix} * & * \\ T & G \end{pmatrix}; \cdots Cov\begin{pmatrix} * & * \\ G, & E \end{pmatrix} = \Omega_{23}$$

and thus covariance matrix is

$$\Omega = \begin{bmatrix} \Omega_{00} & \Omega_{01} & \Omega_{02} & \Omega_{03} \\ & \Omega_{11} & \Omega_{12} & \Omega_{13} \\ & & \Omega_{22} & \Omega_{23} \\ & & & \Omega_{33} \end{bmatrix}$$

Hence we have

$$Var(U) = \lambda^2 \vec{a}\vec{\Omega}\vec{a} \tag{4. 39}$$

Thus we have to minimize the expression subject to restraint given in (4.36)

$$\chi = \lambda^2 \vec{a}\vec{\Omega}\vec{a'} + \vec{\phi}\vec{\psi}'\vec{a'} \tag{4. 40}$$

Where $\vec{\phi} = [\phi_1, \phi_2]$ and ϕ_1 and ϕ_2 are the undetermined Lagrangian multipliers. The minimization of χ is obtained by solutions of the equations with $\vec{a}\vec{\psi} = \vec{k}$ yields.

$$\vec{a} = \vec{k}[\vec{\psi}'\vec{\Omega}^{-1}\vec{\psi}]^{-1}\vec{\psi}'\vec{\Omega}^{-1} \tag{4. 41}$$

and

$$Var(U) = \lambda^2 \vec{k}[\vec{\psi}'\vec{\Omega}^{-1}\vec{\psi}]^{-1}k' \tag{4. 42}$$

If $k = [1, 0]$ we get μ^* as an estimate of μ and similarly if $k = [0, 1]$ we get λ^* as an estimate of λ. We may some times be interested in other values of k_1 and k_2.

5. Estimates of μ and λ from the Rayleigh Distribution

The probability density function of the two parameters Rayleigh distribution is

$$f(x) = \frac{2(x-\mu)}{\lambda^2} e^{-(x-\mu)^2/\lambda^2}; \ \mu \le x < \infty \ \lambda > 0 \tag{5. 43}$$

The M.L.E. of λ say $\hat{\lambda}$ is given by

$$\hat{\lambda} = \sqrt{\sum_{i=1}^{n} \frac{(x_i - \hat{\mu})^2}{n}} \tag{5. 44}$$

where $x_1^{(n)} = \hat{\mu}$ the smallest observation.
As the lower range depends upon μ we can not calculate the large sample variances of these estimates.
For the evaluation of the coefficients of μ^* and λ^* by Cubic Coefficient Estimate (CCE) we have

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix} = \begin{bmatrix} \frac{\theta_0}{n} & \frac{2\theta_1}{n^{(2)}} & \frac{3\theta_3}{n^{(3)}} & \frac{4\theta_4}{n^{(4)}} \\ \frac{\{_0}{n} & \frac{2\{_1}{n^{(2)}} & \frac{3\{_3}{n^{(3)}} & \frac{4\{_4}{n^{(4)}} \end{bmatrix} \tag{5. 45}$$

where

$$\begin{bmatrix} \vec{\theta_0'} \\ \cdots \\ \vec{\{_0'} \end{bmatrix} = \begin{bmatrix} \theta_0 & \theta_1 & \theta_2 & \theta_0 \\ \cdots & \cdots & \cdots & \cdots \\ \{_0 & \{_1 & \{_2 & \{_3 \end{bmatrix} = \Omega^{-1}[\vec{1} \vdots \vec{\alpha}] \left[\begin{bmatrix} \vec{1'} \\ \cdots \\ \vec{\alpha'} \end{bmatrix} \Omega^{-1}[\vec{1} \vdots \vec{\alpha}]\right]^{-1} \tag{5. 46}$$

and

$$\alpha' = [\alpha_1^{(1)}, \alpha_2^{(2)}, \alpha_3^{(3)}, \alpha_4^{(4)}] \text{ and } 1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad (5.\ 47)$$

Using the moments of order statistics Hirai [7] from the Rayleigh distribution we have the elements of $\vec{\Omega}$ from the Rayleigh distribution for $n = 5, 6, 7$ in Table 1.

Table 1

The Elements of Ω in Cubic Coefficient Estimate (CCE) from the Rayleigh Distribution for n = 5, 6, 7

$n = 5$	0.049204	0.0509054	0.0545378	0.0567213
		0.0681764	0.0777652	0.0841323
			0.0926029	0.10322768
				0.1180682
$n = 6$	0.357670	0.0424211	0.0454481	0.0472677
		0.564425	0.0641847	0.0693179
			0.0759378	0.0943095
				0.0956505
$n = 7$	0.0306574	0.0363610	0.0389555	0.0405152
		0.0564423	0.0641847	0.0693179
			0.0759378	0.0843095
				0.0956505

Hence the coefficient of μ^* of μ and λ^* of λ in Cubic Coefficient Estimate (CCE) are respectively as

$$a_{11} = a_1;\ a_{12} = a_1 + a_2;\ a_{13} = a_1 + 2a_2 + 2a_3 \cdots a_{1n} = a_1 + (n-1)a_2 + (n-1)(n-2)a_3 + (n-1)(n-2)(n-3)a_4 \quad (5.\ 48)$$

Similarly

$$a_{21} = b_1;\ a_{22} = b_1 + b_2;\ \text{and } a_{2n} = b_1 + (n-1)b_2 + (n-1)(n-2)b_3 + (n-1)(n-2)(n-3)b_4 \quad (5.\ 49)$$

These coefficients are given in Tables 2 and 3 and variance - covariances of these estimates are given in Table 4 for n = 5, 6, 7.

Table 2

Coefficients of μ^* by Cubic Coefficients Estimate (CCE) from the Rayleigh Distribution for $5 \le n \le 7$

n	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}
5	1.243 4932	0.2607 969	−0.058 8164	−0.119 6135	−0.325 8612		
6	1.086 9110	0.3489 273	0.0183 732	−0.078 3446	−0.1138 614	−0.262 2495	
7	0.096 0048	0.3954 130	0.0901 670	−0.039 7372	−0.0783 446	−0.109 7002	−0.21 78499

Table 3

Coefficients of λ^* by Cubic Coefficient Estimate (CCE) from the Rayleigh Distribution for $5 \le n \le 7$

n	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	a_{27}
5	−1.097 3312	−0.116 28640	0.2276 022	0.34 3886	0.642 1268		
6	−0.955 78813	−0.228 8629	−0.109 2469	0.231 7671	0.341 117	0.5317 147	
7	0.8446 933	−0.228 8629	−0.014 704	0.156 7133	0.220 3651	0.2907 3770	0.452 9087

Table 4

Variance and Covariances of μ^* and λ^* from the Rayleigh Distribution for n = 5, 6, 7. Each value should be multiplied by λ^2

n	5	6	7
Var (μ^*)	0.086745	0.06734607	0.0548885
Var(λ^*)	0.1300687	0.1024643	0.08451043
Cov(μ^*, λ^*)	−0.0826940	−0.034785	−0.051375

Hence we conclude that the Cubic Coefficient Estimate (CCE) can replace the best linear unbiased estimate (Lloyd's) method from the efficiency point of view Cubic Coefficient Estimate (CCE) is quite simple even for small samples and involves less calculations. This method is applicable for complete samples and no viable technique is as yet available to extend it to censored data.

6. Asymptotic Properties of Cubic Coefficient Estimate (CCE)

We have from (3.28)

$$\Omega_{ij} = \frac{(i+1)(j+1)}{n^{(i+1)}}[b_{ij}^{0} + (n-j-1)^{(1)} b_{ij}^{(1)} + \cdots + (n-j-1)^{(i)} b_{ij}^{(i)}] \quad (6.\ 50)$$

and this becomes

$$\simeq \frac{(i+1)(j+1)}{n} b_{ij}^{(i)}$$

Let

$$\vec{B} = \frac{1}{n}\begin{bmatrix} b_{00}^{(u)} & 2b_{01}^{(0)} & 3b_{02}^{(0)} & 4b_{03}^{(0)} \\ & 4b_{11}^{(1)} & 6b_{12}^{(1)} & 28b_{13}^{(1)} \\ & & 9b_{22}^{(2)} & 12b_{23}^{(2)} \\ & & & 16b_{33}^{(3)} \end{bmatrix} \quad (6.\ 51)$$

If $|B| \neq 0$ then asymptotic coefficients $\mu^{'*}$ and $\lambda^{'*}$ of μ and λ are given by

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix} = \begin{bmatrix} \frac{\theta_0}{n} & \frac{2\theta_1}{n^{(2)}} & \frac{3\theta_2}{n^{(3)}} & \frac{4\theta_3}{n^{(4)}} \\ \frac{\{_0}{n} & \frac{2\{_1}{n^{(2)}} & \frac{3\{_2}{n^{(3)}} & \frac{4\{_3}{n^{(4)}} \end{bmatrix}$$

where

$$\begin{bmatrix} \vec{\theta_0'} \\ \cdots \\ \vec{\{_0'} \end{bmatrix} = \begin{bmatrix} \theta_0 & \theta_1 & \theta_2 & \theta_3 \\ \cdots & \cdots & \cdots & \cdots \\ \{_0 & \{_1 & \{_2 & \{_3 \end{bmatrix} = \vec{B}^{-1}[\vec{1}\vdots\vec{\alpha}] \left[\begin{bmatrix} \bar{1} \\ \cdots \\ \bar{\alpha} \end{bmatrix} B^{-1}[\vec{1}\vdots\vec{\alpha}] \right]^{-1} \quad (6.\ 52)$$

and the variances of the estimates are calculated while calculating these coefficients.

Table 5

Asymptotic Coefficients of estimate μ'^* and λ'^* from the Rayleigh distribution for $n = 5, 6, 7$ are given in Tables 5 & 6

n	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}
5	1.3693 861	0.3479 291	−0.2367 639	−0.3846 929	−0.0958 579		
6	1.1411 550	0.4601 837	−0.0024 056	−0.2466 129	−0.2724 382	−0.272 4382	
7	0.9784 186	0.4920 105	0.1303 920	−0.1064 369	−0.2184 962	−0.205 7259	−0.068 1860

Table 6

n	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}
5	−1.150 2775	−0.23 4223	0.340 9165	0.575 1389	0.468 4449		
6	−0.958 5644	−0.034 78609	0.092 4344	0.362 3215	0.461 8004	0.390 8711	
7	0.821 6267	0.385 4099	−0.046 5979	−0194 8093	0.338 8177	0.385 4093	0.334 6021

The asymptotic variance covariance of μ'^* and λ'^* from the Rayleigh distribution for n = 5, 6, 7 are given in Table 7. Each value may be multiplied by λ^2.

Table 7

n	5	6	7
Var (μ'^*)	0.0585286	0.0487738	0.0418061
Var(λ'^*)	0.0970063	0.0808386	0.0692902
Cov(μ'^*, λ'^*)	−0.0524101	−0.0436751	−0.0374358

Mostellet [10] has shown that linear combination of order statistics tends to normality for large samples and therefore it can be shown that these estimates also tend to normality as the sample size becomes very large.

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Alternative Approach To The Persistence In A 3-Species Predator-Pray Modal

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Abstract. The ecological notion of system persistence in modeling the interaction of two competing predatory populations living exclusively on a common prey, is investigated.
Freedom and Waltman [3], and El-Owaidy and Ammar [4] have discussed the persistence of such models based upon the assumption of nonexistence of limit cycles. In this paper, however ; the nonexistence of limit cycles is proven, first, by global asymptotic stability of equilibria through the constriction of a suitable Lyapunov function, and second, persistence criteria of the system are obtained.

AMS (MOS) Subject Classification Codes: 92B05.
Key Words: Predator - prey system, Lyapunov function, limit cycle, asymptotic global stability.

1. Introduction

One of the most interesting questions in ecological models concerns the survival of all species of the model. This question of persistence becomes more important when two or more species compete for a single prey species. The significant concepts of permanence and persistence, both exclude extinction of species for all positive initial conditions. In biological terms, persistence means that the density of each population remains, asymptotically, above a positive bound independent of the initial conditions, i.e. all species stay away from extinction.
Mathematically, this may be stated in terms of behaviour of solutions of the modal which represents the biological phenomenon. An ecological differential system.

$$\dot{x}_i = X_i f(X_1, X_2, ..., X_n); \text{ for } i = 1, 2, ..., n$$

is said to be permanent or uniformly persistent if there exists a compact set k in the interior of $R^n = x \in R^n : x_i \geq 0$ for $1 \leq i \geq n$ such that all orbits end up in k. This guarantees that the number of each population $x_i(t)$ is bounded away from zero if $x_i(0) > 0$ for all i.
By the term weak persistence, we mean that for all i, *limsup* $x_i(t) > 0$ whenever $x(0) > 0$. Under this "lim sup" definition of persistence, a population can frequently approach extinction. By the term strong persistence, we mean that for all i, $t \to \infty$ *liminf* $x_i(t) > 0$ whenever $x(0) > 0$. We have applied the latter definition of persistence as used in Freedman and Waltman [3]. This definition of

persistence was reformulated in [3] as follows: "A persistence with initial conditions in the positive cone will persist if there are no ω- limit point s of the solution on the boundary of the positive cone". This means if $\Box\,(X)$ be the orbit through the point $X = (x, y, z)$ with $x > 0,\ y > 0,\ z > 0$, and if $\Omega(X)$ be the $\omega - limit$ set of $\Box\,(X)$, then $\Omega(X)$ be interior to the positive cone. Since, the question of two predator populations competing for a single prey population has occupied an important place in ecological literature, so in this paper, we have considered a system modelling the interactions of two competing predator populations living exclusively on a common prey. For both the predator populations we have taken density - dependent death rates and in this sense, we have extended the result of EL-Owaidy and Ammar [4] with an alternative approach.

Freedman and Waltman [3] in theorem 4.1, obtained persistence criteria by an assumption that for such a system, there are no limit cycles surrounding an interior equilibrium in a co-ordinate plane. We have actually proved the non-existence of limit cycles by constructing Lyapunov functions for the equilibria and have obtained conditions under which the equilibria are globally asymptotically stable. This implies that all the trajectories of the system in the positive cone will spirsl toward this equilibrium point, that is non-existence of limit cycles.

Then, we have applied the persistence criteria as given in theorem 4.1 under reference, to obtain conditions for persistence of our system. thus our result of persistence seems to be more comprehensive than that in theorem 4.1 of Freedman and Waltman [3].

The organization of this paper is as follows:

In section 2, we give the model. Section 3 deals with the global asymptotic stability of the problem. Section 4 deals with persistence results and we illustrate these by an example.

2. The Model And Equilibria

A system modelling the interaction of two competing predator populations $y(t)$ and $z(t)$ living exclusively on a common prey $x(t)$ is given by

$$\begin{aligned} \dot{x} &= xf(x) - yp(x) - zq(x) \\ \dot{y} &= y[-g(y) + c_1 p(x)] \\ \dot{z} &= z[-h(z) + d_1 q(x)] \qquad (2.\ 1) \\ X(0) = x_0 > 0,\quad & Y(0) = y_0 \geq 0,\quad Z(0) = z_0 \geq 0. \end{aligned}$$

where $(. = \frac{d}{dt})$, $c_1,\ d_1$ are positive constants, known as food conversion rates. We make the following assumptions, which are consistent with models of predator - prey systems. :

(H1) : $f(x)$ [specific growth rate of prey x]:

$$f(0) > 0, f'(x) \leq 0 \text{ for all } x < 0.$$

There exists $k < 0$ (the carrying capacity) such that $f(x) > 0$ on

$$0 \leq x < k, f(k) = 0 \text{ and } f(x) < 0 \text{ on } x > k.$$

(H2) $p(x)$: the functional response of the predator y with respect to the prey x and

$$p(0) = 0, p'(x) > 0 \text{ for all } x \geq 0.$$

(H3) $q(x)$: the functional response of the predator z with respect to the prey x and

$$q(0) = 0, q'(x) > 0 \text{ for all } x \geq 0.$$

(H4) : we assume $q(x) = p(x)$, where is a positive constant
(H5) : $g(y)$; density - dependent death rate of the predatory and

$$g(y) := g'(y) \geq 0 \text{ for all } y > 0.$$

(H6) : $h(z)$;density -dependent death rate of the predator z and

$$h(0) > 0, h'(z) > 0 \text{ for all } z > 0.$$

Clearly the system (2.1) has equilibrium point $E_0(0,0,0)$. By assumption (H1), $E_1(k,0,0)$ is also an equilibrium point. We assume that each of the predators y and z can survive on the prey x, that is there exist equilibrium points

$$E^*(x^*, y^*, 0) \text{ and } \hat{E}(\hat{x}, 0, \hat{z}) \text{ such that}$$

$$\begin{aligned} \dot{x}f(\dot{x}) - \dot{y}p(\dot{x}) &= 0 \\ -g(\dot{y}) + c_1 p(\dot{x}) &= 0 \end{aligned} \qquad (2.\ 2)$$

and

$$\begin{aligned} \hat{x}f(\hat{x}g - \hat{z}q(\hat{x}) &= 0 \\ -h(\hat{z}) + d_1 q(\hat{x}) &= 0 \end{aligned} \qquad (2.\ 3)$$

where $x^*, y^*, \hat{x}, \hat{z}, > 0$ and $x^* < k, \hat{x} < k$.

3. Global Asymptotic Stability Of Equilibria

$$E^*(x^*, y^*, 0) \text{ and } E^*(x^*, y^*, 0) \text{ and } \hat{E}(\hat{x}, 0, \hat{z})$$

Lemma 1. *Assume that (H1) -(H6) hold for the system (2.1) and in a neighborhood of $(x^*, y^*, 0)$ in the positive cone, the function $\frac{xf(x)}{p(x)}$ is strictly decreasing. Then the equilibrium point $E^*(x^*, y^*, 0)$ is globally asymptotically stable.*

Proof. Define a Lyapunov function $V(x, y, z)$ as [see [1]]:

$$V(x,y,z) = \int_{x^*}^{x} [c_1(1 - \frac{p(x^*)}{p(w)}) + d_1(1 - \frac{q(x^*)}{q(w)})]dw + \int_{y^*}^{y} \frac{w - y^*}{w} dw + z$$

Now $E^*(x^*, y^*, 0) = 0$ and due to (H2) and (H3), (x, y, z) is positive in the region: $0 < x < x^* < k,\ 0 < y^* < y < \tilde{\beta}_1, 0 < z < \beta_2,$ where $\beta_1,\ \beta_2$ are positive constants.

$$+d_1[q(x) - q(\dot{x})][\frac{xf(x)}{q(x)} - \frac{yp(x)}{q(x)} - z]$$

$$+(y - y^*)[-g(y) + c_1 p(x)] + z[-h(z) + d_1 q(x)]$$

using (2.2), (H4) and with some algebraic manipulations we get

$$\begin{aligned}\dot{V}(x,y,z) &= [c_1(p(x)-p(\dot{x}))+\frac{1}{\alpha}d_1(q(x)-q(\dot{x}))][\frac{xf(x)}{p(x)}-\frac{\dot{x}f(x^*)}{p(x^*)}] \\ &\quad +(y-y^*)[(g(y^*)-g(y))+\frac{d_1}{\alpha}(q(x^*)-q(x))] \\ &\quad +z[\alpha c_1(p(x^*)-p(x))+(h(0)-h(z))]<0\end{aligned}$$

Thus E^* ($x^*, y^*, 0$) is globally asymptotically stable. □

Lemma 2. *Assume that (H1)-(H6) holds for the system(2.1) and in a neighbourhood of* $x(\hat{x},0,\hat{z})$ *in the positive cone, the function* $\frac{xf(x)}{p(x)}$ *is strictly decrictly decreasing. Then the equilibrium point* $\hat{E}(\hat{x},0,\hat{z})$*is globally asymptotically stable.*

Proof. Define Lypunov function (x,y,z) as:

$$V(x,y,z)=\int_{x}^{x}[c_1(1-\frac{p(\hat{x})}{p(w)})+d_1(1-\frac{q(\hat{x})}{q(w)})]dw+y+\int_{\hat{z}}^{x}\frac{w-\hat{z}}{w}dw$$

Rest of the proof follows as in **Lemma** 1. □

Remark 3. We consider equilibrium points $E_0(0,0,0)$ and $E_1(k,0,0)$. The eigenvalues of the variational matrix (E_0) of the system (2.1) about $E_0(0,0,0)$ are:

$$\lambda_1=f(0)>0, \lambda_2=-g(0)<0, \text{ and } \lambda_3=-h(0)<0.$$

Clearly $E_0(0,0,0)$ is a hyperbolic point and is unstable along the x-axis.This implies that the prey population x grows near E_0. The eigenvalues of the variational matrix (E_1) of the system (2.1) about $E_1(k,0,0)$ are:

$$\lambda_1=kf'(k)<0, \lambda_2=-g(0)+c_1p(k),\ \lambda_3=-h(0)+d_1q(k).$$

Thus $E_1(k,0,0)$ is asymptotically stable along the x-axis.This implies that the prey population x remains in neighbourhood of k.

Remark 4. For existence of $E^*(x^*,y^*,0)$ and $\hat{E}(\hat{x},0,\hat{z})$ it is necessary that

$$g(0)+c_1p(k)>0 \text{ and } -h(0)+d_1q(k)>0$$

As it implies increase of predator population and predator population Z.

4. Persistence Criteria

In section 3, we have given necessary conditions for existence of equilibria $E^*(x^*,y^*,0)$ and $\hat{E}(\hat{x},0,\hat{z})$ and criteria for their global asymptotic stability.
In this section we shall assume global stability of these equilibria and obtain persistence criteria for the system (2.1). First,we prove the following two lemmas.

Lemma 5. *The equilibrium* $E^*(x^*,y^*,0)$ *in the interior of the* $x-y$ *plane is unstable in the positive direction orthogonal to* $x-y$ *plane if*

$$-h(0)+d_1q(x^*)>0 \text{ or } -h(0)+d_1\alpha p(x^*)>0$$

Proof. The proof is immediate upon computing the variational matrix (E) of system (2.1) about $E^*(x^*,y^*,0)$. We have:

$$V(E^*)\begin{pmatrix} f(x^*)+x^*f^1(x^*)-y^*p'(x^*) & -p(x^*) & -q(x^*) \\ -gc_1y^*p'(x^*) & (y^*)+c_1p(x^*)-y^*g(y^*) & 0 \\ 0 & 0 & -h(0)+d_1q(x^*)\end{pmatrix}$$

Thus if $-h(0) + d_1 q(x^*) >$ or $-h(0) + \delta_{1\alpha} p(x^*) > 0$ we have the result. □

Lemma 6. *The equilibrium $\hat{E}(\hat{x}, 0, \hat{z})$ in the interior of the $x - z$ plane is unstable in the positive direction orthogonal to $x - z$ plane if $-g(0) + c_1 p(\hat{x}) > 0$.*

Proof. The proof is immediate upon computing the variational matrix $V(E^*)$ of system (2.1) about $\hat{E}(\hat{x}, 0, \hat{z})$. Now, to apply persistence criteria to our system (2.1), we have to check hypotheses $(B1) - (B4)$ of Theorem 4.1, in Freedman and Waltman [3] and boundedness of solutions. We have:

$$\begin{aligned} F(x, y, z) &= f(x) - y\frac{p(x)}{x} - z\frac{q(x)}{x} \\ G_1(x, y, z) &= -g(y) + c_1 p(x) \\ G_2(x, y, z) &= -h(z) + d_1 q(x) \end{aligned}$$

Therefore, condition (B1) is trivially satisfied due to (H1)-(H6). Also notice that $p(0) = 0$ and $p'(x) > 0$ implies $p(x)$ is strictly increasing positive function, similarly $q(x)$. Condition(B2) is true due to (H1).

$$F(0, 0, 0) = f(0) > 0, F(k, 0, 0) = f(k) = 0$$

$$\frac{\partial f}{\partial x}(x, 0, 0) = f'(x) \leq 0,$$

satisfying (B3) there are no equilibria on y-axis or z-axis in $y - z$ plane, for if we suppose that there exists an equilibrium $E(0, y_1, z_1)$ in $y - z$ plane which is given by:

$$g(y_1) = 0 \textit{ and } h(z_1) = 0,$$

then this contradicts (H5) and(H6), and satisfying (B4) each predator can survive on prey, that is, there exist points $E^*(x^*, y^*, 0)$ and $\hat{E}(\hat{x}, 0, \hat{z})$ such that (2.2) and (2.3) hold. Also see Remark 4. Moreover, we require;
5) : Boundedness of solutions of system (2.1).We suppose that the functions $f(x)$, $p(x)$, $q(x)$, $g(y)$ and $h(z)$ are sufficiently smooth so that the solution to initial value problem (2.1) exists, is unique and continuable for positive values of t. Regarding boundedness of solution, see Freedman and Waltman [2] and [3]. □

We state and prove the main theorem.

Theorem 7. *Let (B1)-(B5) hold and $\frac{xf(x)}{p(x)}$ be strictly decreasing function and*

$$-g(0) + c_1 p(x) > 0 \text{ and } -h(0) + d_1 \alpha p(x) > 0.$$

Then the system (2.1) persists.

Proof. By (B5) solutions are bounded. By Remark 3 the equilibrium $E_0(0, 0, 0)$ is unstable along the x-axis and unstable manifold of $E_1(k, 0, 0)$ is two dimensional. Conditions (4.1) follow from Lemmas 5 and 6. Non-existence of limit cycles follows from Lemmas 1 and 2. This completes the proof. □

Remark 8. From (H4), we have

$$\alpha = \frac{q(x)}{p(x)} = \frac{\text{rate of prey consuon per predator } z \text{ at prey density } x}{\text{rate of prey consuon per predator } y \text{ at prey density } x}$$

Thus, if we consider α as parameter, the system (2.1) will persist provided

$$\alpha > \frac{h(0)}{\alpha_1 d_1 p(\dot{x})}$$

Remark 9. We have discussed persistence criteria for a system modeling the interaction of two competing predator populations living exclusively on a common prey. But in the same way, the persistence criteria can be obtained for the system modeling interactions between two prey populations and one predator population, that is, for construction of Lyapunov functions for the systems (4.2) and (4.3) see [1]. System modeling interactions between two prey populations and one predator population, that is,

$$\left.\begin{aligned} \dot{x} &= xf(x) = zq(x) \\ \dot{y} &= yg(y) - zr(y) \\ \dot{z} &= z[-h(z) + d_1 q(x) + d_2 r(y)] \end{aligned}\right\} \quad (4.\ 4)$$

Example 4.1

To illustrate the Theorem 7, consider the system with the Holling type II functional response.

$$\begin{aligned} \dot{x} &= x(1-x) - y\frac{2x}{1+x}z\alpha\frac{2x}{1+x} \\ \dot{y} &= y[-1(1+y) + \frac{33}{16}\frac{2x}{1+x}] \\ \dot{z} &= z[-1(1-z) + \frac{11}{q}\alpha\frac{2x}{1+x}] \end{aligned}$$

Here, $k=1$, $\frac{xf(x)}{p(x)} = \frac{1-x^2}{2}$. Thus $\frac{xf(x)}{p(x)}$ is a strictly decreasing function for $x > 0$.

$$E^*(x^*, y^*, 0) = E^*(\frac{1}{2}, \frac{3}{8}, 0)$$

$$p(x^*) = p(\frac{1}{2}) = \frac{2}{3}$$

$$\alpha > \frac{h(0)}{d_1 p(\dot{x})} = \frac{27}{22}.$$

Thus take $\alpha = 2$

$$\hat{E}(\hat{x}, 0, \hat{z}) = \hat{E}(\frac{1}{3}, 0, \frac{2}{9})$$

To check condition (4.4)

$$-g(0) + c_1 p(\hat{x}) = -1 + \frac{33}{16} \times \frac{1}{2} = \frac{1}{32} > 0$$

$$-h(0) + d_1 \alpha p(x^*) = -1 + \frac{11}{9} \times 2 \times \frac{2}{3} = -1 + \frac{44}{27} = \frac{17}{27} > 0$$

Theorem 4.1 applies and hence the system (4.4) is persistent.

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On Stratification Of L-Fuzzy Topological Structures

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Abstract. We introduce the stratifications of an L-fuzzy topogenous, proximity and uniformity. We investigate the relationships between these structures and the stratifications of them.

Keywords and Phrases: Stratified L-fuzzy topogeneous, proximity, uniformity, stratifications

1. Introduction and preliminaries

In [25] Čsaszar introduce the concept of a syntopogenous structure to develop a unified approach to the three main structures of set-theoretic topology: topologies, uniformities and proximities. This enabled him to evolve a theory including the foundations of three classic theories of topological spaces, uniform spaces and proximity spaces. In the case of the fuzzy structures there are at least two notions of fuzzy syntopogenous structures, the first notion worked out in [13, 10] presents a unified approach to the theories of Chang fuzzy topological spaces [4], Hutteon fuzzy uniform spaces [7], Katsaras fuzzy proximity spaces [9, 8] and Artico fuzzy proximity spaces [1]. The second notion worked out in [11, 12] agree very well with Lowen fuzzy topological spaces [18], Lowen-Höhle fuzzy uniform spaces [19] and Artico-Moresco fuzzy proximity spaces [1].Šostak [26] introduced the notion of (L-)fuzzy topological spaces as a generalization of L-topological spaces(originally called (L-)fuzzy topological spaces by Chang [4]). It is the grade of openness of L-fuzzy set. Badard introduced the concept of smooth structure and gives some rules and shows how such an extension can be realized [2]. In [22], Ramadan introduced the similar definition of Šostak (L-)fuzzy topology [26] under the name "smooth topology". Also, in [21] Badard et al. introduced the concept of smooth preuniform and preproximity spaces.
In this paper, we can obtain the stratified L-fuzzy topogenous(resp. proximity. uniformity) from an L-fuzzy topogenous(resp. proximity. uniformity). It is called straification. We prove that the L-fuzzy topogenous(resp. proximity) associated to a stratified L-fuzzy uniformity U(resp. topogenous N) also are stratified. Moreover the stratification of the L-fuzzy topology(resp. proximity) associated to U (resp.N) coincide with the L-fuzzy topology (resp. proximity) associated to the stratification of U (resp.N).

Throughout this paper, let X be a nonempty set. Let a complete lattice $L = (L, \leq, \vee, \wedge, ')$ be a completely distributive complete lattice with an order-reversing involution on it, and with a smallest element 0 and largest element 1

$(0 \neq 1)$ [27]. Let X be a non-empty set L^X denotes the collection of all mappings from X into L. The elements of L^X are called L-fuzzy sets on X. L^X can be made into a fuzzy lattice by including the order and involution from $L = (L, \leq, \vee, \wedge, ')$. We say that the fuzzy points x_α belongs to a fuzzy set λ i.e., $x_\alpha \in \lambda$ iff $\alpha \leq \lambda(x)$, for $\alpha \in L$, $\underline{\alpha}(x) = \alpha,\ for\ all\ x \in X$, and the set of all fuzzy points in L^X is denoted by Pt (L^X). Let Ω_X denote the family of all mapping $a : L^X \longrightarrow L^X$ with the following properties:
(i) $a(\underline{0}) = \underline{0}$, $\lambda \leq a(\lambda)$ for each $\lambda \in L^X$,
(ii) $a(\bigvee_{i\in\Gamma} \lambda_i) = \bigvee_{i\in\Gamma} a(\lambda_i)$, for $\lambda_i \in L^X$.
For $a \in \Omega_X$, the mapping $a^{-1} \in \Omega_X$ *and* $a \wedge b : L^X \longrightarrow L^X$ are defined by $a^{-1}(\lambda) = \bigwedge\{\mu : a(\mu) \leq \lambda\}$ *and* $(a \wedge b)(\mu) = \bigwedge\{a(\mu_1) \vee b(\mu_2) : \mu_1 \vee \mu_2 = \mu\}$. Then $(a^{-1})^{-1} = a$ and $a_1 \leq a_2$ iff $(a_1)^{-1} \leq (a_2)^{-1}$. For any $a, b \in \Omega_X$ and $\mu \in L^X$, $(a\ o\ b)(\mu) = a(b(\mu))$ [16]. Notation and notion not described in this paper are standard and usual.

Proposition 1. [3] *For each $\alpha \in L$, define the mapping $\widehat{\alpha} : L^X \longrightarrow L^X$ by*

$$\widehat{\alpha}(\lambda) = \begin{cases} \underline{\sup(\lambda)} : \ if\ \sup(\lambda) \leq \alpha, \\ \underline{1}\ :\ o.\omega \end{cases}$$

for each $\lambda \in L^X$, where $\sup(\lambda) = \bigvee_{x\in X} \lambda(x)$. Then
(i) $\widehat{\alpha} \in \Omega_X$ (ii) $\widehat{\alpha}\ o\ \widehat{\alpha} = \widehat{\alpha}$ for each $\alpha \in L$.
(iii) If $\alpha \leq \beta$, then $\widehat{\alpha} \geq \widehat{\beta}$ for each $\alpha, \beta \in L$. (iv) $a \leq \widehat{0}$ for each $a \in \Omega_X$.

Lemma 2. [3] *Let $a_\lambda : L^X \longrightarrow L^X$ be a mapping define as follows:*

$$a_\lambda(\gamma) = \begin{cases} \lambda : if\ \gamma \leq \lambda, \\ \underline{1}\ :\ o.\omega. \end{cases}$$

Then $a_\lambda \in \Omega_X$ and $a_\lambda\ o\ a_\lambda = a_\lambda$ for any $\lambda \in L^X$

Definition 3. [6, 20] A mapping $T : L^X \longrightarrow L$ is called an L-fuzzy topology on X if it satisfies the following conditions:
(O_1) $T(\underline{0}) = T(\underline{1}) = 1$,
(O_2) $T(\lambda \wedge \mu) \geq T(\lambda) \wedge T(\mu)$ *for any* $\lambda,\ \mu \in L^X$,
(O_3) $T(\bigvee_{k\in\Gamma} \lambda_k) \geq \bigwedge_{k\in\Gamma} T(\lambda_k)$ *for any* $\{\lambda_k\}_{k\in\Gamma} \subset L^X$.

The pair (X, T) is called an L-fuzzy topological space (L-fts, for short). An L-fts (X, T) is called stratified if $T(\underline{\alpha}) = 1$ for each $\alpha \in L$ [?].
Let (X, T) and (Y, η) be L-fts's. A mapping $f : (X, T) \longrightarrow (Y, \eta)$ is called L-fuzzy continuous if $T(f^{-1}(\mu)) \geq \eta(\mu)$ for all $\mu \in L^Y$. Let T_1 and T_2 be L-fuzzy topologies on X. We say that T_1 is finer than T_2 (T_2 is coarser than T_1) if $T_2(\lambda) \leq T_1(\lambda)$, for all $\lambda \in L^X$.

Theorem 4. [15] *Let (X, T) be L-fts.Define the mapping $T_s : L^X \longrightarrow L$ as follows: for each $\lambda \in L^X$,*

$$T^{st}(\lambda) = \bigvee_{\{(\lambda_k, \underline{\alpha}_k) | k\in\Gamma\} \in N(\lambda)} \ \bigwedge_{(\lambda_i, \underline{\alpha}_i) \in \{(\lambda_k, \underline{\alpha}_k) | k\in\Gamma\}} T(\lambda_i),$$

where $N(\lambda) = \{\{(\lambda_k, \underline{\alpha}_k) \mid k \in \Gamma\} \mid \lambda = \bigvee_{k\in\Gamma}(\lambda_k \wedge \underline{\alpha}_k)$. Then T_s the coarsest stratified L-fuzzy topology on X which is finer than T. And T_s is called the stratification of an L-fuzzy topology T on X.

Definition 5. [5] A mapping $\delta : L^X \times L^X \longrightarrow L$ is said to be L-fuzzy quasi-proximity on X, which satisfied the following conditions:
(FP1) $\delta(\underline{0}, \underline{1}) = 0$,
(FP2) $\delta(\lambda \vee \rho, \mu) = \delta(\lambda, \mu) \vee \delta(\rho, \mu)$ *and* $\delta(\lambda, \mu \vee \nu) = \delta(\lambda, \mu) \vee \delta(\lambda, \nu)$.
(FP3) *For any* $\lambda, \mu \in L^X$, *there exist* $\rho \in L^X$ such that

$$\delta(\lambda, \mu) \geq \bigwedge_{\rho \in L^X} (\delta(\lambda, \rho) \vee \delta(\rho, \mu)).$$

(FP4) $\delta(\lambda, \mu) \neq 1$, *then* $\lambda \leq \mu$.
The pair (X, δ) is called L-fuzzy quasi-proximity space. An L-fuzzy quasi-proximity space (X, δ) is called L-fuzzy proximity space if (FP) $\delta(\lambda, \mu) = \delta(\mu, \lambda)$ for any $\mu, \lambda \in L^X$. Let (X, δ_1) and (X, δ_2) be L-fuzzy proximity space. A mapping $f : (X, \delta_1) \longrightarrow (Y, \delta_2)$ is L-fuzzy proximity continuous if satisfies $\delta_1(\lambda, \mu) \leq \delta_2(f(\lambda), f(\mu))$, for any $\lambda, \mu \in L^X$.
Theorem 6. [17] *Let* $\{(X_i, \delta_i)\}_{i \in \Gamma}$ *be a family of L-fuzzy quasi-proximity spaces,* $X = \Pi_{i \in \Gamma} X_i$ *a product set and, for each* $i \in \Gamma$. $\pi_i : X \longrightarrow X_i$ *a projection mapping. Define the mapping* $\Pi_{i \in \Gamma} \delta_i : L^X \times L^X \longrightarrow L$ *on* X *by for any* $\lambda, \mu \in L^X$,

$$\Pi_{i \in \Gamma} \delta_i(\lambda, \mu) = \bigwedge \{ \bigvee_{j,k} \bigwedge_{i \in \Gamma} \delta_i(\pi_i(\lambda_j), (\pi_i(\mu_k)) \}$$

Where $\bigwedge$ *is taken over all finite families* $\{\lambda_j : \lambda = \bigvee_j \lambda_j\}$ *and* $\{\mu_k : \mu = \bigvee_k \mu_k\}$. *Then* $\Pi_{i \in \Gamma} \delta_i$ *is the coarsest L-fuzzy quasi-proximity on* X *which all* π_i *is L-fuzzy quasi-proximity continuous.* $\Pi_{i \in \Gamma} \delta_i$ *is called the product L-fuzzy quasi-proximity structure with respect to a family* $\{\pi_i : X \longrightarrow X_i : i \in \Gamma\}$.
Definition 7. [23] A mapping $N : L^X \times L^X \longrightarrow L$ is said to be L-fuzzy topogenous order on X. Which satisfied the following conditions:

$$\begin{array}{ll}
(FN1) & N(\underline{1}, \underline{1}) = N(\underline{0}, \underline{0}) = 1, \\
(FN2) & If\ N(\lambda, \mu) \neq 0, then\ \lambda \leq \mu, \\
(FN3) & If\ \lambda \leq \lambda_1, \mu_1 \leq \mu,\ then\ N(\lambda_1, \mu_1) \leq N(\lambda, \mu), \\
(FN4) & (i)\ N(\lambda_1, \mu_1) \wedge N(\lambda_2, \mu_2) \leq N(\lambda_1 \vee \lambda_2, \mu_1 \vee \mu_2), \\
 & (ii)\ N(\lambda_1, \mu_1) \wedge N(\lambda_2, \mu_2) \leq N(\lambda_1 \wedge \lambda_2, \mu_1 \wedge \mu_2).
\end{array}$$

The pair (X, N) is called L-fuzzy topogenous order space. The L-fuzzy topogenous order N is called
(i) Symmetrical iff $N(\lambda, \mu) = N(\mu, \lambda)$.
(ii) Perfect iff $N(\bigvee_{i \in \Gamma} \lambda_i, \bigvee_{i \in \Gamma} \mu_i) \geq \bigwedge_{i \in \Gamma} N(\lambda_i, \mu_i)$.

Let (X, N_1) and (Y, N_2) be L-fuzzy topogenous order spaces. A mapping $f : (X, N_1) \longrightarrow (Y, N_2)$ is L-fuzzy topogenous continuous if satisfies
$N_1(f^{-1}(\lambda), f^{-1}(\mu)) \geq N_2(\lambda, \mu)$, for any $\lambda, \mu \in L^Y$.
Theorem 8. [24] *(i) Let* N *be a perfect L-fuzzy topogenous structure on* X. *Define a mapping* $T_N : L^X \longrightarrow L$ *by* $T_N(\lambda) = N(\lambda, \lambda)$. *Then* T_N *is L-fuzzy topology on* X.
(ii) Let T *be an L-fuzzy topology on* X. *Define a mapping* $N_T : L^X \times L^X \longrightarrow L$ *by*

$$N_T(\lambda, \mu) = \bigvee \{T(\nu) : \lambda \leq \nu \leq \mu\}.$$

Then N_T is perfect L-fuzzy topogenous order on X.
(iii) Let N be a symmetrical L-fuzzy topogenous on X. Define a mapping $\delta_N : L^X \times L^X \longrightarrow L$ by $\delta_N(\lambda,\mu) = (N(\lambda,\mu))$. Then δ_N is L-fuzzy proximity on X.
(iv) Let δ be an L-fuzzy proximity on X. Define a mapping $N_\delta : L^X \times L^X \longrightarrow L$ by $N_\delta(\lambda,\mu) = (\delta(\lambda,\mu))$. Then N_δ is a symmetrical L-fuzzy topogenous on X.
Theorem 9. [24] *Let $\{(X_i, N_i)\}_{i\in\Gamma}$ be a family of L-fuzzy topogenous order spaces, $X = \Pi_{i\in\Gamma}X_i$ a product set and, for each $i \in \Gamma, \pi_i : X \longrightarrow X_i$ a projection mapping. Define the mapping $\Pi_{i\in\Gamma}N_i : L^X \times L^X \longrightarrow L$ on X by for any $\mu,\lambda \in L^X$,*

$$\Pi_{i\in\Gamma}N_i(\lambda,\mu) = \bigvee\{\bigwedge_{j,k}\bigvee_{i\in\Gamma} N_i(\pi_i(\lambda_j),(\pi_i(\mu_k)))\},$$

Where $\bigvee$ is taken over all finite families $\{\lambda_j : \lambda = \bigvee_j \lambda_j\}$ and $\{\mu_k : \mu = \bigwedge_k \mu_k\}$. Then $\Pi_{i\in\Gamma}N_i$ is the coarsest L-fuzzy topogenous on X which all π_i is L-fuzzy topogenous continuous. $\Pi_{i\in\Gamma}N_i$ is called the product L-fuzzy topogenous structure with respect to a family $\{\pi_i : X \longrightarrow X_i : i \in \Gamma\}$.
Definition 10. [16] A mapping $U : \Omega_X \longrightarrow L$ is said to be an L-fuzzy quasi-uniformity on X if it satisfied the following conditions:

$(FU1)$	*for $a, b \in \Omega_X$, $U(a \wedge b) \geq U(a \wedge b)$,*
$(FU2)$	*for $a \in \Omega_X$, there exists $a_1 \in \Omega_X$ with $a_1 \, o \, a_1 \leq a$ such that $U(a_1) \geq U(a)$,*
$(FU3)$	*If $a \geq a_1$ such that $U(a) \geq U(a_1)$,*
$(FU4)$	*there exists $a \in \Omega_X$ such that $U(a) = 1$.*

The pair (X,U) is said to be an L-fuzzy quasi-uniform space. An L-fuzzy quasi-uniform space (X,U) is called L-fuzzy uniform space if the following is satisfies: (FU) for $a \in \Omega_X$, there exists $a_1 \in \Omega_X$ with $a_1 \leq a^{-1}$ such that $U(a_1) \geq U(a)$.
Theorem 11. [14] *Let (X,U) be an L-fuzzy uniform space . Define $\delta_U(\lambda,\mu) = \bigwedge\{(U(a)) : a(\mu) \leq \lambda\}$. Then (X, δ_U) is L-fuzzy proximity space.*
Theorem 12. [16] *Let (X,U) be an L-fuzzy quasi-uniform space .For each $\alpha \in L$, $\lambda \in L^X, C_U(\lambda,\alpha) = \bigwedge\{a^{-1}(\lambda) : U(a) > \alpha\}$ and $I_U(\lambda,\alpha) = \bigvee\{\mu : a(\mu) \leq \lambda$ for some $U(a) > \alpha\}$. Then $I_U(\lambda,\alpha) = (C_U(\lambda,\alpha))$. for any $\lambda \in L^X$.*
Consider the mapping $T_U : L^X \longrightarrow L$ define by

$$T_U(\lambda) = \bigvee\{\alpha : I_U(\lambda,\alpha) = \lambda\}.$$

Then T_U is an L-fuzzy topology on X.

2. Stratifications of L-fuzzy topogenous order spaces

An L-fuzzy topogenous order $N : L^x \times L^x \rightarrow L$ is said to be stratified L-fuzzy topogenous order on X, iff N satisfied the following conditions: (FNS) $N\ (\underline{\alpha},\underline{\alpha}) = 1$, for each $\alpha \in L$. And the pair (X,N) is called stratified L-fuzzy topogenous order space.

Theorem 13. *Let (X, N) be an L-fuzzy topogenous order space. We define for all* $\mu,\ \lambda \in L^x$,

$$N^s(\lambda,\mu) = \bigvee_{(\lambda_i,\mu_i,\underline{\alpha}_i)|i\in N\}\in M(\lambda,\mu)} \{ \bigwedge_{(\lambda_l,\mu_l,\underline{\alpha}_l)\in\{\lambda_i,\mu_i,\underline{\alpha}_i)|i\in N\}} N(\lambda_l,\mu_l)\},$$

where

$$M(\lambda,\mu) = \{\{(\lambda_i,\mu_i,\underline{\alpha}_i)|i\in N,\ N \text{ finite index set}\}|\lambda\leq \bigvee_{i\in N}(\lambda_i\wedge\underline{\alpha}_i),\ \mu\geq\bigvee_{i\in N}(\mu_i\wedge\underline{\alpha}_i)\}.$$

Then N^s is the coarsest stratified L-fuzzy topogenous order on X which finer than N.

Proof. (FN1), (FN2) and (FN3) Obvious.
(FN4) (i) Suppose that there exist $\mu_1,\mu_2,\lambda_1,\lambda_2 \in L^X$, such that

$$N^s(\lambda_1,\mu_1)\wedge N^s(\lambda_2,\mu_2)\not\leq N^s(\lambda_1\vee\lambda_2,\mu_1\vee\mu_2).$$

By the definition of N^s, there are $\{\lambda_i,\mu_i,\underline{\alpha}_i)|i\in N\}\in M(\lambda_1,\mu_1)$ and $\{(\lambda_j,\mu_j,\underline{\beta}_j)|,\ j\in K\}\in M(\lambda_2,\mu_2)$ such that

$$\begin{aligned} N^s(\lambda_1\vee\lambda_2,\mu_1\vee\mu_2) &\not\geq \left(\bigwedge_{(\lambda_l,\mu_l,\underline{\alpha}_l)\in\{(\lambda_i,\mu_i,\underline{\alpha}_i)|i\in N\}} N(\lambda_l,\mu_l)\right) \\ &\quad\wedge\left(\bigwedge_{(\lambda_n,\mu_n,\underline{\alpha}_n)\in\{(\lambda_j,\mu_j,\underline{\beta}_j)|j\in K\}} N(\lambda_n,\mu_n)\right) \\ &= \bigwedge_{(\lambda_m,\mu_m,\underline{\alpha}_m)\in\{(\lambda_i,\mu_i,\underline{\alpha}_i),(\lambda_j,\mu_j,\underline{\beta}_j)|i\in N,j\in K\}} N(\lambda_m,\mu_m). \end{aligned}$$

Since $\{(\lambda_i,\mu_i,\underline{\alpha}_i)|i\ \in N\}\in M(\lambda_1,\mu_1)$ and $\{(\lambda_j,\mu_j,\underline{\beta}_j)|j\in K\}\in M(\lambda_2,\mu_2)$, then we have

$$\{(\lambda_i,\mu_i,\underline{\alpha}_i),(\lambda_j,\mu_j,\underline{\beta}_j)|i\in N,j\in K\}\quad\in M(\lambda_1\vee\lambda_2,\mu_1\vee\mu_2).$$

By the definition of N^s we have

$$N^s(\lambda_1\vee\lambda_2,\mu_1\vee\mu_2)\geq \bigwedge_{(\lambda_m,\mu_m,\underline{\alpha}_m)\in\{(\lambda_i,\mu_i,\underline{\alpha}_i),(\lambda_j,\mu_j,\underline{\beta}_j)|i\in N,\ j\in K\}} N(\lambda_m,\mu_m).$$

It is a contradiction. Thus $N^s(\lambda_1,\mu_1)\wedge N^s(\lambda_2,\mu_2)\leq N^s(\lambda_1\vee\lambda_2,\mu_1\vee\mu_2)$.
(ii) Similar the proof of (i)
(FNS) since $\underline{\alpha} = \underline{1}\wedge\underline{\alpha}$, then $N^s(\underline{\alpha},\underline{\alpha})\geq N(\underline{1},\underline{1}) = 1$. Thus $N^s(\underline{\alpha},\underline{\alpha}) = 1$, for each $\alpha\in L$.
Second since $\lambda\leq\lambda\wedge\underline{1}, \mu\geq\mu\wedge\underline{1}$, then $N^s(\lambda,\mu)\geq N(\lambda,\mu)$ for any $\mu,\lambda\in L^X$. Thus N^s is the stratified L-fuzzy topogenous order on X which finer than N.
Finally, consider N^s is stratified finer than N, then $N^*(\lambda,\mu)\geq N(\lambda,\mu)$ for any $\mu,\lambda\in L^X$.
We will show $N^*(\lambda,\mu)\geq N^s(\lambda,\mu)$ for any $\mu,\lambda\in L^X$. Suppose there exist $\mu,\lambda\in L^X$ such that

$$N^*(\lambda,\mu)\not\geq N^s(\lambda,\mu).$$

By the definition of N^s, then there exists $\{(\lambda_i, \mu_i, \underline{\alpha}_i) | i \in N\} \in M(\lambda, \mu)$ such that

$$N^*(\lambda, \mu) \not\geq \bigwedge_{(\lambda_l, \mu_l, \underline{\alpha}_l) \in \{(\lambda_i, \mu_i, \underline{\alpha}_i) | i \in N\}} N(\lambda_l, \mu_l).$$

On the other hand, we have

$$\begin{aligned} N^*(\lambda, \mu) &\geq N^*\left(\bigvee_{i\in N} (\lambda_i \wedge \underline{\alpha}_i), \bigvee_{i \in N} (\mu_i \wedge \underline{\alpha}_i)\right) \\ &\geq \bigwedge_{i \in N} N^*((\lambda_i \wedge \underline{\alpha}_i), (\mu_i \wedge \underline{\alpha}_i)) \\ &\geq \bigwedge_{i\in N} (N^*(\lambda_i, \mu_i) \wedge N^*(\underline{\alpha}_i, \underline{\alpha}_i)) \\ &\geq \bigwedge_{(\lambda_l, \mu_l, \underline{\alpha}_l) \in \{(\lambda_i, \mu_i, \alpha_i) | i \in N\}} N(\lambda_l, \mu_l). \end{aligned}$$

It is a contradiction. Thus N^s is the coarsest stratified L-fuzzy topogenous order on X which finer than N. □

Example. Let $X = \{x, y\}$, be set, $L = [0, 1]$. Let $\nu, \gamma \in L^X, \nu(x) = 0.5, \nu(y) = 0.5$ and $\gamma(x) = 0.3, \gamma(y) = 0.7$. Then we define the L-fuzzy topogenous order N on X as follows:
for each $\lambda, \mu \in L^X$,

$$N(\lambda, \mu) = \begin{cases} 1 & : \lambda = \underline{0} \ or \ \mu = \underline{1}, \\ 0.3 & : \lambda \leq \nu \leq \mu, \mu \not\geq \gamma, \\ 0.5 & : \lambda \leq \gamma \leq \mu, \lambda \not\leq \nu, \\ & : \lambda \leq \nu \vee \gamma \leq \mu, \\ 0.7 & : \lambda \leq \nu \vee \gamma \leq \mu, \\ & : \mu \not\geq \gamma, \ \lambda \not\leq \nu, \\ 0.6 & : \lambda \leq \nu \wedge \gamma \leq \mu, , \\ 0 & : O.w. \end{cases}$$

If $\lambda(x) \leq \alpha \leq \mu(x)$ for $0.5 < \alpha < 0.7$ and $\lambda(y) \leq \beta \leq \mu(y)$ for each $\beta \geq 0.7$, since $\lambda \leq (\underline{\alpha} \wedge \underline{1}) \vee (\underline{\beta} \wedge (v \vee \gamma)) = (\underline{\alpha} \wedge \underline{1}) \vee (\underline{\beta} \wedge \gamma) \leq \mu$, we have $N^s(\lambda, \mu) = (N(\underline{1}, \underline{1}) \wedge N(v \vee \gamma, v \vee \gamma)) \vee N(\underline{1}, \underline{1}) \wedge N(\gamma, \gamma) = 0.7$.
If $\lambda(x) \leq \alpha \leq \mu(x)$ for $0.5 < \alpha < 0.7$ and $\lambda(y) \leq \beta \leq \mu(y)$ for $0.5 < \alpha, \beta < 0.7$ and $\alpha < \beta$, since $\lambda \leq (\underline{\alpha} \wedge \underline{1}) \vee (\underline{\beta} \wedge (v \vee \gamma)) = (\underline{\alpha} \wedge \underline{1}) \vee (\underline{\beta} \wedge \gamma) \leq \mu$, we have $N^s(\lambda, \mu) = 0.7$. By a similar method as the above cases, we can obtain the stratifcation N^s as following:

$$N^s(\lambda, \mu) = \begin{cases} 1 & : \lambda = \underline{\alpha} \ or \ \mu = \underline{\alpha}, \\ 0.7 & : \lambda(x) \leq \alpha \leq \mu(x), \lambda(y) \leq \beta \leq \mu(y), \\ & : for \ 0.5 < \alpha, \beta < 0.7 \ and \ \alpha < \beta, \\ 0.5 & : \lambda(x) \leq \alpha \leq \mu(x), \lambda(y) \leq \beta \leq \mu(y), \\ & : for \ 0.3 \leq \alpha < 0.5, 0.5 < \beta \leq 0.7, \\ 0.6 & : \lambda(x) \leq \alpha \leq \mu(x), \lambda(y) \leq \beta \leq \mu(y), \\ & : for \ 0.3 \leq \alpha, \beta \leq 0.5 \ and \ \alpha < \beta, \\ 0 & : O.w. \end{cases}$$

Theorem 14. *(i) If N be an L-fuzzy topogenous order structure which is symmetrical, then the stratification N^s of N also is symmetrical.*
(ii) If N be a perfect L-fuzzy topogenous order structure. T_N the L-fuzzy topology associated to N, then $T_{N^s} = (T_N)^s$.
(iii) Let T be an L-fuzzy topology, N_T the L-fuzzy topogenous associated to T. Then $N_{T^s} = (N_T)^s$ iff $(N_T)^s$ is perfect L-fuzzy topogenous order.

Proof. (i) Left to leader.
(ii) Since $T_N^s(\underline{\alpha}) = N^s(\underline{\alpha}, \underline{\alpha}) = 1$, for each $\alpha \in L$, then T_N^s is stratified finer than T_N, so $T_N^s \geq (T_N)^s$.
Conversly. suppose there exists $\lambda \in L^X$ such that

$$T_{N^s}(\lambda) = N^s(\lambda, \lambda) \nleq (T_N)^s(\lambda).$$

By the definition of N^s, there exists a family $\{(\lambda_i, \lambda_i, \underline{\alpha})|i \in N\} \in M(\lambda, \lambda)$ such that

$$\begin{aligned}(T_N)^s(\lambda) \quad &\ngeq \bigwedge_{(\lambda_l,\lambda_l,\underline{\alpha}_l)\in\{(\lambda_i,\lambda_i,\underline{\alpha}_i)|i\in N\}} N(\lambda_l, \lambda_l) \\ &= \bigwedge_{(\lambda_l,\lambda_l,\underline{\alpha}_l)\in\{(\lambda_i,\lambda_i,\underline{\alpha}_i)|i\in N\}} T_N(\lambda_l)\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(T_N)^s(\lambda) &= (T_N)^s\left(\bigvee_{i\in N}(\lambda_i \wedge \underline{\alpha}_i)\right) \\ &\geq \bigwedge_{i\in N}(T_N)^s(\lambda_i \wedge \underline{\alpha}_i) \\ &\geq \bigwedge_{i\in N}((T_N)^s(\lambda_i) \wedge (T_N)^s(\underline{\alpha}_i)) \\ &\geq \bigwedge_{(\lambda_l,\lambda_l,\underline{\alpha}_l)\in\{(\lambda_i\lambda_i,\underline{\alpha}_i)|i\in N\}} T_N(\lambda_l).\end{aligned}$$

It is a contradiction. Thus $T_{N^s} = (T_N)^s$.
(iii) ($\Leftarrow$) Suppose $(N_T)^s$ is perfect L-fuzzy topogenous order. We will show $N_{T^s} = (N_T)^s$. Since $N_{T^s}(\underline{\alpha}, \underline{\alpha}) \geq T^s(\underline{\alpha}) = 1$, for each $\alpha \in L$, then N_{T^s} is stratified finer than N_T, so $N_{T^s} \geq (N_T)^s$.
Conversely, Suppose there exist $\mu, \lambda \in L^X$ such that

$$N_{T^s}(\lambda, \mu) \leq (N_T)^s(\lambda, \mu).$$

By the definition of T^s, there exists a family $\{(\nu_i, \underline{\alpha}_i)|i \in \Gamma\} \in N(\nu)$, where $\lambda \leq \nu \leq \mu$ such that

$$\bigwedge_{(\nu_l,\underline{\alpha}_l)\in\{(\nu_i,\underline{\alpha}_i)|i\in\Gamma\}} T(\nu_l) \nleq (N_T)^s(\lambda, \mu).$$

Since $N_T(\nu_i, \nu_i) \geq T(\nu_i)$, for each $i \in \Gamma$, then we have

$$\bigwedge_{(\nu_l,\underline{\alpha}_l)\in\{(\nu_i,\underline{\alpha}_i)|i\in\Gamma\}} N_T(\nu_l, \nu_l) \nleq (N_T)^s(\lambda, \mu).$$

On the other hand, since $(N_T)^s$ is perfect L-fuzzy topogenous, then we have

$$\begin{aligned}(N_T)^s(\lambda,\mu) &\geq (N_T)^s(\nu,\nu)\\ &= (N_T)^s\left(\bigvee_{i\in\Gamma}(\nu_i\wedge\underline{\alpha}_i),\bigvee_{i\in\Gamma}(\nu_i\wedge\underline{\alpha}_i)\right)\\ &\geq \bigwedge_{i\in\Gamma}(N_T)^s(\nu_i\wedge\underline{\alpha}_i,\nu_i\wedge\underline{\alpha}_i)\\ &\geq \bigwedge_{i\in\Gamma}((N_T)^s(\nu_i,\nu_i)\wedge(N_T)^s(\underline{\alpha}_i,\underline{\alpha}_i))\\ &\geq \bigwedge_{(\nu_l,\underline{\alpha}_l)\in\{(\nu_i,\underline{\alpha}_i)|i\in\Gamma\}} N_T(\nu_l,\nu_l).\end{aligned}$$

It is a contradiction. Thus $N_{T^s}=(N_T)^s$.
($\Rightarrow$) Obvious from Theorem 8(ii). □

Theorem 15. *Let (X,N_1) and (Y,N_2) be L-fuzzy topogenous order spaces and N_1^s and N_2^s be stratification for N_1 and N_2 respectively. If $f:(X,N_1)\longrightarrow(Y,N_2)$ is L-fuzzy topogenous continuous, then $f:(X,N_1^s)\longrightarrow(Y,N_2^s)$ is L-fuzzy topogenous continuous.*

Proof. Suppose there exist $\lambda,\mu\in L^Y$ such that

$$N_1^s(f^{-1}(\lambda),f^{-1}(\mu))\not\geq N_2^s(\lambda,\mu).$$

From the definition of N_2^s, there exists a family $\{(\lambda_i,\mu_i,\underline{\alpha}_i)i\in N\}\in M(\lambda,\mu)$ such that

$$N_1^s(f^{-1}(\lambda),f^{-1}(\mu))\not\geq \bigwedge_{(\lambda_l,\mu_l,\underline{\alpha}_l)\in\{(\lambda_i,\mu_i,\underline{\alpha}_i)|i\in N\}} N_2(\lambda_l,\mu_l).$$

Since $f:(X,N_1)\to(Y,N_2)$ is an L-fuzzy topogenous continuous, then $N_1(f^{-1}(\lambda_i),f^{-1}(\mu_i))\geq N_2(\lambda_i,\mu_i)$, for each $i\in N$. On the other hand, we have a family $\{(f^{-1}(\lambda_i),f^{-1}(\mu_i),\underline{\alpha}_i)|i\in N\}\in M(f^{-1}(\lambda),f^{-1}(\mu))$, by the definition of N_1^s, we have

$$\begin{aligned}N_1^s(f^{-1}(\lambda),f^{-1}(\mu)) &\geq \bigwedge_{(\lambda_l^*,\mu_l^*,\underline{\alpha}_l)\in\{(f^{-1}(\lambda_i),f^{-1}(\mu_i),\underline{\alpha}_i)\backslash i\in N\}} N_1(\lambda_l^*,\mu_l^*).\\ &\geq \bigwedge_{(\lambda_l,\mu_l,\underline{\alpha}_l)\in\{(\lambda_i,\mu_i,\underline{\alpha}_i)\backslash i\in N\}} N_2(\lambda_l,\mu_l).\end{aligned}$$

It is a contradiction. Thus $f:(X,N_1^s)\to(Y,N_2^s)$ is L-fuzzy topogenous continuous. □

Counterexample. Let X be any set, $L=[0,1]$ and identity mapping $Id_X:(X,N_1)\to(X,N_2)$ define the L-fuzzy topogenous N_1 and N_2 on X as follows:

$$N_1(\lambda,\mu)=\begin{cases}1:\lambda=\underline{0}\ \textit{or}\ \mu=\underline{1},\\ 0:O.w\end{cases}$$

$$N_2(\lambda,\mu)=\begin{cases}1:\lambda=\underline{0}\ \textit{or}\ \mu=\underline{1},\\ \frac{1}{3}:\underline{0}\neq\lambda\leq\underline{0.5}\leq\mu\neq\underline{1},\\ 0:O.w\end{cases}$$

From Theorem we obtain

$$N_1^s(\lambda,\mu)=N_2^s(\lambda,\mu)=\begin{cases}1:\lambda\leq\underline{\alpha}\leq\mu \; for\ each\ \alpha\in L,\\ 0:O.w\end{cases}$$

Clearly $Id_X:(X,N_1^s)\to(X,N_2^s)$ is L-fuzzy topogenous continuous. But $Id_X:(X,N_1)\to(X,N_2)$ is not L-fuzzy topogenous continuous, where $0=N_1(\underline{0.5},\underline{0.5})<N_2(\underline{0.5},\underline{0.5})=\frac{1}{3}$.

Theorem 16. *Let $\{(X_i,N_i)\}_{i\in\Gamma}$ be a family of L-fuzzy topogenous order spaces, $X=\Pi_{i\in\Gamma}$ a set and, for each $i\in\Gamma$, $\pi_i:X\to X_i$ a projecition mapping and N is the product L-fuzzy topogenous order structure with respect to a family $\{\pi_i:X\to X_i:i\in\Gamma\}$. If there exists a $j_0\in\Gamma$ such that $\overline{N}_{j_0}=N^s_{j_0}$ and $\overline{N}_i=N_i$ for each $i\in\Gamma-\{j_0\}$, then:*
(i) $\Pi_{i\in\Gamma}\overline{N}_i$ is the stratified L-fuzzy topogenous structure on X which is finer than N.
(ii) $\Pi_{i\in\Gamma}\overline{N}_i=N^s$.

Proof. (i) By Theorem 9, $\Pi_{i\in\Gamma}\overline{N}_i$ is the L-fuzzy topogenous on X which all π_i is L-fuzzy topogenous continuous. Since $\Pi_{i\in\Gamma}\overline{N}_i(\underline{\alpha},\underline{\alpha})=1$ for each $\alpha\in L$ (Indeed, by the definition of $\Pi_{i\in\Gamma}\overline{N}_i$, there exists a family $\{\underline{\alpha}\}$,

$$\begin{aligned}\Pi_{i\in\Gamma}\overline{N}_i(\underline{\alpha},\underline{\alpha}) &\geq \bigvee_{i\in\Gamma}\overline{N}_i(\pi_i(\underline{\alpha}),1-\pi_i(\underline{1}-\underline{\alpha}))\\ &= \bigvee_{i\in\Gamma}\overline{N}_i(\underline{\alpha},\underline{\alpha})\\ &\geq \overline{N}_{j_0}(\underline{\alpha},\underline{\alpha})=N^s_{j_0}(\underline{\alpha},\underline{\alpha})=1,\end{aligned}$$

then $\Pi_{i\in\Gamma}\overline{N}_i$ is the stratified. But $N^s_{j_0}$ finer than N_{j_0}. Thus $\Pi_{i\in\Gamma}\overline{N}_i$ is the stratified L-fuzzy topogenous order on X which is finer than N.
(ii) Since $\Pi_{i\in\Gamma}\overline{N}_i$ is a stratified, then for any $\mu,\lambda\in L^X$, $N^s(\lambda,\mu)\leq\Pi_{i\in\Gamma}\overline{N}_i(\lambda,\mu)$. Conversely, suppose there exist $\mu,\lambda\in L^X$ such that

$$N^s(\lambda,\mu)\not>\Pi_{i\in\Gamma}\overline{N}_i(\lambda,\mu) \tag{2. 1}$$

From definition of $\Pi_{i\in\Gamma}\overline{N}_i$, there exist finite families $\{\lambda_j:\lambda=\bigvee_j\lambda_j\}$ and $\{\mu_k:\mu=\bigwedge_k\mu_k\}$.

$$N^s(\lambda,\mu)\not\geq\bigwedge_{j,k}\bigvee_{i\in\Gamma}\overline{N}_i(\pi_i(\lambda_j),(\pi_i(\mu_k))).$$

Then there exists $n\in\Gamma$ such that

$$N^s(\lambda,\mu)\not\geq\overline{N}_n(\pi_n(\lambda),(\pi_n(\mu))).$$

On the other hand, if $n\neq j_0$, then by definition of N^s we have

$$\begin{aligned}N^s(\lambda,\mu)\geq N(\lambda,\mu) &\geq \bigwedge_{j,k}\bigvee_{i\in\Gamma}N_i(\pi_i(\lambda_j),(\pi_i(\mu_k)))\\ &= \bigwedge_{j,k}\bigvee_{i\in\Gamma}\overline{N}_i(\pi_i(\lambda_j),(\pi_i(\mu_k)))\\ &\geq \overline{N}_n(\pi_n(\lambda),(\pi_n(\mu))).\end{aligned}$$

It is a contradiction for eq. 2. 1 . If $n = j_0$, $\overline{N}_n = N_n^s$. By the definition of N_n^s, then there exists a family $\{(\lambda_k^*, \mu_k^*, \underline{\alpha}_k) \setminus k \in N\} \in M(\pi_n(\lambda), (\pi_n(\mu)))$ such that

$$N^s(\lambda, \mu) \ngeq \bigwedge_{(\lambda_l, \mu_l, \underline{\alpha}_l) \in \{(\lambda_k^*, \mu_k^*, \underline{\alpha}_k) \setminus k \in N\}} N_n(\lambda_l, \mu_l)$$

Since π_n is L-fuzzy topogenous continuous, $N(\pi_n^{-1}(\lambda_k^*), \pi_n^{-1}(\mu_k^*)) \geq N_n(\lambda_k^*, \mu_k^*)$, for each $k \in N$,

$$N^s(\lambda, \mu) \ngeq \bigwedge_{(\lambda_l, \mu_l, \underline{\alpha}_l) \in \{(\pi_n^{-1}(\lambda_k^*), \pi_n^{-1}(\mu_k^*), \underline{\alpha}_k) \setminus k \in N\}} N(\lambda_l, \mu_l)$$

On the other hand, since $\{(\pi_n^{-1}(\lambda_k^*), \pi_n^{-1}(\mu_k^*), \underline{\alpha}_k) \setminus k \in N\} \in M(\lambda, \mu)$. By the definition of N^s we have

$$N^s(\lambda, \mu) \geq \bigwedge_{(\lambda_l, \mu_l, \underline{\alpha}_l) \in \{(\pi_n^{-1}(\lambda_k^*), \pi_n^{-1}(\mu_k^*), \underline{\alpha}_k) \setminus k \in N\}} N(\lambda_l, \mu_l)$$

Also, it is a contradiction for eq. 2. 1 . Thus $N^s(\lambda, \mu) \geq \prod_{i \in \Gamma} \overline{N}_i(\lambda, \mu)$. □

Corollary 17. *Let (X, N_1) and (X, N_2) be L-fuzzy topogenous order spaces. We define, for any $\mu, \lambda \in L^X$*

$$(N_1 \circledcirc N_2)(\lambda, \mu) = \bigvee \bigwedge_{j,k} \{N_1(\lambda_j, \mu_k) \wedge N_2(\lambda_j, \mu_k)\}$$

where $\bigvee$ is taken over all finite families $\{\lambda_j : \lambda = \bigvee_j \lambda_j\}$ and $\{\mu_k : \mu = \bigwedge_k \mu_k\}$. Then
(i) $N_1^s \circledcirc N_2$ and $N_1^s \circledcirc N_2^s$ are stratified L-fuzzy topogenous on X, which finer than $N_1 \circledcirc N_2$.
(ii) $N_1^s \circledcirc N_2 = N_1^s \circledcirc N_2^s = (N_1^s \circledcirc N_2)^s$.

Example.
Let $X = \{x, y, z\}$, $Y = \{p, q\}$ be sets, $L = [0, 1]$ and a mapping $f : X \to Y$ defined by $f(x) = p$,$f(y) = f(z) = q$. Let $\nu \in L^X$, $\nu(x) = 0.2$, $\nu(y) = 0.4$, $\nu(z) = 0.5$. Then we define the L-fuzzy topogenous order N on X as follows: for each $\lambda, \mu \in L^X$.

$$N(\lambda, \mu) = \begin{cases} 1 : \lambda = \underline{0} \ or \ \mu = \underline{1}, \\ \frac{1}{2} : \underline{0} \neq \lambda \leq \nu \leq \mu \neq \underline{1}, \\ 0 : O.w \end{cases}$$

Let $f(N)$ is the L-fuzzy topogenous order on Y, define by
$f(N)(\gamma, \nu) = N(f^{-1}(\gamma), f^{-1}(\nu))$ for any $\gamma, \nu \in L^Y$. Then we obtain

$$f(N)(\gamma, \theta) = \begin{cases} 1 : \gamma = \underline{0} \ or \ \theta = \underline{1}, \\ 0 : O.w \end{cases}$$

and from Theorem 2. we obtain

$$(f(N))^s(\gamma, \theta) = \begin{cases} 1 : \gamma \leq \underline{\alpha} \leq \theta = \underline{1} \ for \ each \ \alpha \in L, \\ 0 : O.w \end{cases}$$

On the other hand, we have

$$(N)^s(\lambda, \mu) = \begin{cases} 1 : \lambda \leq \underline{\alpha} \leq \mu \ for \ each \ \alpha \in L, \\ \quad : \lambda \neq \nu \wedge \underline{\beta} \ for 0.2 < \beta \leq 1, \\ \frac{1}{2} : \lambda = \nu \wedge \underline{\beta}, \mu \geq \nu \wedge \underline{\beta} \ for \ 0.2 < \beta \leq 1, \\ \quad : \nu \neq \underline{\alpha}, \ for \ each \ \alpha \in L, \\ 0 : O.w. \end{cases}$$

For $\lambda_1 = \nu \wedge \underline{0.3}$, there exist $\gamma_1 \in L^Y$ such that $f^{-1}(\gamma_1) = \lambda_1$ with $\gamma_1(p) = 0.2$, $\gamma_1(q) = 0.3$. Thus we have $f(N^s)(\gamma_1, \gamma_1) = N^s(f^{-1}(\gamma_1), f^{-1}(\gamma_1)) = \frac{1}{2}$, but $(f(N))^s(\gamma_1, \gamma_1) = 0$. Hence $f(N^s) \neq (f(N))^s$.

3. Stratifications of L-fuzzy proximity spaces

An L-fuzzy quasi-proximity (resp. proximity) $\delta : L^X \times L^X \longrightarrow L$ is said to be stratified L-fuzzy quasi-proximity (resp. proximity) on X, iff δ satisfied the following conditions: (FPS) $\delta(\underline{\alpha}, 1 - \underline{\alpha}) = 0$, for each $\alpha \in L$. The pair (X, δ) is called stratified L-fuzzy quasi-proximity (resp. proximity) space.

Theorem 18. *Let (X, δ) be L-fuzzy proximity space. We define for all $\mu, \lambda \in L^X$,*

$$\delta^s(\lambda, \mu) = \bigwedge_{\{(\lambda_i,\mu_i,\underline{\alpha}_i \backslash i\in N)\}\in N(\lambda,\mu)} \{ \bigwedge_{(\lambda_l,\mu_l,\underline{\alpha}_l)\in\{(\lambda_i,\mu_i,\underline{\alpha}_i)\backslash i\in N\}} \delta(\lambda_l, \mu_l)\}.$$

where $W(\lambda, \mu) = \{\{(\lambda_i, \mu_i, \underline{\alpha}_i)\backslash i \in N,\ N \text{ finite index set}\}\backslash \lambda \leq \bigvee_{i\in N}(\lambda_i \wedge \underline{\alpha}_i), \mu \leq \bigwedge_{i\in N}(\mu_i \vee \underline{\alpha}_i)\}$. *Then δ^s is the coarsest stratified L-fuzzy proximity on X which finer than δ.*

Proof. (FP1), (FP2), (FP4) and (FPS) are similar to the proof of Theorem 2. And (FP) Obvious.
(FP3) Suppose there exists $\lambda, \mu \in L^X$ such that

$$\delta^s(\lambda, \mu) \ngeq \bigwedge_{\rho\in L^X} (\delta^s(\lambda, \rho) \vee \delta^s(\rho, \mu)).$$

By the definition of δ^s there exists a family $\{(\lambda_i, \mu_i, \underline{\alpha}_i) \setminus i \in N\} \in W(\lambda, \mu)$, such that

$$\bigvee_{(\lambda_l,\mu_l,\underline{\alpha}_l)\in\{(\lambda_i,\mu_i,\underline{\alpha}_i)\backslash i\in N\}} \delta(\lambda_l, \mu_l) \ngeq \bigwedge_{\rho\in L^X} (\delta^s(\lambda, \rho) \vee \delta^s(\rho, \mu)).$$

Since δ is L-fuzzy proximity. $\delta(\lambda_i, \mu_i) \geq \bigwedge_{\rho_i\in L^X}(\delta(\lambda_i, \rho_i) \vee \delta(\rho_i, \mu_i))$. Then we have

$$\bigvee_{(\lambda_l,\mu_l,\underline{\alpha}_l)\in\{(\lambda_i,\mu_i,\underline{\alpha}_i)\backslash i\in N\}} (\delta(\lambda_l, \rho_l) \vee \delta(\rho_l, \mu_l)) \ngeq \bigwedge_{\rho\in L^X} (\delta^s(\lambda, \rho) \vee \delta^s(\rho, \mu)).$$

On the other hand, put $\rho = \bigwedge_{i\in N}(\rho_i \vee \underline{\alpha}_i)$. Then by the definition of δ^s

$$\begin{aligned} \delta^s(\lambda, \rho) &\leq \bigvee_{(\lambda_l,\rho_l,\underline{\alpha}_l)\in\{(\lambda_i,\rho_i,\underline{\alpha}_i)\backslash i\in N\}} (\delta(\lambda_l, \rho_l), \delta^s(\rho, \mu)) \\ &\leq \bigvee_{(\rho_l,\mu_l,\underline{\alpha}_l)\in\{(\rho_i,\mu_i,\underline{\alpha}_i)\backslash i\in N\}} \delta(\rho_l, \mu_l). \end{aligned}$$

Thus we have

$$\begin{aligned} \bigwedge_{\rho\in L^X} (\delta^s(\lambda, \rho) \vee \delta^s(\rho, \mu)) &\leq \delta^s(\lambda, \rho) \vee \delta^s(\rho, \mu) \\ &\leq \bigvee_{(\lambda_l,\mu_l,\underline{\alpha}_l)\in\{(\lambda_i,\mu_i,\underline{\alpha}_i)\backslash i\in N\}} (\delta(\lambda_l, \rho_l) \vee \delta(\rho_l, \mu_l)). \end{aligned}$$

It is a contradiction. □

Example Let $X = \{x, y\}$, be set, $L = [0,1]$. Let $\nu, \gamma \in L^X$, $\nu(x) = 0.5$, $\nu(y) = 0.5$ and $\gamma(x) = 0.4$, $\gamma(y) = 0.6$. Then we define the L-fuzzy proximity δ on X as follows: for each $\lambda, \mu \in L^X$,

$$\delta(\lambda,\mu) = \begin{cases} 1 \ : \lambda = \underline{0} \ or \ \mu = \underline{1}, \\ 0.7 \ : \lambda \leq \nu, \mu \leq \nu, \mu \nleq \gamma, \\ 0.5 \ : \lambda \leq \gamma, \mu \leq \gamma, \lambda \nleq \nu, \\ 0.3 \ : \lambda \leq \nu \vee \gamma, \mu \leq (\nu \vee \gamma), \\ \ : \lambda \nleq \nu, \ \mu \nleq \gamma, \\ 0.4 \ : \lambda \leq \nu \wedge \gamma, \mu \leq (\nu \wedge \gamma), \\ 0 \ : O.w \end{cases}$$

We can obtain the stratification δ^s as following:

$$N^s(\lambda,\mu) = \begin{cases} 1 \ : \lambda = \underline{\alpha} \ or \ \mu = \underline{\alpha}, \\ 0.3 \ : \lambda(x) \leq \alpha, \mu(x) \leq \alpha, \lambda(y) \leq \beta, \mu(y) \leq \beta, \\ \ : for \ 0.5 < \alpha, \beta < 0.6 \ and \ \alpha < \beta, \\ 0.5 \ : \lambda(x) \leq \alpha, \mu(x) \leq \alpha, \lambda(y) \leq \beta, \mu(y) \leq \beta, \\ \ : for \ 0.4 \leq \alpha < 0.5, 0.5 < \beta \leq 0.6, \\ 0.6 \ : \lambda(x) \leq \alpha, \mu(x) \leq \alpha, \lambda(y) \leq \beta, \mu(y) \leq \beta, \\ \ : for \ 0.4 \leq \alpha, \beta \leq 0.5 \ and \ \alpha < \beta, \\ 0 \ : O.w \end{cases}$$

Theorem 19. *(i) Let δ be an L-fuzzy proximity structure, N_δ symmetrical L-fuzzy topogenous order associated to δ. Then $N_{\delta^s} = (N_\delta)^s$.*
(ii) Let N be a symmetrical L-fuzzy topogenous order structure, δ_N the L-fuzzy proximity associated to N. Then $\delta_{N^s} = (\delta_N)^s$.

Proof. (i) Since $N_{\delta^s}(\underline{\alpha}, \underline{\alpha}) = (\delta^s(\underline{\alpha}, \underline{\alpha})) = 1$, for each $\alpha \in L$, then N_{δ^s} is stratified finer than N_δ, so $N_{\delta^s} \geq (N_\delta)^s$.
Conversely, suppose there exist $\mu, \lambda \in L^X$ and $t \in L - \{0,1\}$ such that

$$N_{\delta^s}(\lambda,\mu) = (\delta^s(\lambda,\mu)) > t > (N_\delta)^s(\lambda,\mu).$$

Since $\delta^s(\lambda,\mu) < t$, then there exists a family $\{(\lambda_i, \mu_i, \underline{\alpha_i}) \setminus i \in N\} \in W(\lambda,\mu)$, such that $\delta(\lambda_i,\mu_i) < t$ for each $i \in N$. Then $N_\delta(\lambda_i,\mu_i) = (\delta(\lambda_i,\mu_i)) > t$ for each $i \in N$. On the other hand, since $\{(\lambda_i,\mu_i,\underline{\alpha_i}) \setminus i \in N\} \in M(\lambda,\mu)$, then

$$(N_\delta)^s(\lambda,\mu) \geq \bigwedge_{(\lambda_l,\mu_l,\underline{\alpha_l}) \in \{(\lambda_i,\mu_i,\underline{\alpha_i}) \setminus i \in N\}} N_\delta(\lambda_l,\mu_l) > t.$$

It is a contradiction. Thus $N_{\delta^s} = (N_\delta)^s$.
(ii) Similar (i). □

Theorem 20. *Let (X, δ_1) and (Y, δ_2) be L-fuzzy quasi-proximity spaces, δ_1^s and δ_2^s be stratification for δ_1 and δ_2 respectively. If $f : (X, \delta_1) \to (Y, \delta_2)$ is L-fuzzy proximity continuous, then $f : (X, \delta_1^s) \to (Y, \delta_2^s)$ is L-fuzzy proximity continuous.*

Proof. Similar of the proof of Theorem 15. □

Theorem 21. *Let $\{(X_i, \delta_i)\}_{i \in \Gamma}$ be a family of L-fuzzy quasi-proximity spaces, $X = \Pi_{i\in\Gamma} X_i$ a set and, for each $i \in \Gamma$, $\pi_i : X \to X_i$ a projection mapping and δ is the product L-fuzzy quasi-proximity structure with respect to a family $\{\pi_i : X \to X_i : i \in \Gamma\}$. If there exists a $j_0 \in \Gamma$ such that $\overline{\delta}_{j_0} = \delta_{j_0}^s$ and $\overline{\delta}_i = \delta_i$ for each $i \in \Gamma - \{j_0\}$, then:*

(i) $\Pi_{i\in\Gamma}\overline{\delta}_i$ *is the stratified L-fuzzy quasi-proximity structure on X which is finer than* δ.
(ii)$\Pi_{i\in\Gamma}\overline{\delta}_i = \delta^s$.

Proof. Similar of the proof of Theorem 16. □

Corollary 22. *Let* (X, δ_1) *and* (X, δ_2) *be L-fuzzy quasi-proximity spaces. We define, for any* $\mu, \lambda \in L^X$

$$(\delta_1 \Cup \delta_2)(\lambda, \mu) = \bigwedge \bigvee_{j,k} \{\delta_1(\lambda_j, \mu_k) \wedge \delta_2(\lambda_j, \mu_k)\},$$

where $\bigwedge$ *is taken over all finite families* $\{\lambda_j : \lambda = \vee_j \lambda_j\}$ *and* $\{\mu_k : \mu = \vee_k \mu_k\}$. *Then:*
(i) $\delta_1^s \Cup \delta_2$ *and* $\delta_1 \Cup \delta_2^s$ *are stratified L-fuzzy quasi-proximity on X, which finer than* $\delta_1 \Cup \delta_2$.
(ii) $\delta_1^s \Cup \delta_2 = \delta_1 \Cup \delta_2^s = (\delta_1 \Cup \delta_2)^s$.

4. Stratifications of L-fuzzy uniform spaces

an L-fuzzy quasi-uniformity (resp. uniformity) $U : \Omega_X \to L$ is said to be stratified L-fuzzy quasi-uniformity (resp. uniformity) on X, iff U satisfied the following conditions: (FUS) $U(\widehat{\alpha}) = 1$, for each $\alpha \in L$. The pair (X, U) is called stratified L-fuzzy quasi-uniform (resp. uniform) space.

Proposition 23. *If* (X, U) *is stratified L-fuzzy uniform space and* δ_U *L-fuzzy proximity associated to U, then* (X, δ_U) *is stratified L-fuzzy proximity space.*

Proof. Since U is stratified L-fuzzy uniformity, then $U(\widehat{\alpha}) = 1$, $\widehat{\alpha}(\underline{\alpha}) = \underline{\alpha}$ for each $\alpha \in L$. By Theorem 11. $\delta_U(\underline{\alpha}, \underline{\alpha}) = \bigwedge\{(U(a)) : a(\underline{\alpha}) \leq \underline{\alpha}\} = 0$ for each $\alpha \in L$. That is δ_U is stratified. □

Theorem 24. *Let* (X, U) *be an L-fuzzy quasi-uniform space. Define for every* $a \in \Omega_X$,

$$U^s(a) = \begin{cases} \bigvee\{U(b) : a \geq b \wedge \widehat{\alpha},\ \alpha \in L\}, \\ 0\ : O.w. \end{cases}$$

Then U^s *is the coarsest stratified L-fuzzy quasi-uniformity finer than* U.

Proof. (FU1) Suppose that there exist $a_1, a_2 \in \Omega_X$ such that

$$U^s(a_1 \wedge a_2) \not\geq U^s(a_1) \wedge U^s(a_2).$$

Form definition of U^s there exist $b_i \in \Omega_X$, $\alpha_i \in L$ with $a_i \geq b_i \wedge \widehat{\alpha}_i, i = 1, 2$ such that

$$U^s(a_1 \wedge a_2) \not\geq U(b_1) \wedge U(b_2).$$

On the other hand, since $a_1 \wedge a_2 \geq (b_1 \wedge b_2) \wedge \widehat{\alpha}_1$ or $a_1 \wedge a_2 \geq (b_1 \wedge b_2) \wedge \widehat{\alpha}_2$, then we have

$$\begin{aligned} U^s(a_1 \wedge a_2) &\geq U(b_1 \wedge b_2) \\ &\geq U(b_1) \wedge U(b_2). \end{aligned}$$

It is a contradication. Thus $U^s(a_1 \wedge a_2) \geq U^s(a_1) \wedge U^s(a_2)$
(FU2) Let $a_1 \in \Omega_X$. Suppose that there exist $a_1 \in \Omega_X$ such that

$$U^s(a) \not\leq U^s(a_1).$$

By the definition of U^s, there exist $b \in \Omega_X$, $\alpha \in L$ with $a \geq b \wedge \widehat{\alpha}$ such that $U(b) \not\leq U^s(a_1)$. Then there exists $c \in \Omega_X$ such that $c\ o\ c \leq b$, $U(c) \geq U(b)$. By Proposition 1(ii),

$$(c \wedge \widehat{\alpha})\ o\ (c \wedge \widehat{\alpha}) \leq (c\ o\ c) \wedge \widehat{\alpha} \leq b \wedge \widehat{\alpha} \leq a.$$

This means that there is $c \wedge \widehat{\alpha} = a_1$, with $a_1\ o\ a_1 \leq a$, $U^s(a_1) \geq U(c) \geq U(b)$. It is a contradiction. Thus $U^s(a) \leq U^s(a_1)$.
(FU3) Obvious.
(FU4) There exists $b \in \Omega_X$ such that $U(b) = 1$. Since $b = b \wedge \widehat{0}$, by Proposition 1(iv), then $U^s(b) = 1$.
(FUS) Since $\widehat{\alpha} \wedge \widehat{0} = \widehat{\alpha}$ for each $\alpha \in L$, then $U^s(\widehat{\alpha}) = 1$ for each $\alpha \in L$. Thus U^s is stratified.
For each $b \in \Omega_X$, $b \wedge \widehat{0} = b$. Then $U^s(b) \geq U(b)$ for any $b \in \Omega_X$. Thus U^s is finer than U.
Finaly, consider U^* be stratified L-fuzzy quasi-uniformity finer than U. Suppose there exists $a \in \Omega_X$,

$$U^*(a) \not\geq U^s(a).$$

from definition of U^s, there exist $b \in \Omega_X$, $\alpha \in L$ with $a \geq b \wedge \widehat{\alpha}$ such that $U^*(a) \not\geq U(b)$. Since U^* is stratified, then

$$\begin{aligned} U^*(a) &\geq U^*(b \wedge \widehat{\alpha}) \geq U^*(b) \wedge U^*(\widehat{\alpha}) \\ &= U^*(b) \geq U(b) \end{aligned}$$

It is a contradiction. Hence $U^*(a) \geq U^s(a)$ for any $a \in \Omega_X$. Thus U^s is the coarsest stratified L-fuzzy quasi-uniformity finer than U. U^s is called the stratification of an L-fuzzy quasi-uniformity of U on X. □

Example. Let X be any set, $L = [0, 1]$, define the L-fuzzy quasi-uniformity U on X as follows:

$$U(a) = \begin{cases} 1 : a = a_{\underline{0}}, \\ \frac{1}{4} : a_{\underline{\alpha}} \leq a < a_{\underline{0}}, \\ 0 : O.w \end{cases}$$

where $a_\lambda : L^X \longrightarrow L^X$ defined by Theorem 6. Since $a_{\underline{\alpha}} = \widehat{\alpha}$ for any $\alpha \in L$, we obtain

$$U_2(a) = \begin{cases} 1 : a \geq \widehat{\alpha}, for\ each\ \alpha\ \varepsilon\ L \\ 0 : O.w \end{cases}$$

Theorem 25. *Let (X, U) be an L-fuzzy uniform space, δ_U the L-fuzzy proximity associated to U. Then $(\delta_U)^s = \delta_{U^s}$.*

Proof. Since U^s is stratified finer than U, then δ_{U^s} is stratified finer than δ_U. Thus $\delta_{U^s}(\lambda, \mu) \leq (\delta_U)^s(\lambda, \mu)$. for any $\mu, \lambda \in L^X$.
Conversely, suppose there exist $\mu, \lambda \in L^X$ and $t \in L - \{0, 1\}$ such that

$$\delta_{U^s}(\lambda, \mu) < t < (\delta_U)^s(\lambda, \mu).$$

By Theorem 11. there exists $a \in \Omega_X$ such that $\delta_{U^s}(\lambda, \mu) \leq (U^s(a))' < t$ and $a(\mu) \leq \lambda'$. Then there exists $b \in \Omega_X$ with $a \geq b \wedge \widehat{\alpha}$ such that $U^s(a) \geq U(b) > t'$

and $(b \wedge \widehat{\alpha})(\mu) \leq \lambda'$. Hence

$$\lambda \leq \Big(\bigwedge_{\mu=\mu_1 \vee \mu_2} (b(\mu_1) \vee \widehat{\alpha}(\mu_2))\Big) = \bigvee_{i \in N} (\lambda_i \wedge \underline{\alpha}_i)$$

where $(\widehat{\alpha}(\mu_2)) = \underline{\alpha}_i$, $(b(\mu_1)) = \lambda_i$ for $i \in N = \{1, 2, 3, ..., n\}$. Put $\mu_1 = \nu_i$ for each $i \in N$, then $\mu \leq \bigwedge_{i \in N}(\nu_i \vee (\underline{\alpha}_i))$ and hence $b(\nu_i) \leq \lambda_i$, for each $i \in N$. Then we have

$$\delta_U(\lambda_i, \nu_i) \leq (U(b)) < t, \ for\ each\ i \in N.$$

On the other hand, from definition of $(\delta_U)^s$ we have

$$(\delta_U)^s(\lambda, \mu) \leq \bigwedge_{(\lambda_l, \nu_l, \underline{\alpha}_l) \in \{(\lambda_i, \nu_i, \underline{\alpha}_i) \setminus i \in N\}} \delta_U(\lambda_l, \nu_l) < t.$$

It is a contradiction. Thus $\delta_{U^s}(\lambda, \mu) \geq (\delta_U)^s(\lambda, \mu)$. □

Theorem 26. *Let (X, U) be an L-fuzzy quasi-uniform space and T_U be an L-fuzzy topology associated to U. Then the L-fuzzy topology T_{U^s} associated to U^s coincides with the stratification of T_U.*

Proof. We will show $T_{U^s} = (T_U)^s$. Since $U^s(\widehat{\alpha}) = 1$, $\widehat{\alpha}(\underline{\alpha}) = \underline{\alpha}$, for each $\alpha \in L$, then

$$I_{U^s}(\underline{\alpha}, \beta) = \bigvee\{\mu : a(\mu) \leq \lambda\ for\ some\ U^s(a) > \beta\} = \underline{\alpha},$$

for each $\beta, \alpha \in L$. Hence $T_{U^s}(\underline{\alpha}) = 1$, for each $\alpha \in L$ i.e T_{U^s} is stratified. By Theorem 24, U^s is finer than U. Then T_{U^s} is stratified finer than T_U. Thus $T_{U^s} \geq (T_U)^s$.
Conversely, Suppose there exists $\lambda \in L^X$ such that

$$T_{U^s}(\lambda) \nleq (T_U)^s(\lambda).$$

Then from definition of T_{U^s}, there exists $\alpha \in L$, $T_{U^s}(\lambda) \geq \alpha$. That is

$$I_{U^s}(\lambda, \alpha) = \lambda = \bigvee\{\mu : a(\mu) \leq \lambda, U^s(a) > \alpha\}.$$

Since $U^s(a) > \alpha$, then there exist $b \in \Omega_X$, $\beta \in L$ with $a \geq b \wedge \widehat{\beta}$ such that

$$U^s(a) \geq U(b) > \alpha.$$

On the other hand, by Proposition 1. $\widehat{\beta}(\mu) = \underline{1}$ for each $\mu \in L^X$ with $\sup(\mu) \nleq \beta$. Then $b(\mu) \leq \lambda$ for each μ with $\sup(\mu) \nleq \beta$ and therefore

$$\begin{aligned} \lambda &= \bigvee\{\mu : b(\mu) \wedge \widehat{\beta}(\mu) \leq \lambda, U(b) > \alpha\} \\ &\leq \bigvee\{\mu : b(\mu) \leq \lambda, \sup(\mu) \nleq \alpha, U(b) > \alpha\} \\ &\leq I_U(\lambda, \alpha). \end{aligned}$$

Hence $I_U(\lambda, \alpha) = \lambda$ i.e $T_U(\lambda) \geq \alpha$. Hence $(T_U)^s(\lambda) \geq \alpha$. It is a contradiction. Thus $T_{U^s} \leq (T_U)^s$. □

Theorem 27. *Let (X, T) be a stratified L-fts. Define for each $a \in \Omega_X$,*

$$U(a) = \vee \bigwedge_{i \in N} \{T(\lambda_i) : a \geq \bigwedge_{i \in N} a_{\lambda_i}\},$$

where $\bigvee$ is taken over all finite families $\{\lambda_i : a \geq \bigwedge_{i \in N} a_{\lambda_i}\}$. Then
(i) U is stratified L-fuzzy quasi-uniformity on X. (ii) $T_U = T$.

Proof. (i) (FU1) Suppose there exist $a, b \in \Omega_X$ such that

$$U(a \wedge b) \not\geq U(a) \wedge U(b).$$

Then by the definition of U, there exist two finite families $\{\lambda_i : a \geq \bigwedge_{i\in N} a_{\lambda_i}\}$, $\{\gamma_i : b \geq \bigwedge_{j\in K} a_{\gamma_j}\}$, such that

$$U(a \wedge b) \not\geq \bigwedge_{i\in N} T(\lambda_i) \wedge \bigwedge_{j\in K} T(\gamma_j).$$

On the other hand, we have

$$a \wedge b \geq \left(\bigwedge_{i\in N} a_{\lambda_i}\right) \wedge \left(\bigwedge_{j\in K} a_{\gamma_j}\right).$$

Then by definition of U,

$$U(a \wedge b) \geq \left(\bigwedge_{i\in N} T(\lambda_i)\right) \wedge \left(\bigwedge_{j\in K} T(\gamma_j)\right).$$

It is a contradiction.
(FU2) Since $a_\lambda \ o \ a_\lambda = a_\lambda$ by Lemma 2, then (FU2) holed.
(FU3) Obivous.
(FU4) There exists $a(say) = a_{\underline{1}} \in \Omega_X$ such that $U(a) \geq T(\underline{1}) = 1$.
(FUS) Since $a_{\underline{\alpha}} = \widehat{\alpha}$, then $U(\widehat{\alpha}) \geq T(\underline{\alpha}) = 1$. Hence $U(\widehat{\alpha}) = \underline{1}$ for each $\alpha \in L$. Thus U is stratified.
(ii) Obvious $T_U \leq T$, by definition of U.
Conversely, suppose there exist $\lambda \in L^X$ and $\alpha \in L - \{0, 1\}$ such that

$$T_U(\lambda) < \alpha < T(\lambda).$$

Since $a_\lambda(\lambda) = \lambda$, then $U(a_\lambda) \geq T(\lambda) > \alpha$ and then $I_U(\lambda, \alpha) = \bigvee\{\mu : a(\mu) \leq \lambda, for\ some\ U(a) > \alpha\} = \lambda$. Thus $T_U(\lambda) \geq \alpha$. It is a contradiction. Thus $T_U \geq T$. □

Theorem 28. *Let (X, U_1) and (Y, U_2) be L-fuzzy quasi-uniform spaces and U_1^s and U_2^s be stratification for U_1 and U_2 respectively. If $f : (X, U_1) \to (Y, U_2)$ is L-fuzzy uniformly continuous, then $f : (X, U_1^s) \to (Y, U_2^s)$ is L-fuzzy uniformly continuous.*

Proof. We will show $U_1^s(f^l(b)) \geq U_2^s(b)$ for each $b \in \Omega_Y$, If $U_2^s(b) = 0$, trivial. If $U_2^s \neq 0$, suppose that

$$U_1^s(f^l(b)) \not\geq U_2^s(b)$$

From definition of U_2^s, there exist $c \in \Omega_Y$, $\alpha \in L$ with $b \geq c \wedge \widehat{\alpha}$,

$$U_1^s(f^l(b)) \not\geq U_2(c).$$

Since $f : (X, U_1) \to (Y, U_2)$ is L-fuzzy uniformly continuous, $U_1(f^l(c)) \geq U_2(c)$. From the definition of U_1^s, we have

$$U_1^s(f^l(b)) \geq U_1(f^l(c)) \geq U_2(c)$$

It is a contradiction. Hence $U_1^s(f^l(b)) \geq U_2^s(b)$ for any $b \ \varepsilon \ \Omega_Y$. □

Counterexample.
Let X be any set, $L = [0,1]$ and identity mapping $Id_X : (X, U_1) \to (Y, U_2)$, define the L-fuzzy quasi-uniformity U_1 and U_2 on X as follows:

$$U_1(a) = \begin{cases} 1 : a = a_{\underline{0}}, \\ 0 : O.w \end{cases}$$

$$U_2(a) = \begin{cases} 1 : a = a_{\underline{0}}, \\ \frac{1}{3} : a_{\underline{0.5}} \leq a < a_{\underline{0}}, \\ 0 : O.w \end{cases}$$

where $a_\lambda : L^X \to L^X$ defined by Theorem 6. Since $a_{\underline{\alpha}} = \widehat{\alpha}$ for any $\alpha \in L$, then from Theorem 2, we obtain

$$U_1^s(a) = U_2^s(a) = \begin{cases} 1 : a \geq \widehat{\alpha}, for\ each\ \alpha \in L \\ 0 : O.w \end{cases}$$

Clearly $Id_X : (X, U_1^s) \to (X, U_2^s)$, is L-fuzzy uniformly continuous. But $Id_X : (X, U_1^s) \to (X, U_2^s)$, is not L-fuzzy uniformly continuous, where $0 = U_1(f^l(a_{\underline{0.5}})) < U_1(a_{\underline{0.5}}) = \frac{1}{3}$.

Theorem 29. *Let $\{(X_i, U_i)\}_{i \in \Gamma}$ be a family of L-fuzzy uniform spaces, X a set and, for each $i \in \Gamma$, $f_i : X \longrightarrow X_i$ a mapping. We define, for each $a \in \Omega_X$*

$$U(a) = \bigvee \bigwedge_{j \in N} \{U_j(b_j) : a \geq \bigwedge_{j \in N} f_j^i(b_j)\},$$

where $\bigvee$ is taken over all finite index $N \subset \Gamma$. Then
(i) U is the coarsest L-fuzzy uniformity on X for which all f_i, are L-fuzzy uniformity continuous.
(ii) A mapping $f : (X^*, U^*) \longrightarrow (X, U)$ is L-fuzzy uniformity continuous iff for each $i \in \Gamma$, f_i o f is L-fuzzy uniformity continuous.

Proof. (i) (FU1) Suppose that there exist $a, b \in \Omega_X$ such that

$$U(a \wedge b) \ngeq U(a) \wedge U(b).$$

By the definition of U there exist two finite families $\{a_n : a \geq \bigwedge_{n \in N} f_n^l(a_n)\}$, $\{b_j : b \geq \bigwedge_{j \in K} f_j^l(b_j)\}$ such that

$$U(a \wedge b) \ngeq (\bigwedge_{n \in N} U_n(a_n)) \wedge (\bigwedge_{j \in K} U_j(b_j)).$$

On the other hand, since $a \wedge b \geq (\bigwedge_{n \in N} f_n^l(a_n)) \wedge (\bigwedge_{j \in K} f_j^l(b_j))$, then

$$U(a \wedge b) \geq (\bigwedge_{n \in N} U_n(a_n)) \wedge (\bigwedge_{j \in J} U_j(b_j)).$$

It is a contradiction. Thus $U(a \wedge b) \geq U(a) \wedge U(b)$.
(FU2) Let $a \in \Omega_X$. Suppose that there exist $b \in \Omega_X$ such that

$$U(a) \nleq U(b).$$

By the definition of U, there exists finite family $\{a_n : a \geq \bigwedge_{n \in N} f_n^l(a_n)\}$ such that

$$\bigwedge_{n \in N} U_n(a_n) \nleq U(b).$$

Since $U_n(a_n) > 0$, for each $n \in N$, then there exists $b_n \in \Omega_{X_n}$, $b_n \ o \ b_n \leq a_n$ such that $U_n(b_n) \geq U_n(a_n)$. Put $b = \bigwedge_{n\in N} f_n^l(b_n)$. Hence

$$\begin{aligned} b \ o \ b &= \bigwedge_{n\in N} (f_n^l(b_n) \ o \ f_n^l(b_n)) \\ &\leq \bigwedge_{n\in N} f_n^l(b_n \ o \ b_n) \\ &\leq \bigwedge_{n\in N} f_n^l(a_n) \leq a. \end{aligned}$$

Then there exists $b \in \Omega_X$ such that $b \ o \ b \leq a$ and

$$U(b) \geq \bigwedge_{n\in N} U_n(b_n) \geq \bigwedge_{n\in N} U_n(a_n).$$

It is a contradiction. Thus $U(b) \geq U(a)$.
(FU3) Obvious.
(FU4) There exists $b \in \Omega_{X_j}, j \in \Gamma$ such that $U_j(b) = 1$, put $f_j^l(b) = a$. Then $U(a) = 1$.
(FU) Let $a \in \Omega_X$. Suppose that there exist $b \in \Omega_X$ such that

$$U(a) \not\leq U(b).$$

By the definition of U, there exists finite family $\{a_n : a \geq \bigwedge_{n\in N} f_n^l(a_n)\}$ such that

$$\bigwedge_{n\in N} U_n(a_n) \not\leq U(b).$$

Since $U_n(a_n) > 0$, for each $n \in N$, then there exists $b_n \in \Omega_{X_n}, b_n \leq a_n^{-1}$ such that $U_n(b_n) \geq U_n(a_n)$ for each $n \in N$. Put $b = \bigwedge_{n\in N} f_n^l(b_n)$. Hence

$$\begin{aligned} b^{-1} &= (\bigwedge_{n\in N} f_n^l(b_n))^{-1} \leq \bigwedge_{n\in N} (f_n^l(b_n))^{-1} \leq \bigwedge_{n\in N} f_n^l(b_n^{-1}) \\ &\leq \bigwedge_{n\in N} f_n^l((a_n^{-1})^{-1}) = \bigwedge_{n\in N} f_n^l(a_n) \\ &\leq a. \end{aligned}$$

Then there exists $b \in \Omega_X$ such that $b \leq a^{-1}$ and

$$U(b) \geq \bigwedge_{n\in N} U_n(b_n) \geq \bigwedge_{n\in N} U_n(a_n)$$

It is a contradiction. Thus $U(b) \geq U(a)$.
Second, it is easily proved that, by the definition of U, for all $i \in \Gamma$,

$$U(f_i^l(b)) \geq U_i(b) \ for \ each \ b \in \Omega_{X_i}.$$

Hence, for each $f_i : X \to X_i$ is L-fuzzy uniformity continuous.
Finally, if $f_i : (X, U^*) \to (X_i, U_i)$ is L-fuzzy uniformity continuous, then $U^*(f_j^l(b)) \geq$

$U_i(b)$ for each $j \in \Gamma$. We will show $U^* \geq U$, from the following:

$$\begin{aligned} U(a) &= \vee \bigwedge_{j\in N}\{U_j(b_j) : a \geq \bigwedge_{j\in N} f_j^l(b_j)\} \\ &\leq \vee \bigwedge_{j\in N}\{U^*(f_j^l(b)) : a \geq \bigwedge_{j\in N} f_j^l(b_j)\} \\ &\leq \vee\{U^*(\bigwedge_{j\in N} f_j^l(b)) : a \geq \bigwedge_{j\in N} f_j^l(b_j)\} \\ &\leq U^*(a). \end{aligned}$$

(ii) Obvious.

□

Definition 30. In a above theorem. U is called the initial L-fuzzy uniformity structure on X with respect to a family $\{(X_i, U_i)\}_{i\in\Gamma}$.

Corollary 31. *Let $\{(X_i, U_i)\}_{i\in\Gamma}$ be a family of L-fuzzy uniform spaces, $X = \Pi_{i\in\Gamma}X_i$ a set and, for each $i \in \Gamma, \pi_i : X \to X_i$ a projection mapping . We define, for each $a \in \Omega_X$*

$$\Pi_{i\in\Gamma}U_i(a) = \vee \bigwedge_{j\in N}\{U_j(b_j) : a \geq \bigwedge_{j\in N} \pi_j^l(b_j)\},$$

Where $\bigwedge$ is taken over all finite index $N \subset \Gamma$. Then
(i) $\Pi_{i\in\Gamma}U_i$ is the coarsest L-fuzzy uniformity on X for which all π_i, are L-fuzzy uniformity continuous.
(ii) A mapping $f : (X^*, U^*) \to (X, \Pi_{i\in\Gamma}U_i)$ is L-fuzzy uniformity continuous iff for each $i \in \Gamma$, π_i o $f$is L-fuzzy uniformity continuous.
The initial L-fuzzy uniformity uniformity structure $\Pi_{i\in\Gamma}U_i$is called the product L-fuzzy uniformity structure of a family $\{(X_i, U_i)\}_{i\in\Gamma}$,and $(X, \Pi_{i\in\Gamma}U_i)$is called product uniform space.

Theorem 32. *Let U be a product L-fuzzy uniformity structure of a family $\{(X_i, U_i)\}_{i\in\Gamma}$ and $X = \Pi_{i\in\Gamma}X_i$. If there exists a $j_0 \in \Gamma$ such that $\overline{U}_{j_0} = U^s_{j_0}$ and $\overline{U}_i = U_i$for each $i \in \Gamma - \{j_0\}$, then:*
(i) $\Pi_{i\in\Gamma}\overline{U}_i$ is the stratified L-fuzzy quasi -uniformity on X which is finer than U.
(ii) $\Pi_{i\in\Gamma}\overline{U}_i = U^s$.

Proof. (i) From corollary 31, $\Pi_{i\in\Gamma}\overline{U}_i$ is L-fuzzy quasi -uniformity on X for which all $\pi_i, i \in \Gamma$are L-fuzzy uniformity continuous. For each $\alpha \in L$, we have

$$\Pi_{i\in\Gamma}\overline{U}_i(\widehat{\alpha}) \geq \overline{U}_{j_0}(\widehat{\alpha}) = U^s_{j_0}(\widehat{\alpha}) = 1.$$

Thus $\Pi_{i\in\Gamma}\overline{U}_i(\widehat{\alpha}) = 1$. Moreover, since $U^s_{j_0}$is finer than $\Pi_{i\in\Gamma}\overline{U}_i$ is the stratified L-fuzzy quasi-uniformity on X which is finer than U.
(ii) Since $\Pi_{i\in\Gamma}\overline{U}_i$is stratified, then by Theorem 24, $U^s \preceq \Pi_{i\in\Gamma}\overline{U}_i$.
Conversely, suppose there exist $a \in \Omega_X$ such that

$$U^s(a) \not\geq \Pi_{i\in\Gamma}\overline{U}_i(a). \tag{4. 2}$$

By the definition of $\Pi_{i\in\Gamma}\overline{U}_i$, there exists a finite family $\{a_n\ : a \geq \bigwedge_{n\in N} \pi^l_n(a_n)\}$such that

$$U^s(a) \not\geq \bigwedge_{n\in N} \overline{U}_n(a_n).$$

On the other hand, if $j_0 \notin N$, then

$$U^s(a) \geq U(a) \geq \bigwedge_{n\in N} U_n(a_n) = \bigwedge_{n\in N} \overline{U}_n(a_n).$$

It is a contradiction for Eq. 4. 2 . If $j_0 \in N$, then $\overline{U}_{j_0}(a_{j_0}) = U^s_{j_0}(a_{j_0})$. From definition of $U^s_{j_0}$, there exists $b_{j_0} \in \Omega_{X_{j_0}}$ with $a_{j_0} \geq b_{j_0} \wedge \widehat{\alpha}_{j_0}$ such that $U^s_{j_0}(a_{j_0}) \geq U_{j_0}(b_{j_0})$. Since $\pi_{j_0} : X \to X_{j_0}$is L-fuzzy uniformity continuous, then $U(\pi^l_{j_0}(b_{j_0})) \geq U_{j_0}(b_{j_0})$. Now,

$$\begin{aligned} a &\geq \pi^l_{j_0}(a_{j_0}) \wedge (\bigwedge_{n\in N-\{j_0\}} \pi^l_n(a_n)) \\ &\geq \pi^l_{j_0}(b_{j_0} \wedge \widehat{\alpha}_{j_0}) \wedge (\bigwedge_{n\in N-\{j_0\}} \pi^l_n(a_n)) \\ &\geq (\pi^l_{j_0}(b_{j_0}) \wedge (\bigwedge_{n\in N-\{j_0\}} \pi^l_n(a_n))) \wedge \widehat{\alpha}_{j_0} \end{aligned}$$

Then we have

$$\begin{aligned} U^s(a) &\geq U(\pi^l_{j_0}(b_{j_0}) \wedge (\bigwedge_{n\in N-\{j_0\}} \pi^l_n(a_n))) \\ &\geq U(\pi^l_{j_0}(b_{j_0})) \wedge (\bigwedge_{n\in N-\{j_0\}} U(\pi^l_n(a_n))) \\ &\geq U_{j_0}(b_{j_0}) \wedge (\bigwedge_{n\in N-\{j_0\}} \overline{U}_n(a_n)) \\ &= \bigwedge_{n\in N} \overline{U}_n(a_n). \end{aligned}$$

Also, it is a contradiction for Eq. 4. 2 . Thus $U^s \succeq \Pi_{i\in\Gamma}\overline{U}_i$. □

Corollary 33. *Let (X, U_1) and (X, U_2) be L-fuzzy quasi-uniform spaces. We define, for each $a \in \Omega_X$*

$$(U_1 \Cap U_2)(a) = \vee\{U_1(b_1) \wedge U_2(b_2) : a \geq b_1 \wedge b_2\}.$$

Then: (i) $U_1 \Cap U^s_2$ and $U^s_1 \Cap U_2$are stratified L-fuzzy quasi-uniformity on X, which finer than $U_1 \Cap U_2$.
(ii) $U_1 \Cap U^s_2 = U^s_1 \Cap U_2 = (U_1 \Cap U_2)^s$
(iii) $T_{U_1\ U^s_2} = T_{U^s_1\ U_2} = T_{(U_1\ U_2)^s} = (T_{U_1\ U_2})^s$.

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