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Local Convergence for Multistep Simplified Newton-like Methods

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Abstract. In this paper we provide a local convergence analysis for multistep Newton-like method (1.3) in order to approximate a solution of the nonlinear equation (1.1) in a Banach space setting. A refined and more flexible than before local [4]-[7] local convergence analysis of multistep simplified Newton-like methods for approximating solutions of nonlinear operator equations in Banach space is provided, by approximating not only the differentiable (see [4]-[7]) but also the non differentiable part (see also [1],[2]). A numerical example is used where our results compare favorably with earlier ones [4]-[7].

AMS (MOS) Subject Classification Codes: 65H10, 65J15, 47H17, 49M15.

Key Words: Local convergence, Banach space, radius of convergence, Fréchet derivative, Multi-step simplified Newton-like method.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution of equation

$$F(x) = f(x) + g(x) = 0, \tag{1.1}$$

where f is a Fréchet-differentiable operator, g a continuous operator both defined on an open convex subset D of a Banach space X with values in a Banach space Y.

Newton-like (single step) method of the form

$$x^{n+1} = x^n - A(x^n)^{-1}F(x^n) \quad (n \geq 0 \tag{1.2}$$

has been used by several authors to approximate x^* [1]-[6]. With the exception of the works in [1]-[3] the authors take $A(x) \in L(X, Y)$ (the space of bounded linear operators from X into Y) to be a conscious approximation to the Fréchet-derivative $F'(x)$ of operator F. A survey of local and semilocal convergence results for method (1.2) can be found in [2].

However as already stated in [1], [3] there are several advantages (see Remark 3) if A is related not only to F' but also to the difference $g(x) - g(y)$. Here we extend these advantages (in the local convergence case) following some ideas in [5].

In order to compute each iterate in method (1.2) we solve the linear system $A(x^n)z = -F(x^n)$ and then set $x^{n+1} = x^n + z$ $(n \geq 0)$. The computation of $A(x^n)$ may be very expensive or impossible in general (for every $n \geq 0$). In practice we wish to use $A(x^n)$ instead of $A(x^{n-1}), ..., A(x^{n+m})$ to minimize the computational cost. That is why in [5] the multistep simplified Newton-like method was introduced for $x_0 \in D$ in the form:

$$\begin{aligned} x^{n,0} &= x^n \\ x^{n,i} &= x^{n,i-1} - A(x^n)^{-1}F(x^{n,i-1}), \quad i = 1, 2, ..., m \\ x^{n+1} &= x^{n,m} \quad (n \geq 0), \end{aligned} \tag{1.3}$$

where m is a natural number. Note that for $m = 1$ method (1.3) reduces to (1.2) which includes the so called simplified Newton-like method

$$x^{n+1} = x^n - A^{-1}F(x^n) \quad (n \geq 0), \tag{1.4}$$

with a constant linear operator A.

If $m = +\infty$ in (1.3) then the sequence $\{x^{0,i}\}$ also coincides with the one generated by (1.4) with $A = A(x^0)$. That is why in this study we assume m is finite. Local convergence results for method (1.3) were given in [5] for the interesting case $g \neq 0$ and $m > 1$. Here we show that under weaker hypotheses and the same computational cost the results in [5] can be improved (see more precisely Remark 3).

A numerical example is provided to justify the advantages of our approach over the ones in [5].

2. Local Convergence Analysis Of Simplified Newton-Like Method (1.3)

Suppose that equation (1.1) has a solution $x^* \in D$. We assume that there exists positive constants r_0, K, q, η and nonnegative constants c, e and an invertible linear operator L, such that for any

$$x, y \in U(x^*, r_0) = \{x \in X \,|\, \|x - x^*\| < r_0\} \subseteq D,$$

$A_1, A_2 \in L(Y, X)$, $A = A_1 + A_2$,

$$A(x)^{-1} \in L(X, Y)$$

such that

$$\begin{aligned} \left\|A(x)^{-1}L\right\| &\leq q, \\ \left\|A(x)^{-1}F(x)\right\| &\leq \eta, \\ \left\|L^{-1}(f'(x) - A_1(y))\right\| &\leq K\|x - y\| + c, \\ \left\|L^{-1}[g(x) - g(y) - A_2(x)(x - y)]\right\| &\leq e\|x - y\|. \end{aligned}$$

Define the scalar sequence $\{t_{n,i}\}$ by

$$t_{n,0} = 0, \; t_{n,i} = s_n(t_{n,i-1}), \; i = 1, ..., m+1, n \geq 0$$

where

$$\begin{aligned} s_n(t) &= q\left(\tfrac{K}{2}t + c + e\right)t + \eta_n, \\ \eta_0 &= \eta, \; \eta_n = t_{n-1,m+1} - t_{n-1,m} \; n \geq 1. \end{aligned}$$

Clearly $s_n(t)$ is an increasing function of $t \geq 0$. Therefore we have $t_{n,i} \leq t_{n,i+1}$. Further, define

$$t^* \geq \min(\max_n t_{n,m-1}, 2r_0),$$

$$b = q\left(\frac{Kt^*}{2} + c + e\right),$$

$$r_1 = \frac{2(1-b)}{qK},$$

and

$$a = \frac{qK}{2}.$$

We can state and show the local convergence theorem for Newton-like method (1.3).

Theorem 1. *Under the above assumptions, set* $r^* = \min\{r_0, r_1\}$. *If* $b \in [0,1)$, *then* $U(x^*, r^*)$ *is a convergence ball for (1.3). Moreover the following estimate holds for all* $n \geq 0$:

$$\left\|x^{n+1} - x^*\right\| \leq a(\|x^n - x^*\| + b)^m \|x^n - x^*\| \leq p^m \|x^n - x^*\|, \tag{2.5}$$

where,

$$p = a\left\|x^0 - x^*\right\| + b \in [0,1).$$

Proof. Let $x^0 \in U(x^*, r^*)$. Then we have

$$p < ar^* + b \leq ar_1 + b = 1$$

We shall prove the first inequality in (2.5) using induction on $k \geq 0$. We must show

$$\left\|x^{k,i} - x^{k,i-1}\right\| \leq t_{k,i} - t_{k,i-1} \;\; i = 1.....m \tag{2.6}$$

and

$$\left\|x^{k,i} - x^*\right\| \leq (a\left\|x^k - x^*\right\| + b)^i \left\|x^k - x^*\right\|, \; i = 1.....m \tag{2.7}$$

For $k = 0$, we have

$$\left\|x^{0,1} - x^{0,0}\right\| = \left\|x^{0,1} - x^0\right\| = \left\|A(x^0)^{-1}F(x^0)\right\| \leq \eta - t_{0,1} = t_{0,1} - t_{0,0}$$

and

$$\begin{aligned}
\left\|x^{0,1} - x^*\right\| &= \left\|-A(x^0)^{-1}(F(x^0) - F(x^*) - A(x^0)(x^0 - x^*))\right\| \\
&\leq q\left\|\int_0^1 L^{-1}(f'(x^* + t(x^0 - x^*)) - A_1(x^0))dt(x^0 - x^*)\right\| \\
&+ q\left\|L^{-1}(g(x^0) - g(x^*) - A_2(x^0)(x^0 - x^*))\right\| \\
&\leq q\left(\frac{K}{2}\left\|x^0 - x^*\right\| + c + e\right)\left\|x^0 - x^*\right\| \\
&\leq (a\left\|x^0 - x^*\right\| + b)\left\|x^0 - x^*\right\|
\end{aligned}$$

This implies that if $m = 1$, then (2.6) and (2.7) hold for $k = 0$. If $m \geq 2$, then we have by induction on i

$$\begin{aligned}
\left\|x^{0,i} - x^0\right\| &\leq \min\left\{\sum_{j=1}^{i}(t_{0,j} - t_{0,j-1}), \left\|x^{0,i} - x^*\right\| + \left\|x^0 - x^*\right\|\right\} \\
&\leq \min(t_{0,i}, 2r_0) \leq \min(t_{0,m-1}, 2r_0) \leq t^* \\
\left\|x^{0,i-1} - x^{0,i}\right\| &\leq \left\|L^{-1}(F(x^{0,i}) - A(x^0)(x^{0,i} - x^{0,i-1}) - F(x^{0,i-1}))\right\| \\
&\leq q\left(K\int_0^1 \left\|t(x^{0,i} - x^0) + (1-t)(x^{0,i-1} - x^0)\right\| dt + c + e\right) \\
&\quad \times \left\|x^{0,i} - x^{0,i-1}\right\| \\
&\leq q\left(\tfrac{K}{2}\left(t_{0,i} - t_{0,i-1}\right) + c + e\right)\left(t_{0,i} - t_{0,i-1}\right) = t_{0,i+1} - t_{0,i}
\end{aligned}$$

and

$$\begin{aligned}
\left\|x^{0,i+1} - x^*\right\| &= \left\|-A(x^0)^{-1}(F(x^{0,i}) - F(x^*) - A(x^0)(x^{0,i} - x^*))\right\| \\
&\leq q\left(\tfrac{K}{2}(\left\|x^0 - x^*\right\| + \left\|x^{0,i} - x^0\right\|) + c + e\right)\left\|x^{0,i} - x^*\right\| \\
&\leq q\left(\tfrac{K}{2}(\left\|x^0 - x^*\right\| + t^*) + c + e\right)\left\|x^{0,i} - x^*\right\| \\
&\leq (a\left\|x^0 - x^*\right\| + b)(a\left\|x^0 - x^*\right\| + b)^i\left\|x^0 - x^*\right\| \\
&= (a\left\|x^0 - x^*\right\| + b)^{i-1}\left\|x^0 - x^*\right\|.
\end{aligned}$$

This proves (2.6) and (1.1) for the case $k = 0$.

Assume now that (2.6) and (1.1) hold for some k. Then we have

$$x^{k+1,0} = x^{k-1} = x^{k,m} \in U(x^*, r)$$

and

$$\begin{aligned}
&\left\|x^{k+1,1} - x^{k+1,0}\right\| \\
&= \left\|x^{k+1.1} - x^{k+1}\right\| \\
&\leq \left\|A(x^{k+1})^{-1}L\right\|\left\|L^{-1}(F(x^{k,m}) - A(x^k)(x^{k,m} - x^{k,m-1}) - F(x^{k,m-1}))\right\| \\
&\leq q\left(\tfrac{K}{2}(\left\|x^{k,m} - x^k\right\| + \left\|x^{k,m-1} - x^k\right\|) + c + e\right)\left\|x^{k,m} - x^{k,m-1})\right)\right\| \\
&\leq q\left(\tfrac{K}{2}(t_{k,m} + t_{k,m-1}) + e + c\right)\left(t_{k,m} - t_{k,m-1}\right) = t_{k,m+1} - t_{k,m} = \eta_{k-1}.
\end{aligned}$$

By the same argument as for $k = 0$, we can prove that (2.6) and (2.7) hold for $k + 1$. This completes the induction and the proof of the theorem. □

Setting $L = A(x^*)$ in Theorem 1, we obtain the following:

Corollary 2. *Assume that $A(x^*)$ is nonsingular and for any $x \in D$, the following hold:*

$$\begin{aligned}
\left\|A(x^*)^{-1}(f'(x) - A_1(y))\right\| &\leq K\left\|x - y\right\| + c \\
\left\|A(x^*)^{-1}(A(x) - A(x^*))\right\| &\leq L\left\|x - x^*\right\| + d \\
\left\|A(x^*)^{-1}[g(x) - g(x^*) - A_2(x)(x - x^*)]\right\| &\leq e\left\|x - x^*\right\| \\
p &= c + d + e < 1
\end{aligned}$$

Then

(i) *The ball $U(x^*, r^*)$ with $r^* = 2(1-p)/(3K+2L)$ is a convergence ball for the iterative method (1.3) with any m, provided that $U(x^*, r^*) \subset D$. The speed of convergence is estimated as follows:*

$$\left\|x^{n+1} - x^*\right\| = \|x^{n,m} - x^*\| \le (a\|x^n - x^*\| + b)^m \|x^n - x^*\| \le p^m \|x^n - x^*\|$$

where

$$a = \tfrac{3K}{2(1-Lr-d)}, \quad b = \tfrac{c+e}{1-Lr-d}$$
$$p = a\left\|x^0 - x^*\right\| + b < 1$$

(ii) *The ball $U(x^*, r^*)$ with $r^* = 2(1-p)/(K+2L)$ is convergence ball for the iteration (1.4) and*

$$\left\|x^{n+1} - x^*\right\| \le \frac{1}{1 - Lr - d}\left(\tfrac{K}{2}\|x^n - x^*\| + c + e\right)\|x^n - x^*\|$$

provided that $U(x^*, r^*) \subset D$.

Proof. (see Corollary 1 in [5, p.19]). □

Remark 3. If we set

$$A_2 = 0 \quad \text{and } A_1 = A \tag{2.8}$$

our results reduce to the corresponding ones in [4]. Otherwise our results have the following advantages over the ones in [4]: more flexible choices of operator A (i.e A_1 and A_2); finer error bounds on the distances $\left\|x^{n+1} - x^*\right\|$;and a larger radius of r^*. That is we can obtain a desired error tolerance ε with fewer computations, a larger m can be used and there is a wider choice of initial guesses x^0 available. Such an information is important in computational mathematics and scientific computing [1], [2]. In what follows we provide an example. For simplicity we take $m = 1$, and $A(x) = L$.

Example 4. *Let $X = Y = (\mathbf{R}^2, \|\cdot\|_\infty)$. Consider the system* [3]*:*

$$3x^2y + y^2 - 1 + |x-1| = 0 \tag{2.9}$$
$$x^4 + xy^3 - 1 + |y| = 0.$$

It can easily be seen that the solution of (2.9) is given by

$$x^* = (.8946553334687, \ .327826521746298) \tag{2.10}$$

Set for $v = (v_1, v_2), \|v\|_\infty = \|(v_1, v_2)\|_\infty =$ $\max\{|v_1|, |v_2|\}$, *$F(v) = f(v) + g(v)$, $f(v) = (f_1, f_2)\,, g(v) = (g_1, g_2)\,.$*

Define

$$f_1(v) = 3v_1^2v_2 + v_2^2 - 1, \quad f_2(v) = v_1^4 + v_1v_2^3 - 1,$$
$$g_1(v) = |v_1 - 1|\,, g_2(v) = |v_2|\,.$$

We shall take divided differences of order one $[x, y; f]\,, [x, y; g] \in M_{2\times 2}(\mathbf{R})$ to be for $w = (w_1, w_2)$:

$$[v, w; f]_{i,1} = \frac{f_i(w_1, w_2) - f_i(v_1, w_2)}{w_1 - v_1}$$
$$[v, w; f]_{i,2} = \frac{f_i(v_1, w_2) - f_i(v_1, v_2)}{w_2 - v_2}$$

provided that $w_1 \neq v_1$ *and* $w_2 \neq v_2$. *If* $w_1 = v_1$ *or* $w_2 = v_2$ *replace* $[x, y, f]$ *by* f'. *Similarly we define*

$$[v, w; g]_{i,1} = \frac{g_i(w_1, w_2) - g_i(v_1, w_2)}{w_1 - v_1}$$
$$[v, w; g]_{i,2} = \frac{g_i(v_1, w_2) - g_i(v_1, v_2)}{w_2 - v_2}$$

for $w_1 \neq v_1$ *and* $w_2 \neq v_2$. *If* $w_1 = v_1$ *or* $w_2 = v_2$ *replace* $[x, y; g]$ *by the zero* 2×2 *matrix in* $M_{2\times 2}(\mathbf{R})$. *We consider a possible choice for operator* A *as suggested by the hypotheses in* [5]*:*

$$A(v) = A_1(v) = F'(v), \text{ and } A_2 = 0.$$

Then, using Newton's method (1.2) in this case for $x^0 = (1, 0)$, *we obtain Table 1. Moreover, if we choose:* $A(v, w) = A_1(v, w) = [v, w; g]$, *and* $A_2 = 0$, *i.e. the method of Chord or Secant method (1.2), we obtain Table 2, for* $x^{-1} = (5, 5)$, *and* $x^0 = (1, 0)$. *Furthermore if we choose:* $A = A_1 + A_2$, *where* $A_1(v, v) = F'(v) = [v, v; f]$, *and* $A_2(v, w) = [v, w; g]$ *for* $x^{-1} = (5, 5)$, *and* $x^0 = (1, 0)$ *our method (1.2) provides Table 3. Tables 2 and 3 show the superiority of the results obtained here, over the results in* [5] *using Table 1. Finally, although the superiority of our results over the ones in* [5] *has already been established, we note that if e.g., we let* $x^{-1} = x_7$, $x^0 = x_8$ *(chosen from Table 3), then hypotheses of Theorem 1 hold for* $K = q = 1, e = .25, c = 0, \eta = r_0 = 1.077E - 14, r^* = r_0$, *and* $t^* = 2r_0$.

TABLE 1.

n	$x_n^{(1)}$	$x_n^{(2)}$	$\lVert x_n - x_{n-1} \rVert$
0	1	0	
1	1	0.333333333333333	3.333E-1
2	0.906550218340611	0.354002911208151	9.344E-2
3	0.885328400663412	0.338027276361322	2.122E-2
4	0.891329556832800	0.326613976593566	1.141E-2
5	0.895238815463844	0.326406852843625	3.909E-3
6	0.8951546711372635	0.327730334045043	1.323E-3
7	0.894673743471137	0.327979154372032	4.809E-4
8	0.894598908977448	0.327865059348755	1.140E-4
9	0.894643228355865	0.327815039208286	5.002E-5
10	0.894659993615645	0.327819889264891	1.676E-5
11	0.894657640195329	0.327826728208560	6.838E-6
12	0.894655219565091	0.327827351826856	2.420E-6
13	0.894655074977661	0.327826643198819	7.086E-7
...			
39	0.89455373334687	0.327826521746298	5.149E-19

TABLE 2.

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x_n - x_{n-1}\|$
-1	5	5	
0	1	0	5.000E+00
1	0.989800874210782	0.021627489072365	1.262E-02
2	0.921814765493287	0.307939916152262	2.953E-01
3	0.900073765669214	0.325927010697792	2.174E-02
4	0.894939851625105	0.327725437396226	5.133E-03
5	0.894658420586013	0.327825363500783	2.814E-04
6	0.894655375077418	0.327826521051833	3.045E-04
7	0.894655373334698	0.327826521746293	1.742E-09
8	0.894655373334687	0.327826521746298	1.076E-14
9	0.894655373334687	0.327826521746298	5.421E-20

TABLE 3.

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x_n - x_{n-1}\|$
-1	5	5	
0	1	0	5
1	0.909090909090909	0.363636363636364	3.636E-01
2	0.894886945874111	0.329098638203090	3.453E-02
3	0.894655531991499	0.327827544745569	1.271E-03
4	0.894655373334793	0.327826521746906	1.022E-06
5	0.894655373334687	0.327826521746298	6.089E-13
6	0.894655373334687	0.327826521746298	2.710E-E20

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Numerical Solution of the Riemann–Hilbert Problem

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Abstract. This paper presents an implementation of the integral equations with the generalized Neumann kernel to solve numerically the uniquely and the non-uniquely solvable Riemann-Hilbert problems in Jordan regions with smooth boundaries. The non-uniquely solvable problems are made uniquely solvable by requiring their solutions to satisfy additional constraints. Two type of constraints are presented. Various test numerical examples are presented. The computational efficiency appears significantly excellent.

AMS (MOS) Subject Classification Codes: 30E25; 45B05; 45P05; 65R20.

Key Words: Riemann-Hilbert problem, Generalized Neumann kernel, Fredholm integral equation, Nyström method.

1. Introduction

The boundary integral equation method is an inexpensive, flexible technique to solve the elliptic boundary value problems on a simply connected region in the plane Ω. The reformulation of the boundary value problem as an equivalent integral equation over the boundary of Ω reduces the dimensionality of the problem which makes the method an efficient tool for complicated engineering problems.

Riemann–Hilbert problems on Ω are the prototypical examples of elliptic systems of differential equations in the plane (see e.g., Wendland [15]). The Dirichlet problem for Laplace's equation, which is one of the classical elliptic boundary value problems, is a special case of the Riemann-Hilbert problem.

The boundary integral equation method is a classical method for solving the Dirichlet problem (see e.g., Atkinson [1, Ch. 7] and Henrici [4, §15.9]). When the Dirichlet problem solved by the double layer potential representation, a boundary integral equation with a continuous kernel results. The kernel is known as the Neumann kernel [4, pp. 282–286].

The Riemann-Hilbert problem can also be solved using boundary integral equation [11, 8, 6, 7, 14]. Sherman [11] used a generalization of the double layer potential representation to derive a boundary integral equation with a continuous kernel for the interior Riemann-Hilbert problem (see e.g., Gakhov [3, p. 400]). Using a similar approach, a boundary integral equation can be derived for the exterior Riemann-Hilbert problem. The kernel of the both boundary integral equations is, as expected, a generalization of the Neumann kernel.

Recently, Murid and Nasser [8, 6, 7] used a different approach to derive two new boundary integral equations for the interior and the exterior Riemann-Hilbert problems. The kernel of these integral equations is the same kernel of the integral equations derived in [11]. This kernel was called in [7] the *generalized Neumann kernel*. The properties of the generalized Neumann kernel has been studied in [7] and more extensively in [14].

The boundary integral equation method is a popular method for solving the Dirichlet problem [1]. However, this is not the case for the Riemann-Hilbert problem. Riemann-Hilbert problems in Jordan regions are often solved by conformal mapping of the region to the unit disk where the problem can be solved in a closed form using the harmonic conjugation [2, 3, 4]. Gakhov introduced the concept of *regularizing factor* to reduce the Riemann-Hilbert problem to three Dirichlet problem in the same region of consideration (see e.g., Gakhov [3, p. 222] and Begehr [2, pp. 45–69]).

The current paper extends the results of the previous papers [8, 6, 7, 14, 10]. The papers [8, 6, 7, 14] were concentrated on the deriving and studying the solvability of the integral equations. Although, the paper [10] consider the numerical solution of the Riemann-Hilbert problems, only the integral equations derived in [7] were used. The non-uniquely solvable Riemann-Hilbert problem was solved by requiring their solution to satisfy additional constraints. Only one type of constraints was present in [14, 10].

The main propose of this paper is to present the numerical treatment of the integral equations with generalized Neumann kernel and the applications of the integral equations to solve the Riemann-Hilbert problems. We shall consider the integral equations derived in [11] as well as the integral equations derived in [8, 6, 7]. For the non-uniquely solvable Riemann-Hilbert problems, we present two type of constraints to reduce the problems to uniquely solvable problems. Some results of the original work [7, 14, 10] are included to allow this paper to be read independently.

This paper is organized as follows. In §2 we review some auxiliary results. The Riemann–Hilbert problems and the integral equations for the Riemann–Hilbert problems will be reviewed in §3. The solutions of the interior and the exterior Riemann–Hilbert problems in terms of the solution of the integral equations will be given in §4 and §5, respectively. The numerical implementations will be given in §6. Some numerical examples will be given in §7 and a short conclusion will be given in §8.

2. Auxiliary Material

Let Ω be a bounded simply connected Jordan region with $0 \in \Omega$. The boundary $\Gamma := \partial\Omega$ is assumed to have a positively oriented parametrization $\eta(s)$ where $\eta(s)$ is a 2π-periodic twice continuously differentiable function with $\dot{\eta}(s) = \frac{d\eta}{ds} \neq 0$. The exterior of Γ is denoted by Ω^-.

For a fixed α with $0 < \alpha < 1$, the Hölder space H^α consists of all 2π-periodic real functions which are uniformly Hölder continuous with exponent α. A Hölder continuous function $\hat{h}$ on Γ can be interpreted via $h(s) := \hat{h}(\eta(s))$ as a Hölder continuous function h of the parameter s and vice versa.

Let $A(s)$ be a complex continuously differentiable 2π-periodic function with $A \neq 0$. With $\gamma, \mu \in H^{\alpha}$, let the function $\Phi(z)$ be defined by

$$\Phi(z) := \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{\gamma + \mathrm{i}\mu}{A} \frac{d\eta}{\eta - z}, \quad z \notin \Gamma. \tag{2.1}$$

Then $\Phi(z)$ is analytic in Ω as well as in Ω^- and the boundary values Φ^+ from inside and Φ^- from outside belong to H^{α} and can be calculated by Plemelj's formula

$$\Phi^{\pm}(\zeta) = \pm \frac{1}{2} \frac{\gamma(\zeta) + \mathrm{i}\mu(\zeta)}{A(\zeta)} + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{\gamma(\eta) + \mathrm{i}\mu(\eta)}{A(\eta)} \frac{d\eta}{\eta - \zeta}, \quad \zeta \in \Gamma. \tag{2.2}$$

The integral in (2.2) is a Cauchy principal value integral. The boundary values satisfy the jump relation

$$A\Phi^+ - A\Phi^- = \gamma + \mathrm{i}\mu. \tag{2.3}$$

We define two real functions N and M by

$$N(s,t) := \frac{1}{\pi} \mathrm{Im} \left(\frac{A(s)}{A(t)} \frac{\dot{\eta}(t)}{\eta(t) - \eta(s)} \right), \tag{2.4}$$

$$M(s,t) := \frac{1}{2\pi} \cot \frac{s-t}{2} + \frac{1}{\pi} \mathrm{Re} \left(\frac{A(s)}{A(t)} \frac{\dot{\eta}(t)}{\eta(t) - \eta(s)} \right). \tag{2.5}$$

The function $N(s,t)$ is called the *generalized Neumann kernel* formed with A and η [7, 14].

Lemma 1 ([14]). (a) *The kernel $N(s,t)$ is continuous with*

$$N(t,t) = \frac{1}{\pi} \mathrm{Im} \left(\frac{1}{2} \frac{\ddot{\eta}(t)}{\dot{\eta}(t)} - \frac{\dot{A}(t)}{A(t)} \right). \tag{2.6}$$

(b) *The kernel $M(s,t)$ is continuous with*

$$M(t,t) = \frac{1}{\pi} \mathrm{Re} \left(\frac{1}{2} \frac{\ddot{\eta}(t)}{\dot{\eta}(t)} - \frac{\dot{A}(t)}{A(t)} \right). \tag{2.7}$$

The integral operators with the kernels N and M will be denoted by $\mathcal{N}$ and $\mathcal{M}$, i.e.,

$$(\mathcal{N}\mu)(s) := \int_0^{2\pi} N(s,t)\mu(t)dt, \tag{2.8}$$

$$(\mathcal{M}\mu)(s) := \int_0^{2\pi} M(s,t)\mu(t)dt. \tag{2.9}$$

The eigenvalues of the operator $\mathcal{N}$ have been studied in [7, 14]. It turns out that the dimensions of the spaces $Null(\mathcal{I} \pm \mathcal{N})$ depend upon the index of the function A which is defined as the winding number of A with respect to 0

$$\kappa := ind(A) := \frac{1}{2\pi} \arg(A)|_0^{2\pi} \tag{2.10}$$

i.e., the increment of the argument of A in traversing the curve Γ in the positive sense divided by 2π.

Theorem 2 ([14]).

$$\dim(Null(\mathcal{I}-\mathcal{N})) = \max(0,-2\kappa+1), \tag{2.11}$$
$$\dim(Null(\mathcal{I}+\mathcal{N})) = \max(0,2\kappa-1). \tag{2.12}$$

Let the complex-valued functions $\tilde{A}(t)$ be defined by

$$\tilde{A}(t) = \dot{\eta}(t)/A(t). \tag{2.13}$$

and let $\tilde{N}(s,t)$ be the generalized Neumann kernel formed with $\tilde{A}$ and η. Then the adjoint kernel $N^*(s,t)$ of the generalized Neumann kernel $N(s,t)$ can be written as

$$\begin{aligned} N^*(s,t) &= N(t,s) = \frac{1}{\pi}\mathrm{Im}\left(\frac{A(t)}{A(s)}\frac{\dot{\eta}(s)}{\eta(s)-\eta(t)}\right) \\ &= -\frac{1}{\pi}\mathrm{Im}\left(\frac{\tilde{A}(s)}{\tilde{A}(t)}\frac{\dot{\eta}(t)}{\eta(t)-\eta(s)}\right) = -\tilde{N}(s,t). \end{aligned} \tag{2.14}$$

Let also the complex-valued functions $A_0(s)$ and $A_1(s)$ be defined by

$$A_0(s) := \eta^{-\kappa}(s)A(s), \tag{2.15}$$
$$A_1(s) := \eta^{1-\kappa}(s)A(s). \tag{2.16}$$

Then the generalized Neumann kernel formed with A_i and η will be denoted by $N_i(s,t)$, $i=0,1$. Similarly, the continuous kernel $M_i(s,t)$ is defined as in (2.5) with A replaced by A_i, $i=0,1$. The integral operators with the kernels $\tilde{N}$, N_0, M_0, N_1 and M_1 are denoted by $\tilde{\mathcal{N}}$, $\mathcal{N}_0$, $\mathcal{M}_0$, $\mathcal{N}_1$ and $\mathcal{M}_1$, respectively.

The conjugation operator $\mathcal{K}$ is defined by

$$(\mathcal{K}\mu)(s) := \frac{1}{2\pi}\int_0^{2\pi} \mu(t)\cot\frac{s-t}{2}dt. \tag{2.17}$$

The operator $\mathcal{K}$ is also known as the Hilbert transform (see e.g., Henrici [4, p. 107] and [5]).

Let $z_0 = \alpha_0 - \mathrm{i}\beta_0$ be a given point. Other assumptions on z_0 will be given latter. We define the continuous real-valued functions $a(t)$, $b(t)$, $P(t)$ and $Q(t)$ by

$$a(t)+\mathrm{i}b(t) := \frac{1}{2\pi\mathrm{i}}\frac{\dot{\eta}(t)}{\eta(t)A_0(t)}, \tag{2.18}$$
$$P(t) := \alpha_0 a(t) + \beta_0 b(t), \tag{2.19}$$
$$Q(t) := \beta_0 a(t) - \alpha_0 b(t). \tag{2.20}$$

Then we define the integral operators $\mathcal{P}$ and $\mathcal{Q}$ by

$$(\mathcal{P}\mu)(s) := \int_0^{2\pi} P(t)\mu(t)dt, \tag{2.21}$$
$$(\mathcal{Q}\mu)(s) := \int_0^{2\pi} Q(t)\mu(t)dt. \tag{2.22}$$

The functions $\mathcal{P}\mu$ and $\mathcal{Q}\mu$ are constants on $[0,2\pi]$.

For a given set of points $r_i \in [0, 2\pi)$, $i = 1, 2, \ldots, m$, $m > 0$, we define the integral operator $\mathcal{R}_m$ by

$$(\mathcal{R}_m\mu)(s) = \begin{pmatrix} \int_0^{2\pi} N(r_1,t)\mu(t)dt \\ \int_0^{2\pi} N(r_2,t)\mu(t)dt \\ \vdots \\ \int_0^{2\pi} N(r_m,t)\mu(t)dt \end{pmatrix}. \tag{2.23}$$

3. Integral Equations For The Riemann-Hilbert Problems

3.1. The Riemann-Hilbert Problems. For $\gamma \in H^\alpha$, the Riemann-Hilbert (RH) problems are defined as follows:
Interior RH problem: Given functions A and γ, it is required to find a function f analytic in Ω and continuous on the closure $\overline{\Omega}$ such that the boundary values f^+ satisfy

$$\mathrm{Re}[A(s)f^+(\eta(s))] = \gamma(s) \quad \text{for all } s. \tag{3.24}$$

Exterior RH problem: Given functions A and γ, it is required to find a function g analytic in Ω^- and continuous on the closure $\overline{\Omega^-}$ with $g(\infty) = 0$ such that the boundary values g^- satisfy

$$\mathrm{Re}[A(s)g^-(\eta(s))] = \gamma(s) \quad \text{for all }, s. \tag{3.25}$$

3.2. The Solvability Of The Rh Problems. The solvability of the RH problems depend upon the index $\kappa = ind(A)$ [3]. The interior RH problem (3.24) is not necessary solvable for $\kappa > 0$. It is solvable only if

$$\gamma \in Null(\mathcal{I} - \tilde{\mathcal{N}})^\perp. \tag{3.26}$$

Similarly, the exterior RH problem (3.25) is solvable for $\kappa \le 0$ only if

$$\gamma \in Null(\mathcal{I} + \tilde{\mathcal{N}})^\perp. \tag{3.27}$$

If γ satisfies these conditions, then the RH problems are uniquely solvable (see e.g, [14]).

The interior RH problem for $\kappa \le 0$ and the exterior RH problem for $\kappa > 0$ are non-uniquely solvable. The general solution of the interior RH problem contains $-2\kappa+1$ arbitrary real constants and the general solution of the exterior RH problem contains $2\kappa - 1$ arbitrary real constants.

To reduce the non-uniquely solvable RH problems to uniquely solvable problems, we need to define the following two analytic functions Y and Z. Suppose that $\gamma \in H^\alpha$. The Schwarz operator $\mathcal{S}_i$ for the region Ω is an operator which determines the the unique analytic function $F(z) := (\mathcal{S}_i\gamma)(z)$ in Ω such that $F(0)$ is real and $\mathrm{Re}F^+ = \gamma$. Similarly, the Schwarz operator $\mathcal{S}_e$ for the region Ω^- is an operator which determines the the unique analytic function $G(z) := (\mathcal{S}_e\gamma)(z)$ in $\Omega^- \cup \{\infty\}$ such that $G(\infty)$ is real and $\mathrm{Re}G^- = \gamma$ (see e.g., [3]). Using the Schwarz operators $\mathcal{S}_i$ and $\mathcal{S}_e$, we define the two analytical functions Y in Ω and Z in $\Omega^- \cup \{\infty\}$ as follows

$$Y(z) := (\mathcal{S}_i \arg A_0)(z), \quad z \in \Omega, \tag{3.28}$$
$$Z(z) := (\mathcal{S}_e \arg A_0)(z), \quad z \in \Omega^-. \tag{3.29}$$

The interior RH problem (3.24) can be made uniquely solvable for $\kappa \le 0$ by two methods as in the following two lemmas.

Lemma 3 ([14]). *Suppose that $\kappa \le 0$, the point z_0 satisfies* $\text{Re}[z_0 e^{-\text{i}Y(0)}] \ne 0$, *$e_j$ ($j = 0, 1, ..., |\kappa| - 1$) are given complex numbers and $e_{|\kappa|}$ is a given real number. Then the interior RH problem* (3.24) *with the constraints*

$$\text{Im}[z_0 f^{(|\kappa|)}(0)] = e_{|\kappa|} \quad f^{(j)}(0) = e_j \ (j = 0, 1, ..., |\kappa| - 1) \tag{3.30}$$

is uniquely solvable.

Lemma 4 ([2]). *Suppose that $\kappa \le 0$, r_j are given distinct real numbers in* $[0, 2\pi)$ *and d_j are prescribed real numbers, $j = 1, 2, ..., 2|\kappa| + 1$,. Then the interior RH problem* (3.24) *with the side conditions*

$$\text{Im}[A(r_j) f^+(\eta(r_j))] = d_j \quad (j = 1, 2, ..., 2|\kappa| + 1) \tag{3.31}$$

is uniquely solvable.

Similarly, the exterior RH problem (3.25) can be made uniquely solvable for $\kappa > 0$ as in the following two lemmas.

Lemma 5 ([14]). *Suppose that $\kappa > 0$, the point z_0 satisfies* $\text{Re}[z_0 e^{-\text{i}Z(\infty)}] \ne 0$, *$e_j$ ($j = 1, 2, ..., \kappa - 1$) are given complex numbers and e_κ is a given real number. Then the exterior RH problem* (3.25) *with the constraints*

$$\text{Im}\left[\frac{z_0}{2\pi\text{i}} \int_\Gamma \eta^{\kappa-1} g(\eta) d\eta\right] = e_\kappa, \ \frac{1}{2\pi\text{i}} \int_\Gamma \eta^{j-1} g(\eta) d\eta = e_j \ (j = 1, 2, ..., \kappa - 1) \tag{3.32}$$

is uniquely solvable.

Lemma 6. *Suppose that $\kappa > 0$, r_j are given distinct real numbers in* $[0, 2\pi)$ *and d_j are prescribed real numbers, $j = 1, 2, ..., 2\kappa - 1$. Then the exterior RH problem* (3.25) *with the side conditions*

$$\text{Im}[A(r_j) g^-(\eta(r_j))] = d_j \quad (j = 1, 2, ..., 2\kappa - 1) \tag{3.33}$$

is uniquely solvable.

Proof.
This lemma can be proved with the same arguments as Lemma 4 in [2, p. 55]. □

3.3. **The Integral Equations.** For given functions $\gamma, \mu \in H^\alpha$, let the function $\Phi(z)$ be defined by (2.1). Based on the application of the Plemelj's formula, two boundary integral equations with the generalized Neumann kernel have been derived for the interior and the exterior RH problems by Murid and Nasser [8, 6, 7] and Wegmann et al [14] as in the following two lemmas.

Lemma 7 ([14]). *Suppose that $\gamma \in Null(\mathcal{I} - \tilde{\mathcal{N}})^\perp$ for $\kappa > 0$ and γ is arbitrary for $\kappa \le 0$. Then the function $f(z) := \Phi(z)$ is a solution of the interior RH problem* (3.24) *with the boundary values*

$$A(t) f^+(\eta(t)) = \gamma(t) + \text{i}\mu(t) \tag{3.34}$$

if and only if μ is a solution of the integral equation

$$(\mathcal{I} - \mathcal{N})\mu = -(\mathcal{M} - \mathcal{K})\gamma. \tag{3.35}$$

Lemma 8 ([14]). *Suppose that $\gamma \in Null(\mathcal{I} + \tilde{\mathcal{N}})^\perp$ for $\kappa \le 0$ and γ is arbitrary for $\kappa > 0$. Then the function $g(z) := -\Phi(z)$ is a solution of the exterior RH problem* (3.25) *with the boundary values*

$$A(t) g^-(\eta(t)) = \gamma(t) + \text{i}\mu(t) \tag{3.36}$$

if and only if μ is a solution of the integral equation

$$(\mathcal{I}+\mathcal{N})\mu = (\mathcal{M}-\mathcal{K})\gamma. \tag{3.37}$$

Another two boundary integral equations with the generalized Neumann kernel can be derived for the interior and the exterior RH problems (Sherman [11] and Gakhov [3, p. 400]). The derivation of these integral equations based on using a generalization of the double layer potential representation and on using the Plemelj's formula.

Let $f(z)$ be a solution of the interior RH problem (3.24), then there exists a real function h, and for $\kappa \le 0$, there exist complex constants c_k $(k = 0, 1, \dots, -\kappa)$ such that $f(z)$ can be written as [3, 9]

$$f(z) = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{h}{A}\frac{d\eta}{\eta - z} + \sum_{k=0}^{-\kappa} c_k z^k. \tag{3.38}$$

The constant $c_{-\kappa}$ has the form (see [3, p. 299])

$$c_{-\kappa} = (\hat\alpha + \mathrm{i}\hat\beta)\exp(-\mathrm{i}Z(\infty))$$

where $\hat\beta$ is a real constant depending on A and f, and $\hat\alpha$ is arbitrary real constant. In [3, p. 299], the arbitrary constant $\hat\alpha$ is chosen such that the constant $c_{-\kappa}$ can be written in the form

$$c_{-\kappa} = \mathrm{i}c\overline{z_0} \tag{3.39}$$

where c is a real constant and $z_0 = 1$ or $z_0 = \mathrm{i}$. In this paper, we shall choose $\hat\alpha$ such that $c_{-\kappa}$ can be written in the form (3.39) where z_0 satisfies the conditions

$$\mathrm{Re}[z_0 e^{-\mathrm{i}Y(0)}] \ne 0, \quad \mathrm{Re}[z_0 e^{-\mathrm{i}Z(\infty)}] \ne 0, \quad |z_0| = 1. \tag{3.40}$$

If $\kappa > 0$, the term containing the summation in (3.38) is replaced by zero. Then, by using the Plemelj's formula, we can prove the following lemma.

Lemma 9. *The function f given by* (3.38) *is a solution of the interior RH problem* (3.24) *if and only if the function h is a solution of the integral equation*

$$(\mathcal{I}+\mathcal{N})h + 2\sum_{k=0}^{-\kappa}\mathrm{Re}[c_k A\eta^k] = 2\gamma. \tag{3.41}$$

Similarly, if $g(z)$ is a solution of the exterior RH problem (3.25), then $g(z)$ can be written as [3, 9]

$$g(z) = -\frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{h}{A}\frac{d\eta}{\eta - z} + \sum_{k=1}^{\kappa}\frac{c_k}{z^k} \tag{3.42}$$

where h is a real function, and for $\kappa > 0$, c_k $(k = 1, 2, \dots, \kappa)$ are complex constants. The constant c_κ has the form

$$c_\kappa = (\hat\alpha + \mathrm{i}\hat\beta)\exp(-\mathrm{i}Y(0))$$

where $\hat\beta$ is a real constant depending on A and g, and $\hat\alpha$ is arbitrary real constant. The constant $\hat\alpha$ will be chosen such that the constant c_κ can be written in the form

$$c_\kappa = \mathrm{i}c\overline{z_0}$$

where c is a real constant and z_0 satisfies the conditions (3.40). If $\kappa \le 0$, the term containing the summation in (3.42) is replaced by zero. Then using the Plemelj's formula, we can prove the following lemma.

Lemma 10. *The function* g *defined by* (3.42) *is a solution of the exterior RH problem* (3.25) *if and only if the function* h *is a solution of the integral equation*

$$(\mathcal{I}-\mathcal{N})h + 2\sum_{k=1}^{\kappa} \mathrm{Re}[c_k A\eta^{-k}] = 2\gamma. \tag{3.43}$$

4. Solving The Interior RH Problem

In this section, we shall use the integral equations (3.35) and (3.41) to give the solutions for the interior RH problem (3.24). We shall assume the RH problems are uniquely solvable, i.e., we shall assume the right-hand side γ satisfies the condition (3.26) for $\kappa > 0$ and the solutions of the RH problems satisfy the constraints (3.30) or (3.31) for $\kappa \leq 0$.

4.1. The Interior Rh Problem (3.24) with the Condition (3.26).

For the interior RH problem (3.24) with the condition (3.26), we have $\kappa > 0$ and $\gamma \in (\mathcal{I}-\tilde{\mathcal{N}})^{\perp}$. Hence, Theorem 2 implies that $\dim Null(\mathcal{I}-\mathcal{N}) = 0$ and $\dim Null(\mathcal{I}+\mathcal{N}) = 2\kappa - 1 > 0$. Thus, by the Fredholm alternative theorem, the integral equation (3.35) is uniquely solvable. Moreover, the integral equation (3.41) becomes

$$(\mathcal{I}+\mathcal{N})h = 2\gamma. \tag{4.44}$$

Since $N^* = -\tilde{N}$ and $\gamma \in Null(\mathcal{I}-\tilde{\mathcal{N}})^{\perp}$, then the Fredholm alternative theorem implies that the integral equation (4.44) is solvable. However, it is non-uniquely solvable.

Since, it is not easy to solve numerically the non-uniquely solvable integral equations. Hence, in this paper, we shall use only the uniquely solvable integral equation (3.35) to solve the interior RH problem (3.24) with the condition (3.26).

Let μ be the unique solution of the integral equation (3.35) and let $\Phi(z)$ be defined by (2.1), then by Lemma 7, the unique solution of the interior RH problem (3.24) with the condition (3.26) is given by $f(z) := \Phi(z)$. The boundary values of the function $f(z)$ are given by (3.34).

4.2. The Interior RH Problem (3.24) with the Constraints (3.30).

For this case, we have $\kappa \leq 0$. Let $f(z)$ be the unique solution of the interior RH problem (3.24) with the constraints (3.30), then $f(z)$ can be written as

$$f(z) = \sum_{j=0}^{|\kappa|-1} \frac{e_j}{j!} z^j + z^{|\kappa|} f_0(z), \quad z \in \Omega, \tag{4.45}$$

where $f_0(z)$ is the unique solution of the interior RH problem

$$\mathrm{Re}[A_0(s) f_0^+(\eta(s))] = \gamma_0(s), \tag{4.46}$$

subject to the constraint

$$\mathrm{Im}[z_0 f_0(0)] = l_0 \tag{4.47}$$

with $l_0 := e_{|\kappa|}/|\kappa|!$ and $\gamma_0(s) := \gamma(s) - \mathrm{Re}\left[A(s)\sum_{j=0}^{|\kappa|-1} e_j \eta(s)^j/j!\right]$.

We shall present two methods for solving the the interior RH problem (4.46) with zero index. The first method is based on an integral equation obtained by modifying the integral equation (3.35). In the second method, we develop a new method based on an integral equation obtained by modifying the integral equation (3.41).

4.2.1. *The Method 1.* Since $\kappa_0 = \text{ind}(A_0) = 0$, then Lemma 7 implies that the function

$$f_0(z) := \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\gamma_0 + \mathrm{i}\mu}{A_0} \frac{d\eta}{\eta - z} \tag{4.48}$$

is a solution of the interior RH problem (4.46) with boundary values

$$A_0(s) f_0^+(\eta(s)) = \gamma_0(s) + \mathrm{i}\mu(s) \tag{4.49}$$

if and only if μ is a solution of the integral equation

$$\mu - \mathcal{N}_0 \mu = -(\mathcal{M}_0 - \mathcal{K})\gamma_0. \tag{4.50}$$

By the definitions of a and b, we have

$$f_0(0) = \frac{1}{2\pi i} \int_\Gamma \frac{\gamma_0 + i\mu}{A_0} \frac{d\eta}{\eta} = \int_0^{2\pi} (\gamma_0(s) + \mathrm{i}\mu(s))(a(s) + \mathrm{i}b(s))ds. \tag{4.51}$$

Since the function $f_0(z)$ satisfies the constraint (4.47), then (4.51) and (4.47) imply that $\mu(s)$ satisfies

$$\mathcal{P}\mu = \mathcal{Q}\gamma_0 + l_0 \tag{4.52}$$

where the integral operators $\mathcal{P}$ and $\mathcal{Q}$ are as in (2.21) and (2.22). By adding (4.52) to (4.50), we obtain the following integral equation for the determination of μ,

$$(\mathcal{I} - \mathcal{N}_0 + \mathcal{P})\mu = -(\mathcal{M}_0 - \mathcal{Q} - \mathcal{K})\gamma_0 + l_0. \tag{4.53}$$

Lemma 11. *Suppose that* $\kappa \leq 0$ *and* z_0 *satisfies* $\text{Re}[z_0 e^{-\mathrm{i}Y(0)}] \neq 0$, *then the integral equation* (4.53) *is uniquely solvable.*

Proof.
Suppose that $\mu \in Null(\mathcal{I} - \mathcal{N}_0 + \mathcal{P})$, i.e.,

$$\mu(s) - \int_0^{2\pi} N_0(s,t)\mu(t)dt + \int_0^{2\pi} P(t)\mu(t)dt = 0. \tag{4.54}$$

Since $\kappa_0 = \text{ind}(A_0) = 0$, then by the Fredholm alternative theorem and by Theorem 2, $\dim Null(\mathcal{I} - \mathcal{N}_0^*) = \dim Null(\mathcal{I} - \mathcal{N}_0) = 1$. Let $\phi \in Null(\mathcal{I} - \mathcal{N}_0^*)$. By Theorem 7 in [14], we can assume ϕ to be a strictly positive function. Multiplying both sides of (4.54) by $\phi(s)$ then integrating with respect to s, we obtain

$$\int_0^{2\pi} P(t)\mu(t)dt \int_0^{2\pi} \phi(s)ds = 0.$$

Since $\phi(s)$ is a positive real function, we obtain $\mathcal{P}\mu = 0$¿ Hence (4.54) implies that $\mu \in Null(\mathcal{I} - \mathcal{N}_0)$. Let $F(z) := \Phi(z)$ where $\Phi(z)$ be formed with $\gamma = 0$ and μ by (2.1), then Lemma 7 implies that the function $F(z)$ is a solution of the homogeneous interior RH problem

$$\text{Re}[A_0(s)F^+(\eta(s))] = 0,$$

with $A_0(s)F^+(\eta(s)) = \mathrm{i}\mu(s)$. By the definition of the functions a and b, we have $F(0) = \int_0^{2\pi} \mathrm{i}\mu(s)(a(s) + \mathrm{i}b(s))ds$ which implies in view of $\mathcal{P}\mu = 0$ that $\text{Im}[z_0 F(0)] = 0$. Since $\text{Re}[z_0 e^{-\mathrm{i}Y(0)}] \neq 0$, it follows from Lemma 3 that $F(z) = 0$ for all $z \in \Omega$. Hence, $\mu = 0$, which implies in view of the Fredholm alternative theorem that the integral equation (4.53) is uniquely solvable. □

Consequently, by the solving the uniquely solvable integral equation (4.53) for μ, the unique solution of the interior RH problem (3.24) with the constraints (3.30) is given by (4.45) where f_0 is given by (4.48). The boundary values of the function $f(z)$ can be calculated from (4.49) and (4.45).

4.2.2. *The Method 2.* Since $\kappa_0 = \text{ind}(A_0) = 0$, then Lemma 9 implies that the solution $f_0(z)$ of the interior RH problem (4.46) withe the constraint (4.47) is given by

$$f_0(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{h}{A_0} \frac{d\eta}{\eta - z} + \mathrm{i}c\overline{z_0} \tag{4.55}$$

where c is a real constant, z_0 satisfies the conditions (3.40), and h is a solution of the integral equation

$$(\mathcal{I} + \mathcal{N}_0)h - 2c\text{Im}[\overline{z_0}A_0] = 2\gamma_0. \tag{4.56}$$

By the Cauchy integral formula and by the definition of the functions a and b, we have

$$f_0(0) = \int_0^{2\pi} h(t)(a(t) + \mathrm{i}b(t))dt + \mathrm{i}c\overline{z_0}.$$

Since $f_0(z)$ satisfies the constraint (4.47) and $|z_0| = 1$, the constant c is given by

$$c = l_0 + \mathcal{Q}h. \tag{4.57}$$

By substituting (4.57) into (4.56), we obtain

$$(\mathcal{I} + \mathcal{N}_2)h = 2l_0\text{Im}[\overline{z_0}A_0] + 2\gamma_0, \tag{4.58}$$

where $\mathcal{N}_2$ is the integral operator with the kernel

$$N_2(s,t) := N_0(s,t) - 2Q(t)\text{Im}[\overline{z_0}A_0(s)]. \tag{4.59}$$

Lemma 12. *Suppose that $\kappa \le 0$ and z_0 satisfies the conditions* (3.40), *then the integral equation* (4.58) *is uniquely solvable.*

Proof.
Let $h \in Null(\mathcal{I} - \mathcal{N}_2)$, i.e.,

$$h(s) + \int_0^{2\pi} \left(N_0(s,t) - 2Q(t)\text{Im}[\overline{z_0}A_0(s)]\right) h(t)dt = 0, \tag{4.60}$$

and let the function $F(z)$, $z \notin \Gamma$, be defined by

$$F(z) := \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{h}{A_0} \frac{d\eta}{\eta - z} + \mathrm{i}\overline{z_0} \int_0^{2\pi} Q(t)h(t)dt. \tag{4.61}$$

Then by mean of the Plemelj's formula, (4.60) implies that $F(z)$ is a solution of the homogeneous interior RH problem

$$\text{Re}[A_0(s)F^+(\eta(s))] = 0. \tag{4.62}$$

By the definition of the functions P and Q, we have

$$z_0F(0) = \int_0^{2\pi} P(s)h(s)ds \tag{4.63}$$

which implies that $F(z)$ satisfies the constraint $\text{Im}[z_0F(0)] = 0$.

Since $\kappa_0 = 0$, Lemma 3 implies that $F(z) = 0$ for all $z \in \Omega$. Consequently, we have

$$\frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{h}{A_0} \frac{d\eta}{\eta - z} = -\mathrm{i}\overline{z_0} \int_0^{2\pi} Q(t)h(t)dt, \quad z \in \Omega, \tag{4.64}$$

which implies that $F(z)$ is analytic in Ω^- with $A_0(s)F^-(\eta(s)) = -h(s)$ and

$$F(\infty) = \mathrm{i}\overline{z_0} \int_0^{2\pi} Q(t)h(t)dt. \tag{4.65}$$

Let $G(z) := -\mathrm{i}F(z)/z$, then G is analytic in Ω^- with $A_1(s)G^-(\eta(s)) = \mathrm{i}h(s)$ and satisfies the homogeneous exterior RH problem

$$\mathrm{Re}[A_1(s)G^-(\eta(s))] = 0. \tag{4.66}$$

By the definition of the function G and by (4.65), we have

$$\frac{1}{2\pi\mathrm{i}} \int_\Gamma G^-(\eta)d\eta = -\mathrm{i}\frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{F^-(\eta)}{\eta} d\eta = -\mathrm{i}F(\infty) = \overline{z_0} \int_0^{2\pi} Q(t)h(t)dt, \tag{4.67}$$

which implies that the function G satisfies
$\mathrm{Im}\left[z_0 \frac{1}{2\pi\mathrm{i}} \int_\Gamma G^-(\eta)d\eta\right] = 0$.

Since $\mathrm{Re}[z_0 e^{-\mathrm{i}Z(\infty)}] \neq 0$, then by Lemma 5, $G(z) = 0$ for all $z \in \Omega^-$. Hence $h = 0$. Then the Fredholm alternative theorem implies that the integral equation (4.58) is uniquely solvable. □

Let μ be the unique solution of the uniquely solvable integral equation (4.58) and let f_0 be given by (4.55), then the unique solution of the interior RH problem (3.24) with the constraints (3.30) is given by (4.45).

In the above two methods, the values of the unique solution f of the interior RH problem (3.24) with the constraints (3.30) at any point $z \in \Omega$ can be calculated using the Cauchy integral formula. However, on the one hand, the method 1 provides us with the boundary values f^+ of the unique solution f without any extra calculations as we need for the method 2. This is an advantage of the method 1 over the method 2. On the other hand, the right-hand side of the integral equation (4.58) is given explicitly and the right-hand side of the integral equation (4.53) requires extra calculations which is an advantage of the method 2 over the method 1. This remark is true of the methods 1 and 2 which will be given in the next section of the exterior RH problem (3.25) with the constraints (3.32).

4.3. **The Interior RH Problem (3.24) with the Side Conditions (3.31).** Let $\kappa \leq 0$ and let $f(z)$ be the unique solution of the interior RH problem (3.24) with the side conditions (3.31) and let μ be a solution of the integral equation (3.35), then Lemma 7 implies that $f(z) = \Phi(z)$ where $\Phi(z)$ is defied by (2.1). Furthermore, the boundary values of $f(z)$ are given by (3.34).

From (3.34), the side conditions (3.31) on the solution $f(z)$ of the interior RH problem (3.24) require the solution of the integral equation (3.35) to satisfy the constraints

$$\mu(r_i) = d_i, \quad i = 1, 2, \ldots, 2|\kappa| + 1. \tag{4.68}$$

Substituting (4.68) into (3.35) implies that

$$(\mathcal{R}_{2|\kappa|+1}\mu)(t) = \mathbf{c}_{2|\kappa|+1}, \tag{4.69}$$

where $\mathbf{c}_{2|\kappa|+1}$ is the $2|\kappa| + 1 \times 1$ vector with the elements

$$(\mathbf{c}_{2|\kappa|+1})_i = d_i + (\mathcal{M}\gamma)(r_i) - (\mathcal{K}\gamma)(r_i), \quad i = 1, 2, \ldots, 2|\kappa| + 1.$$

Consequently, the function μ satisfies the system of integral equations

$$\begin{pmatrix} \mathcal{R}_{2|\kappa|+1} \\ \mathcal{I} - \mathcal{N} \end{pmatrix} \mu = \begin{pmatrix} \mathbf{c}_{2|\kappa|+1} \\ -(\mathcal{M} - \mathcal{K})\gamma \end{pmatrix}. \tag{4.70}$$

Lemma 13. *The system of integral equations* (4.70) *is uniquely solvable.*

Proof.
Since the interior RH problem (3.24) with the side conditions (3.31) is uniquely solvable, then the system of integral equations (4.70) is solvable. Thus to show that (4.70) is uniquely solvable, it is sufficient to show that the homogenous system

$$\begin{pmatrix} \mathcal{R}_{2|\kappa|+1} \\ \mathcal{I}-\mathcal{N} \end{pmatrix} \mu_0 = 0 \tag{4.71}$$

has only the trivial solution $\mu_0 = 0$. Let μ_0 by any solution of the homogenous system (4.71), then $\mu_0 = 0$ satisfies

$$\mu_0 - \mathcal{N}\mu_0 = 0 \quad \text{and} \quad \mathcal{R}_{2|\kappa|+1}\mu_0 = 0.$$

Hence $\mu_0(r_i) = (\mathcal{N}\mu_0)(r_i) = 0$ for $i = 1, 2, \dots, 2|\kappa| + 1$. Let $f_0(z) := \Phi(z)$ where Φ formed with $\gamma = 0$ and μ_0 as in (2.1), then Lemma 7 implies that f_0 is a solution of the homogenous interior RH problem

$$\mathrm{Re}[A(s) f_0(\eta(s))] = 0 \tag{4.72}$$

with $A(t)f_0(\eta(t)) = \mathrm{i}\mu_0(t)$. Since $\mu_0(r_i) = 0$, hence

$$\mathrm{Im}[A(r_i) f_0(\eta(r_i))] = \mu_0(r_i) = 0, \quad i = 1, 2, \dots, 2|\kappa| + 1. \tag{4.73}$$

Then by Lemma 5, the homogenous interior RH problem (4.72) with the side conditions (4.73) has the unique solution $f_0 = 0$ which implies that $\mu_0 = 0$. □

Let μ be the unique solution of the system of integral equations (4.70) and let $\Phi(z)$ be defined by (2.1), then the unique solution of the interior RH problem (3.24) with the side conditions (3.31) is given by $f(z) := \Phi(z)$. The boundary values of the function $f(z)$ are given by (3.34).

5. Solving the Exterior RH Problems

In this section, we shall give formulas for the solutions of the exterior RH problem (3.25) in terms of the solution of the integral equations (3.37) and (3.43). We shall assume the right-hand side γ satisfies the condition (3.27) for $\kappa \le 0$ and the solutions of the exterior RH problem satisfy the constraints (3.32) or (3.33) for $\kappa > 0$ so the problem is always uniquely solvable.

5.1. **The Exterior RH Problem (3.24) with the Condition (3.26).** Since $\kappa \le 0$ and $\gamma \in Null(\mathcal{I} + \tilde{\mathcal{N}})^{\perp}$, then Theorem 2 implies that $\dim Null(\mathcal{I} + \mathcal{N}) = 0$ and $\dim Null(\mathcal{I} - \mathcal{N}) = 2|\kappa| + 1 > 0$. Hence, the integral equation (3.43) becomes

$$(\mathcal{I} - \mathcal{N})h = 2\gamma. \tag{5.74}$$

Since $N^* = -\tilde{N}$ and $\gamma \in Null(\mathcal{I} + \tilde{\mathcal{N}})^{\perp}$, then the Fredholm alternative theorem implies that the integral equation (5.74) is non-uniquely solvable.

However, by the Fredholm alternative theorem, the integral equation (3.37) is uniquely solvable. Hence, we shall use only the uniquely solvable integral equation (3.37) to solve the exterior RH problem (3.25) with the condition (3.27).

By Lemma 8, the unique solution of the exterior RH problem (3.25) with the condition (3.27) is given by $g(z) := -\Phi(z)$ where $\Phi(z)$ is defined by (2.1) with μ being the unique solution of the integral equation (3.37). The boundary values of the function $g(z)$ are given by (3.36).

5.2. **The Exterior RH Problem (3.25) with the Constraints (3.32).** Let $\kappa > 0$ and let $g(z)$ be the unique solution of the exterior RH problem (3.25) with the constraints (3.32). Then $g(z)$ can be written as

$$g(z) = \sum_{j=1}^{\kappa-1} \frac{e_j}{z^j} + \frac{g_1(z)}{z^{\kappa-1}}, \quad z \in \Omega, \tag{5.75}$$

where $g_1(z)$ is the unique solution of the exterior RH problem

$$\mathrm{Re}[A_1(s) g_1^-(\eta(s))] = \gamma_1(s) \tag{5.76}$$

subject to the constraint

$$\mathrm{Im}\left[\frac{z_0}{2\pi \mathrm{i}} \int_\Gamma g_1(\eta) d\eta\right] = l_1 \tag{5.77}$$

with $l_1 := e_\kappa$ and $\gamma_1(s) := \gamma(s) - \mathrm{Re}\left[A(s) \sum_{j=1}^{\kappa-1} e_j / \eta(s)^j\right]$.

As for the interior RH problem, we shall present two methods for solving the the exterior RH problem (5.76) with the constraint (5.77). The first method is based on a uniquely solvable integral equation obtained by modifying the integral equation (3.37). In the second method, we modify the integral equation (3.43) to obtain a new uniquely solvable integral equation.

5.2.1. *The Method 1.* Since $\kappa_1 = \mathrm{ind}(A_1) = 1$, then Lemma 8 implies that the function

$$g_1(z) := -\frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\gamma_1 + \mathrm{i}\mu}{A_1} \frac{d\eta}{\eta - z} \tag{5.78}$$

is a solution of the exterior RH problem (5.76) with boundary values

$$A_1(s) g_1^-(\eta(s)) = \gamma_1(s) + \mathrm{i}\mu(s) \tag{5.79}$$

if and only if μ is a solution of the integral equation

$$(\mathcal{I} + \mathcal{N}_1)\mu = (\mathcal{M}_1 - \mathcal{K})\gamma_1. \tag{5.80}$$

By the definition of a and b, we have

$$\frac{1}{2\pi \mathrm{i}} \int_\Gamma g_1^-(\eta) d\eta = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\gamma_1 + \mathrm{i}\mu}{A_1} d\eta = \int_0^{2\pi} (\gamma_1(s) + \mathrm{i}\mu(s))(a(s) + ib(s)) ds.$$

Since the function $g_1(z)$ satisfies the constrain (5.77), then (5.77) implies that $\mu(s)$ satisfies

$$\mathcal{P}\mu = \mathcal{Q}\gamma_1 + l_1. \tag{5.81}$$

By adding (5.81) to (5.80), we obtain the following integral equation for the determination of μ,

$$(\mathcal{I} + \mathcal{N}_1 + \mathcal{P})\mu = (\mathcal{M}_1 + \mathcal{Q} - \mathcal{K})\gamma_1 + l_1. \tag{5.82}$$

The following lemma can be proved along the same lines as Lemma 11.

Lemma 14. *Suppose that* $\kappa > 0$ *and* z_0 *satisfies* $\mathrm{Re}[z_0 e^{-\mathrm{i}Z(\infty)}] \neq 0$, *then the integral equation* (5.82) *is uniquely solvable.*

By the solving the uniquely solvable integral equation (5.82) for μ, the unique solution of the exterior RH problem (3.25) with the constraints (3.32) can be calculated from (5.75) where g_1 is given by (5.78). The boundary values of the function $g(z)$ can be calculated from (5.79) and (5.75).

5.2.2. *The Method 2.* By Lemma 10 and since $\kappa_1 = \text{ind}(A_1) = 1$, the solution of the exterior RH problem (5.76) with the constraint (5.77) is given by

$$g_1(z) = -\frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{h}{A_1}\frac{d\eta}{\eta - z} + \frac{\mathrm{i}c\overline{z_0}}{z}, \tag{5.83}$$

where c is a real constant, z_0 satisfies the conditions (3.40) and h is a solution of the integral equation

$$(\mathcal{I} - \mathcal{N}_1)h - 2c\mathrm{Im}[\overline{z_0}A_0] = 2\gamma_1. \tag{5.84}$$

The function g_1 can be written in the form

$$g_1(z) = \frac{1}{z}\left(-\frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{h}{A_0}\frac{d\eta}{\eta - z} + \int_0^{2\pi}(a(s) + \mathrm{i}b(s))h(s)ds + \mathrm{i}c\overline{z_0}\right). \tag{5.85}$$

Hence

$$\frac{1}{2\pi\mathrm{i}}\int_\Gamma g_1^-(\eta)d\eta = \int_0^{2\pi}(a(s) + \mathrm{i}b(s))h(s)ds + \mathrm{i}c\overline{z_0}. \tag{5.86}$$

Since g_1 satisfies the constraint (5.77) and $|z_0| = 1$, then (5.86) implies that the constant c is given by

$$c = l_1 + \mathcal{Q}h. \tag{5.87}$$

By substituting (5.87) into (5.84), we obtain

$$(\mathcal{I} - \mathcal{N}_3)h = 2l_1\mathrm{Im}[\overline{z_0}A_0] + 2\gamma_1, \tag{5.88}$$

where $\mathcal{N}_3$ is the integral operator with the kernel

$$N_3(s,t) = N_1(s,t) + 2Q(t)\mathrm{Im}[\overline{z_0}A_0(s)]. \tag{5.89}$$

Lemma 15. *Suppose that $\kappa > 0$ and z_0 satisfies the conditions* (3.40)*, then the integral equation* (5.88) *is uniquely solvable.*

Proof.
Let h be a solution of the homogenous equation

$$h(s) - \int_0^{2\pi}\left(N_1(s,t) + 2Q(t)\mathrm{Im}[\overline{z_0}A_0(s)]\right)h(t)dt = 0, \tag{5.90}$$

and let the function $G(z)$, $z \notin \Gamma$, be defined by

$$G(z) := \frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{h}{A_1}\frac{d\eta}{\eta - z} - \frac{\mathrm{i}\overline{z_0}}{z}\int_0^{2\pi}Q(t)h(t)dt. \tag{5.91}$$

Then it follows from the jump relation (2.3) that

$$G^+(\eta(s)) - G^-(\eta(s)) = \frac{h(s)}{A_1(s)} - \frac{\mathrm{i}\overline{z_0}}{\eta(s)}\int_0^{2\pi}Q(t)h(t)dt. \tag{5.92}$$

By mean of the Plemelj's formula, (5.90) implies that the function G is a solution of the homogeneous exterior RH problem

$$\mathrm{Re}[A_1(s)G^-(\eta(s))] = 0. \tag{5.93}$$

By the definition of the function a and b, we have

$$\frac{z_0}{2\pi\mathrm{i}}\int_\Gamma G^-(\eta)d\eta = -\int_0^{2\pi}a(s)h(s)ds, \tag{5.94}$$

which implies that G satisfies the constraint $\mathrm{Im}\left[z_0 \frac{1}{2\pi \mathrm{i}} \int_\Gamma G^-(\eta) d\eta\right] = 0$. Since $\kappa_1 = 1$, Lemma 5 implies that $G(z) = 0$ on Ω^-. Hence, $G^-(\eta) = 0$ on Γ, which implies in view of (5.92) that the function $G(z)$ is analytic in Ω with

$$G^+(\eta(s)) = \frac{h(s)}{A_1(s)} - \frac{\mathrm{i}\overline{z_0}}{\eta(s)} \int_0^{2\pi} Q(t)h(t)dt. \tag{5.95}$$

Hence the function

$$F(z) := \mathrm{i}zG(z) - \overline{z_0} \int_0^{2\pi} Q(t)h(t)dt, \tag{5.96}$$

is a solution of the homogeneous interior RH problem

$$\mathrm{Re}[A_0(s)F^+(\eta(s))] = 0, \tag{5.97}$$

with the condition $\mathrm{Im}[z_0F(0)] = 0$. Since $\kappa_0 = 0$, it follows from Lemma 3 that $F(z) = 0$ for all $z \in \Omega$. Consequently,

$$G(z) = \frac{\overline{z_0}}{\mathrm{i}z} \int_0^{2\pi} Q(t)h(t)dt. \tag{5.98}$$

Since $G(z)$ is analytic in Ω, it follows from (5.98) that $\int_0^{2\pi} Q(t)h(t)dt = 0$ which implies that $G(z) = 0$ on Ω. In view of (5.95), we obtain $\mu = 0$. Hence, by the Fredholm alternative theorem, the integral equation (5.88) is uniquely solvable. □

Let μ be the solution of the uniquely solvable integral equation (5.88), then unique solution of the exterior RH problem (3.25) with the constraints (3.32) can be calculated from (5.75) where g_1 is given by (5.83).

5.3. **The Exterior RH Problem (3.25) with the Side Conditions (3.33).** Let $\kappa \le 0$ and let $g(z)$ be the unique solution of the exterior RH problem (3.25) with the side conditions (3.33), then Lemma 8 implies that $g(z) = -\Phi(z)$ where $\Phi(z)$ defied by (2.1) with μ is a solution of the integral equation (3.37). Furthermore, the boundary values of $g(z)$ are given by (3.36).

By (3.36), the side conditions (3.33) on the solution $g(z)$ of the exterior RH problem (3.25) require the solution $\mu(t)$ of the integral equation (3.37) to satisfy the constraints

$$\mu(r_i) = d_i, \quad i = 1, 2, \ldots, 2\kappa - 1. \tag{5.99}$$

Substituting (5.99) into (3.37) implies that

$$(\mathcal{R}_{2\kappa-1}\mu)(t) = \mathbf{d}_{2\kappa-1}, \tag{5.100}$$

where $\mathbf{d}_{2\kappa-1}$ is the $2\kappa - 1 \times 1$ vector with the elements

$$(\mathbf{d}_{2\kappa-1})_i = -d_i + (\mathcal{M}\gamma)(r_i) - (\mathcal{K}\gamma)(r_i), \quad i = 1, 2, \ldots, 2\kappa - 1.$$

Consequently, the function μ satisfies the system of integral equations

$$\begin{pmatrix} \mathcal{R}_{2\kappa-1} \\ \mathcal{I} + \mathcal{N} \end{pmatrix} \mu = \begin{pmatrix} \mathbf{d}_{2\kappa-1} \\ (\mathcal{M} - \mathcal{K})\gamma \end{pmatrix}. \tag{5.101}$$

Lemma 16. *The the system of integral equations* (5.101) *is uniquely solvable.*

Proof.
This lemma can be proved with the same arguments as Lemma 13. □

By solving the uniquely solvable system of integral equations (5.101) for μ, the unique solution of the exterior RH problem (3.25) with the side conditions (3.33) is given by $g(z) := -\Phi(z)$ where $\Phi(z)$ is defined by (2.1). The boundary values of the function $g(z)$ are given by (3.36).

6. The Numerical Implementations

Since the integrals in the integral equations of this paper are over $2\pi-$periodic functions, they can be best discretized on an equidistant grid by the trapezoidal rule, i.e., the integral equations are solved by the Nyström method with the trapezoidal rule as the quadrature rule (Atkinson [1]).

Suppose that n is an even integer and define the the n equidistant collocation points t_j by

$$t_j := (j-1)\frac{2\pi}{n}, \quad j = 1, 2, \ldots, n. \tag{6.102}$$

For a $2\pi-$periodic function h, then the trapezoidal rule approximate the integral $I(h) := \int_0^{2\pi} h(t)dt$ by

$$I_n(h) := \frac{2\pi}{n}\sum_{j=1}^{n} h(t_j). \tag{6.103}$$

Then the trapezoidal rule (6.103) with the grid (6.102) will be used to discretize the integrals in the integral equations (3.35) for $\kappa > 0$, (4.53), (4.58), (3.37) for $\kappa \le 0$, (5.82), (5.88) and the system of integral equations (4.70) for $\kappa \le 0$ and (5.101) for $\kappa > 0$.

The discretization operator $\mathcal{N}_n$ of the operator $\mathcal{N}$ is given by

$$(\mathcal{N}_n h)(s) := \frac{2\pi}{n}\sum_{j=1}^{n} N(s, t_j)h(t_j). \tag{6.104}$$

Then we define the matrix $\mathbf{N}$ to be the $n \times n$ matrix with the elements

$$\mathbf{N}_{ij} := \frac{2\pi}{n} N(t_i, t_j), \quad i, j = 1, 2, \ldots n. \tag{6.105}$$

The discretization of the operators $\mathcal{N}_0$, $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, $\mathcal{M}$, $\mathcal{M}_0$ and $\mathcal{M}_1$ is defined as in (6.104) and will be denoted by $\mathcal{N}_{0,n}$, $\mathcal{N}_{1,n}$, $\mathcal{N}_{2,n}$, $\mathcal{N}_{3,n}$, $\mathcal{M}_n$, $\mathcal{M}_{0,n}$ and $\mathcal{M}_{1,n}$, respectively. Similarly, we define the matrices $\mathbf{N}_0$, $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$, $\mathbf{M}$, $\mathbf{M}_0$ and $\mathbf{M}_1$ as in (6.105) with N replaced by N_0, N_1, N_2, N_3, M, M_0 and M_1, respectively.

The discretization operator $\mathcal{P}_n$ of the operator $\mathcal{P}$ is defined by

$$(\mathcal{Q}_n h)(s) = \frac{2\pi}{n}\sum_{j=1}^{n} Q(t_j)h(t_j). \tag{6.106}$$

The discretization operator $\mathcal{Q}_n$ of the operator $\mathcal{Q}$ is defined as in (6.106). Then we define the matrices $\mathbf{P}$ and $\mathbf{Q}$ to be the $n \times n$ matrices with the elements

$$\mathbf{P}_{ij} := \frac{2\pi}{n} P(t_j), \quad \mathbf{Q}_{ij} := \frac{2\pi}{n} Q(t_j), \quad i, j = 1, 2, \ldots n. \tag{6.107}$$

For $\kappa \le 0$, we define the $(2|\kappa| + 1) \times n$ matrix $\mathbf{R}_1$ and the $(2|\kappa| + 1) \times 1$ vector $\tilde{\mathbf{c}}$ for $i = 1, 2, \ldots 2|\kappa| + 1$ and $j = 1, 2, \ldots, n$ by

$$\mathbf{R}_{1,ij} := \frac{2\pi}{n} N(r_i, t_j), \quad \tilde{\mathbf{c}}_i = d_i + (\mathcal{M}\gamma)(r_i) - (\mathcal{K}\gamma)(r_i).$$

where r_i and d_i are as in Lemma 4. Similarly, for $\kappa > 0$, we define the $(2\kappa - 1) \times n$ matrix $\mathbf{R}_2$ and the $(2\kappa - 1) \times 1$ vector $\tilde{\mathbf{d}}$ by

$$\mathbf{R}_{2,ij} := \frac{2\pi}{n} N(r_i, t_j), \quad \tilde{\mathbf{d}}_i = -d_i + (\mathcal{M}\gamma)(r_i) - (\mathcal{K}\gamma)(r_i)$$

where r_i and d_i are as in Lemma 6, $i = 1, 2, \ldots 2\kappa - 1$ and $j = 1, 2, \ldots, n$.

We denote by μ_n to the approximate solution of the integral equations (3.35) for $\kappa > 0$, (4.53), (4.58), (3.37) for $\kappa \le 0$, (5.82), (5.88) and the system of integral equations (4.70) for $\kappa \le 0$ and (5.101) for $\kappa > 0$. Then we define the $n \times 1$ vector $\mathbf{x}$ by

$$\mathbf{x}_i := \mu_n(t_i), \quad i = 1, 2, \ldots, n.$$

We define also the $n \times 1$ vectors $\mathbf{y}$, $\mathbf{y}_0$ and $\mathbf{y}_1$ by

$$\mathbf{y}_i := \gamma(t_i), \quad \mathbf{y}_{0,i} := \gamma_0(t_i), \quad \mathbf{y}_{1,i} := \gamma_1(t_i), \quad \mathbf{y}_{2,i} := \mathrm{Im}[\overline{z_0} A_0(t_i)].$$

Let $\mathcal{K}_n$ be the discretization of the operator $\mathcal{K}$, then we can calculate $(\mathcal{K}_n h)(t)$ efficiently for all $t \in [0, 2\pi]$ using the FFT. For the collocation points t_i ($i = 1, 2, \ldots, n$), if $\mathbf{y}_3$ is the $n \times 1$ vector with the elements

$$\mathbf{y}_{3,i} = (\mathcal{K}_n h)(t_i),$$

then

$$\mathbf{y}_3 = \mathbf{K}\mathbf{x}$$

where the matrix $\mathbf{K}$ is known as the Wittich's matrix and is given for $i, j = 1, 2, \ldots, n$ by

$$\mathbf{K}_{ij} = \begin{cases} 0, & \text{if } j - i \text{ is even} \\ \frac{2}{n} \cot \frac{(i-j)\pi}{n}, & \text{if } j - i \text{ is odd.} \end{cases}$$

Let $\mathbf{I}$ be the $n \times n$ identity matrix and let $\mathbf{I}_1$ be the $n \times 1$ vector whose elements are all ones. Hence, the applying of the Nyström method to the uniquely solvable integral equations (3.35) for $\kappa > 0$, (4.53), (4.58), (3.37) for $\kappa \le 0$, (5.82) and (5.88) leads, respectively, to the following linear systems

$$(\mathbf{I} - \mathbf{N})\mathbf{x} = -(\mathbf{M} - \mathbf{K})\mathbf{y}, \tag{6.108}$$

$$(\mathbf{I} - \mathbf{N}_0 + \mathbf{P})\mathbf{x} = -(\mathbf{M}_0 - \mathbf{Q} - \mathbf{K})\mathbf{y}_0 + l_0\mathbf{I}_1, \tag{6.109}$$

$$(\mathbf{I} + \mathbf{N}_2)\mathbf{x} = 2l_0\mathbf{y}_2 + 2\mathbf{y}_0, \tag{6.110}$$

$$(\mathbf{I} + \mathbf{N})\mathbf{x} = (\mathbf{M} - \mathbf{K})\mathbf{y}, \tag{6.111}$$

$$(\mathbf{I} + \mathbf{N}_1 + \mathbf{P})\mathbf{x} = (\mathbf{M}_1 + \mathbf{Q} - \mathbf{K})\mathbf{y}_1 + l_1\mathbf{I}_1, \tag{6.112}$$

$$(\mathbf{I} - \mathbf{N}_3)\mathbf{x} = 2l_1\mathbf{y}_2 + 2\mathbf{y}_1. \tag{6.113}$$

Since the integral equations are uniquely solvable, then the resulting linear system (6.108)–(6.113) are uniquely solvable for sufficiently large n [1, p. 107].

By using the trapezoidal rule (6.103) to discretize the integrals in the system of integral equations (4.70) for $\kappa \le 0$ and (5.101) for $\kappa > 0$ then collocating at the node points (6.102), we obtain, respectively, the following over-determined linear system

$$\begin{pmatrix} \mathbf{R}_1 \\ \mathbf{I} - \mathbf{N} \end{pmatrix} \mathbf{x} = \begin{pmatrix} \tilde{\mathbf{c}} \\ -(\mathbf{M} - \mathbf{K})\mathbf{y} \end{pmatrix}, \tag{6.114}$$

$$\begin{pmatrix} \mathbf{R}_2 \\ \mathbf{I} + \mathbf{N} \end{pmatrix} \mathbf{x} = \begin{pmatrix} \tilde{\mathbf{d}} \\ (\mathbf{M} - \mathbf{K})\mathbf{y} \end{pmatrix}. \tag{6.115}$$

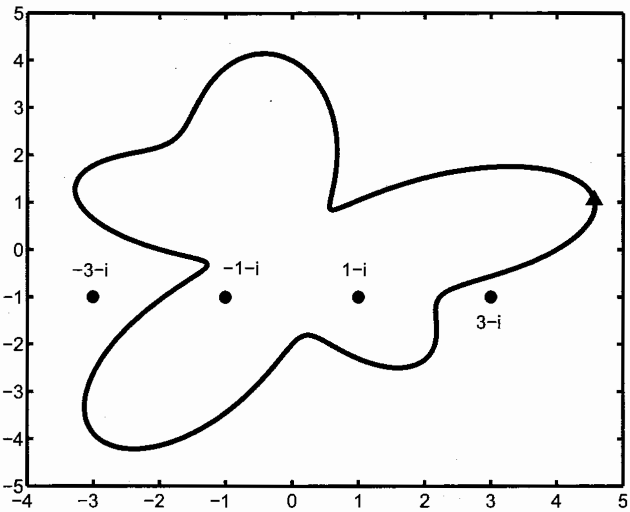

FIGURE 1. The Curve Γ.

The above linear systems (6.108)–(6.115) are either uniquely solvable square linear systems or over-determined linear systems. In our numerical calculations, both type of linear systems are solved using the MATLAB's \ operator that makes use of the Gauss elimination method for square systems and the QR factorization with column pivoting method for over-determined systems [12].

By solving the linear systems, we obtain the solutions of the integral equations at the collocation points t_i, $i = 1, 2, \ldots, n$. Then the Nyström interpolating formula provides us with approximate solutions $\mu_n(t)$ to the integral equations for all $t \in [0, 2\pi]$. The approximate solutions $\mu_n(t)$ of these integral equations can be then used to obtain approximate solutions to the RH problems.

7. Numerical Examples

In this section we apply the proposed method to six examples contain three interior RH problems (Examples 1–3) and three exterior RH problems (Examples 4–6) in the interior and the exterior of the smooth Jordan curve Γ with the parameterization $\eta(s) = (3 + \cos 3s + \sin 5s)e^{is}$, $0 \le s \le 2\pi$. The graphs of Γ is shown in Figure 1.

Tables 1–8 show the values of the approximate solutions of the RH problems at the test points $z_1 = -1 - \mathrm{i}$ and $z_2 = 1 - \mathrm{i}$ for the interior problems and at the test points $z_3 = -3 - \mathrm{i}$ and $z_4 = 3 - \mathrm{i}$ for the exterior problems.

Tables 1, 2, 4–6 and 8 list also the sup-norm error $\|f^+ - f_n^+\|_\infty$ and $\|g^+ - g_n^+\|_\infty$ where f^+, g^+ are the boundary values of the exact solutions and f_n^+, g_n^+ are the boundary values of the approximate solutions. The sup-norm is computed numerically by comparing $f^+(\eta(t))$, $g^+(\eta(t))$ and $f_n^+(\eta(t))$, $g_n^+(\eta(t))$ at 100 equally spaced points in $[0, 2\pi]$, most of which are not collocation points.

The exact solution of the RH problem in Example 1 is $f(z) = z$ and the exact solution of the RH problem in Example 4 is $g(z) = 1/z$. The exact solutions of the RH problems in the remaining examples are not known. For this case, we consider the approximate solution obtained with $n = 1024$ as the exact solution.

TABLE 1. The numerical results for Example 17.

n	$\|f - f_n\|_\infty$	$f(z_1)$	$f(z_2)$
32	2.67(−01)	−1.030373 − 1.001340i	1.001033 − 0.966912i
64	3.95(−02)	−1.002061 − 0.998145i	0.999868 − 0.996886i
128	1.13(−03)	−0.999988 − 1.000000i	1.000008 − 1.000012i
256	4.92(−07)	−1.000000 − 1.000000i	1.000000 − 1.000000i
512	4.14(−11)	−1.000000 − 1.000000i	1.000000 − 1.000000i

TABLE 2. The numerical results for Example 18 using method 1.

n	$\|f - f_n\|_\infty$	$f(z_1)$	$f(z_2)$
32	1.64(−01)	0.121506 + 0.388527i	−0.082456 − 0.044672i
64	2.92(−02)	0.139595 + 0.405296i	−0.069501 − 0.056723i
128	5.88(−04)	0.139656 + 0.405110i	−0.068900 − 0.058102i
256	1.88(−07)	0.139653 + 0.405115i	−0.068910 − 0.058077i
512	8.81(−12)	0.139653 + 0.405115i	−0.068910 − 0.058077i

TABLE 3. The numerical results for Example 18 using method 2.

n	$f(z_1)$	$f(z_2)$
32	0.116989 + 0.289777i	−0.125253 + 0.013495i
64	0.142597 + 0.405528i	−0.068479 − 0.058130i
128	0.139671 + 0.405097i	−0.068861 − 0.058195i
256	0.139653 + 0.405115i	−0.068910 − 0.058077i
512	0.139653 + 0.405115i	−0.068910 − 0.058077i

TABLE 4. The numerical results for Example 19.

n	$\|f - f_n\|_\infty$	$f(z_1)$	$f(z_2)$
32	2.44(−01)	0.049024 − 0.649521i	−0.608792 − 0.403835i
64	4.15(−02)	0.077166 − 0.672586i	−0.564633 − 0.384698i
128	6.31(−04)	0.086361 − 0.679775i	−0.555738 − 0.381776i
256	3.15(−07)	0.086348 − 0.679774i	−0.555758 − 0.381788i
512	1.99(−11)	0.086348 − 0.679774i	−0.555758 − 0.381788i

Example 17. $A(s) = e^{\mathrm{i}s}/R(s)$ $(\kappa = 1)$ *and* $\gamma(s) = \cos 2s$.

Example 18. $A(s) = e^{-\mathrm{i}s}$ $(\kappa = -1)$, $e_0 = e_1 = 0$, $z_0 = 1$ *and* $\gamma(s) = \cos 2s$.

Example 19. $A(s) = e^{-\mathrm{i}s}$ $(\kappa = -1)$, $r_1 = 0$, $r_2 = \pi/2$, $r_3 = \pi$, $d_1 = d_2 = d_3 = 0$ *and* $\gamma(s) = \cos 2s$.

Example 20. $A(s) = R(s)e^{-is}$ $(\kappa = -1)$ *and* $\gamma(s) = \cos 2s$.

Example 21. $A(s) = e^{\mathrm{i}s}$ $(\kappa = 1)$, $e_1 = 0$, $z_0 = 1$ *and* $\gamma(s) = \cos 2s$.

Example 22. $A(s) = e^{\mathrm{i}s}$ $(\kappa = 1)$, $r_1 = 0$, $d_1 = 0$ *and* $\gamma(s) = \cos 2s$.

TABLE 5. The numerical results for Example 20.

n	$\|g-g_n\|_\infty$	$g(z_3)$	$g(z_4)$
32	———	−0.356640 + 0.123182i	0.347841 + 0.186617i
64	2.97(−02)	−0.301208 + 0.099487i	0.301949 + 0.103685i
128	8.74(−04)	−0.299973 + 0.099983i	0.299980 + 0.099960i
256	1.71(−07)	−0.300000 + 0.100000i	0.300000 + 0.100000i
512	3.66(−12)	−0.300000 + 0.100000i	0.300000 + 0.100000i

TABLE 6. The numerical results for Example 21 using method 1.

n	$\|g-g_n\|_\infty$	$g(z_3)$	$g(z_4)$
32	———	−0.113352 + 0.559966i	0.465106 + 1.480249i
64	2.85(−01)	−0.252018 + 0.533466i	0.259034 + 1.379282i
128	1.16(−02)	−0.251439 + 0.535743i	0.264041 + 1.372368i
256	4.21(−06)	−0.251351 + 0.535805i	0.264285 + 1.372376i
512	3.49(−11)	−0.251351 + 0.535805i	0.264285 + 1.372376i

TABLE 7. The numerical results for Example 21 using method 2.

n	$g(z_3)$	$g(z_4)$
32	−0.179432 + 0.544865i	0.633549 + 1.580277i
64	−0.248657 + 0.538546i	0.286340 + 1.396616i
128	−0.251359 + 0.535816i	0.264296 + 1.372618i
256	−0.251351 + 0.535805i	0.264285 + 1.372376i
512	−0.251351 + 0.535805i	0.264285 + 1.372376i

TABLE 8. The numerical results for Example 22.

n	$\|g-g_n\|_\infty$	$g(z_3)$	$g(z_4)$
32	———	−0.447336 − 0.075625i	0.103461 + 1.831099i
64	3.42(−01)	−0.034616 + 1.192131i	0.482947 + 0.717498i
128	1.20(−02)	−0.030617 + 1.198096i	0.484756 + 0.710105i
256	4.58(−06)	−0.030675 + 1.197833i	0.484962 + 0.710348i
512	7.02(−11)	−0.030675 + 1.197833i	0.484961 + 0.710348i

8. CONCLUSIONS

We developed a numerical method for solving numerically the interior and the exterior RH problems. The method is based on the boundary integral equations with the generalized Neumann kernel that have been derived in [11, 8, 6, 7, 14].

The uniquely solvable RH problems were solved using only the integral equations derived in [8, 6, 7, 14] because they are uniquely solvable and the integral equations derived in [11] are non-uniquely solvable.

The non-uniquely solvable RH problems with additional constraints (at $z = 0$ or at $z = \infty$) were solved using two methods, the method 1 based on the boundary integral equation of [7, 14] and the method 2 based on the boundary integral equation of [11]. For the two methods, the solutions of the RH problems were calculated using the Cauchy integral formula. The advantage of method 1 is that it provides us with the boundary values of the solutions of the RH problems without any extra calculations as we need for the method 2. However, the right-hand side of the integral equation of [11] is given explicitly and the right-hand side of the integral equation of [7, 14] requires extra calculations which is an advantage of method 2 over method 1.

Several RH problems were solved using the developed method. The numerical examples show clearly that the developed method gives results of high accuracy.

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Common Fixed Point Theorems In Ultra Metric Spaces

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Abstract. The purpose of this paper is to prove some common fixed point theorems for a pair of maps of Jungck type on a spherically complete metric space.

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Key Words: Ultra metric space, Spherically complete, Common fixed point.

1. Introduction

Generally to prove fixed or common fixed point theorems for maps satisfying strictly contractive conditions, one has to assume the continuity of maps and compact metric spaces. In spherically complete ultra metric spaces, the continuity of maps are not necessary to obtain fixed points. First we state some known definitions.

Definition 1. ([3]): Let (X, d) be a metric space. If the metric d satisfies strong triangle inequality:
$d(x, y) \leq max\{d(x, z), d(z, y)\} \forall x, y, z \in X$
then d is called an ultra metric on X and the pair (X, d) is called an ultra metric space.

Definition 2. ([3]): An ultra metric space (X, d) is said to be spherically complete if every shrinking collection of balls in X has a non empty intersection.

Recently Gajic [1] proved the following

Theorem 3. *(Theorem 1, [1]): Let (X, d) be a spherically complete ultra metric space. If $T : X \longrightarrow X$ is a mapping such that*
$d(Tx, Ty) < max\{d(x, y), d(x, Tx), d(y, Ty)\} \forall x, y \in X, x \neq y$
then T has a unique fixed point in X.

Now we extend this Theorem for a pair of maps of Jungck type.

2. Main Results

Theorem 4. *Let (X, d) be a spherically complete ultra metric space. If f and T are self maps on X satisfying*

$$T(X) \subseteq f(X), \tag{2.1}$$

$$d(Tx,Ty) < max\{d(fx,fy), d(fx,Tx), d(fy,Ty)\} \forall x, y \in X, x \neq y \tag{2.2}$$

then there exists $z \in X$ such that$fz = Tz$.
Further if f and T are coincidentally commuting at z then z is the unique common fixed point of f and T.

Proof. Let $B_a = (fa; d(fa, Ta))$ denote the closed sphere centered at fa with the radius $d(fa, Ta)$ and let A be the collection of these spheres for all $a \in X$. Then the relation $B_a \leq B_b$ iff $B_b \subseteq B_a$ is a partial order on A. Let A_1 be a totally ordered sub family of A. Since (X, d) is spherically complete , we have $\bigcap_{B_a \in A_1} B_a = B \neq \phi$.
Let $fb \in B$ and $B_a \in A_1$. Then $fb \in B_a$. Hence
$d(fb, fa) \leq d(fa, Ta) \cdots\cdots$ (i)
If $a = b$ then $B_a = B_b$. Assume that $a \neq b$.
Let $x \in B_b$. Then

$$\begin{aligned} d(x, fb) &\leq d(fb, Tb) \\ &\leq \max\{d(fb, fa), d(fa, Ta), d(Ta, Tb)\} \\ &= \max\{d(fa, Ta), d(Ta, Tb)\} \quad from(i) \\ &< \max\{d(fa, fb), d(fa, Ta), d(fb, Tb)\} \quad from(2.2) \\ &= d(fa, Ta)........(ii) \end{aligned}$$

Now , $d(x, fa) \leq \max\{d(x, fb), d(fb, fa)\} \leq d(fa, Ta)$ from(i) and (ii).

Thus $x \in B_a$. Hence $B_b \subseteq B_a$ for any $B_a \in A_1$. Thus B_b is an upper bound in A for the family A_1 and hence by Zorn's Lemma, A has a maximal element,say $B_z, z \in X$.
Suppose $fz \neq Tz$. Since $Tz \in T(X) \subseteq f(X)$, there exists $w \in X$such that $Tz = fw$. Clearly $z \neq w$. Now from (2.2) we have

$$\begin{aligned} d(fw, Tw) &= d(Tz, Tw) \\ &< \max\{d(fz, fw), d(fz, Tz), d(fw, Tw)\} \quad from(2.2) \\ &= d(fz, fw) \end{aligned}$$

Thus $fz \notin B_w$. Hence $B_z \not\subseteq B_w$. It is a contradiction to the maximality of B_z.
Hence $fz = Tz$.
Further assume that f and T are coincidentally commuting at z .
Then $f^2z = f(fz) = fTz = Tfz = T(Tz) = T^2z$.
Suppose $fz \neq z$. Now from (2.2), we have

$$\begin{aligned} d(Tfz, Tz) &< \max\{d(f^2z, fz), d(f^2z, Tfz), d(fz, Tz)\} \\ &= d(Tfz, Tz). \end{aligned}$$

Hence $fz = z$. Thus $z = fz = Tz$. Uniqueness of common fixed point of f and T follows easily from(2.2). □

Now we give an example to illustrate our Theorem 4.

Example 5. . *Let* $X = \boldsymbol{R}$,

$$d(x,y) = \begin{cases} 0 & if \quad x = y \\ 1 & if \quad x \neq y \end{cases}$$

Define $T, f : X \longrightarrow X$ *as* $Tx = 1$ *and* $fx = \frac{x+1}{2}$, $\forall x \in X$.
All conditions of Theorem 4 are satisfied. Clearly 1 is the unique common fixed point of T *and* f.

Corollary 6. *Theorem 4 holds if the inequality (2.2) is replaced by*

$$d(Tx,Ty) \quad < \quad max\{d(fx,fy), d(fx,Tx), d(fy,Ty), d(fx,Ty), d(fy,Tx)\}$$
$$\forall x, y \in X, x \neq y \tag{2.3}$$

Proof. Since $d(fx,Ty) \leq max\{d(fx,fy), d(fy,Ty)\}$ and
$d(fy,Tx) \leq max\{d(fy,fx), d(fx,Tx)\}$ it follows that (2.3) implies that (2.2). □

Corollary 7. *Taking* $f = I$*(Identity map) in Theorem 4 , we obtain Theorem 1 of [1].*
Now we generalize Theorem 4 when T *is a multi-valued map. Let* $C(X)$ *denote the class of all non empty compact subsets of* X*. For* $A, B \in C(X)$*, the Hausdorff metric is defined as*

$$H(A,B) = \max\left\{ \sup_{x\in A} d(x,B), \quad \sup_{y\in B} d(y,A) \right\}$$

where $d(x,A) = inf\{d(x,a) : a \in A\}$.

Definition 8. Let (X,d) be an ultra metric space, $f : X \longrightarrow X$ and $T : X \longrightarrow C(X)$. f and T are said to be coincidentally commuting at $z \in X$ if $fz \in Tz$ implies $fTz \subseteq Tfz$.

Theorem 9. *Let* (X,d) *be a spherically complete ultra metric space. Let* $f : X \longrightarrow X$ *and* $T : X \longrightarrow C(X)$ *be satisfying*

$$Tx \subseteq f(X), \ \forall x \in X, \tag{2.4}$$
$$H(Tx,Ty) < max\{d(fx,fy), d(fx,Tx), d(fy,Ty)\} \forall x, y \in X, x \neq y. \tag{2.5}$$

Then there exists $z \in X$ *such that* $fz \in Tz$.
Further assume that

$$d(fx,fu) \leq H(Tfy,Tu) \forall x, y, u \in X with fx \in Ty \tag{2.6}$$

and

$$f and\ T \text{ are coincidentally commuting at} z. \tag{2.7}$$

Then fz *is the unique common fixed point of* f *and* T.

Proof. Let $B_a = (fa; d(fa,Ta))$ denote the closed sphere centered at fa with the radius $d(fa,Ta)$ and let A be the collection of these spheres for all $a \in X$. Then the relation $B_a \leq B_b$ iff $B_b \subseteq B_a$ is a partial order on A. Let A_1 be a totally ordered sub family of A. Since (X,d) is spherically complete , we have $\bigcap_{B_a \in A_1} B_a = B \neq \phi$.

Let $fb \in B$ and $B_a \in A_1$. Then $fb \in B_a$.
Hence $d(fb,fa) \leq d(fa,Ta) \cdots$ (i)
If $a = b$ then $B_a = B_b$. Assume that $a \neq b$.
Let $x \in B_b$. Then $d(x,fb) \leq d(fb,Tb)$.

Since Ta is compact, there exists $u \in Ta$ such that $d(fa, u) = d(fa, Ta) \cdots$ (ii)
Consider

$$\begin{aligned} d(fb, Tb) &= \inf_{c \in Tb} d(fb, c) \\ &\leq \max\{d(fb, fa), d(fa, u), \inf_{c \in Tb} d(u, c)\} \\ &\leq \max\{d(fa, Ta), d(Ta, Tb)\} \quad from (i) and (ii) \\ &< \max\{d(fa, Ta), d(fb, Tb)\} \quad from (i) and (2.5) \end{aligned}$$

Thus $d(fb, Tb) < d(fa, Ta) \cdots$ (iii)
Now,

$$\begin{aligned} d(x, fa) &\leq \max\{d(x, fb), d(fb, fa)\} \\ &\leq d(fa, Ta) \quad from (i) and (iii) \end{aligned}$$

Thus $x \in B_a$ and $B_b \subseteq B_a$ for any $B_a \in A_1$.Thus B_b is an upper bound in A for the family A_1 and hence by Zorn's Lemma, A has a maximal element,say $B_z, z \in X$.
Suppose $fz \notin Tz$.
Since Tz is compact, there exists $k \in Tz$ such that $d(fz, Tz) = d(fz, k)$. From (2.4), there exists $w \in X$ such that $k = fw$.
Thus $d(fz, Tz) = d(fz, fw) \cdots$ (iv)
Clearly $z \neq w$. Now,

$$\begin{aligned} d(fw, Tw) &\leq H(Tz, Tw) \\ &< \max\{d(fz, fw), d(fz, Tz), d(fw, Tw)\} \\ &= d(fz, fw) \quad from (iv). \end{aligned}$$

Hence $fz \notin B_w$. Thus $B_z \not\subseteq B_w$.
It is a contradiction to the maximality of B_z. Hence $fz \in Tz$.
Further assume (2.6) and (2.7).
Write $fz = p$. Then $p \in Tz$. From (2.6),
$d(p, fp) = d(fz, fp) \leq H(Tfz, Tp) = H(Tp, Tp) = 0$. This implies that $fp = p$. From (2.7), $p = fp \in fTz \subseteq Tfz = Tp$. Thus $fz = p$ is a common fixed point of f and T.
Suppose $q \in X, q \neq p$ is such that $q = fq \in Tq$. From (2.5) and (2.6) we have

$$\begin{aligned} d(p, q) = d(fp, fq) &\leq H(Tfp, Tq) \\ &= H(Tp, Tq) \\ &< \max\{d(fp, fq), d(fp, Tp), d(fq, Tq)\} \\ &= d(p, q). \end{aligned}$$

This implies that $p = q$. Thus $p = fz$ is the unique common fixed point of f and T. □

Remark 10. If $f = I$ (Identity map) then the first part of Theorem 9 is the main theorem of Gajic [2].

Acknowledgement

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Stability Analysis of Predator-Prey Population Model with Time Delay and Constant Rate of Harvesting

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Abstract. This paper studies the effect of time delay and harvesting on the dynamics of the predator - prey model with a time delay in the growth rate of the prey equation. The predator and prey are then harvested with constant rates. The constant rates may drive the model to one, two, or none positive equilibrium points. When there exist two positive equilibrium points, one of them is possibly stable. In the case of the constant rates are quite small and the equilibrium point is not stable, an asymptotically stable limit cycle occurs. The result showed that the time delay can induce instability of the stable equilibrium point, Hopf bifurcation and stability switches.

Key Words: Predator-prey, Limit cycle, Time delay, Harvesting rate, Hopf bifurcation.

1. Introduction

The Lotka-Volterra model is one of the earliest predator-prey models to be based on sound mathematical principles. It forms the basis of many models used today in the analysis of population dynamics and is one of the most popular models in mathematical ecology. In both the analysis and experiment, the predator and prey can coexist by reducing the frequency of contact between them, Luckinbill [13]. In the context of predator-prey interaction, some studies that treat population can be extended by considering harvesting, stocking, diffusion, and time delay. In the model with harvesting, some studies relate the population to the economic problems. The time delay is considered into the population dynamics when the rate of change of the population is not only a function of the present population but also depends on the past population.

One predator-one prey system in Hogart et al. [10] where both the predator and prey are harvested with constant yield has been considered and the stability at maximum sustainable yield is established. Martin and Ruan [14] have analyzed generalized Gause predator-prey models where the prey is harvested with constant rate while Kar [12] considered the predator-prey model with the predator harvested and suggested that it is ideal to study the combined harvesting of predator and prey population models. The effect of constant rate of harvesting has been studied by Holmberg [11] and the results showed that the constant catch quota can lead to both oscillations and chaos and an increased risk for over exploitation. While the effects on population size and yield of different levels of harvesting of a predator in a predator-prey system have been explored by Matsuda and Abrams [15] and showed that the predator may increase in population size with increasing £shing effort.

Brauer and Soudack [3] have analyzed the global behavior of a predator-prey system under constant rate predator harvesting. They showed how to classify the possibilities and determine the region of stability. They found that if the equilibrium point is asymptotically stable, which is determined by a local linearization, then every solution whose initial value is in some neighborhood of the stable equilibrium point tends to it as the time approaches in£nity. There exists an asymptotically stable limit cycle when the constant rate is small and the equilibrium point is unstable. A predator-prey model with Holling type using harvesting efforts as control has been presented by Srinivasu et al. [17] and showed that with harvesting, it is possible to break the cyclic behavior of the system and introduces a globally stable limit cycle in the system.

The effect of constant rate of harvesting on the dynamics of predator-prey systems has been investigated by many authors, see, for example, Brauer and Soudack [2, 4], Myerscough et al. [16], Dai and Tang [7], Xiao and Ruan [18]. Some interesting dynamical behaviors have been observed such as the stability of the equilibria, existence of Hopf bifurcation and limit cycles. It is also observed that in some cases, before a catastrophic harvest rate is reached the effect of harvesting is to stabilize the equilibrium point of the population system. In this paper we present a deterministic and continuous model for predator - prey population based on Lotka - Volterra model which is extended by incorporating time delay and constant rates of harvesting of both populations. The objective of this paper is to study the combined effects of harvesting and time delay on the dynamics of predator-prey model.

2. The Predator - Prey Population Model

We consider a predator - prey model based on Lotka - Volterra model with one predator and one prey populations. The model for the rate of change of prey population (x) and predator population (y) is

$$\begin{aligned} \frac{dx}{dt} &= rx(1-\frac{x}{K}) - \alpha xy \\ \frac{dy}{dt} &= -cy + \beta xy. \end{aligned} \tag{2.1}$$

The model includes parameter K, the carrying capacity, for the prey population in the absence of the predator. The parameter r is the intrinsic growth rate of the prey, c is the mortality rate if the predator without prey, α measures the rate of consumption of prey by the predator, β measures the conversion of prey consumed into the predator reproduction rate. All the parameters are assumed to be positive.

The equilibrium points of model (2.1) are $(0,0)$, $(K,0)$ and $E^* = (x^*, y^*) = (\frac{c}{b}, \frac{r(K\beta-c)}{\alpha\beta K})$. In order to get a positive equilibrium point we assume that $K\beta - c > 0$. The Jacobian matrix of model (2.1) takes the form

$$J = \begin{pmatrix} r - \frac{2rx}{K} - \alpha y & -\alpha x \\ \beta y & -c + bx \end{pmatrix}.$$

The characteristic equation of the Jacobian matrix J at the equilibrium point E^* is $f(\lambda) = \lambda^2 + \frac{cr}{\beta K}\lambda + \frac{cr}{\beta K}(\beta K - c)$ and the eigenvalues have negative real parts. It means that the equilibrium points E^* is locally asymptotically stable. Furthermore, since $K\beta - c > 0$ then the equilibrium point E^* is also globally asymptotically stable, see Ho and Ou [9].

3. The Predator-Prey Model with Time Delay and Constant Rate of Harvesting

We consider the predator and prey populations of model (2.1) where both populations are subjected to a constant rate of harvesting. Before we go to the model with time delay, we need to analyze the stability of the equilibrium point of the model without time delay. The model without time delay is

$$\begin{aligned} \frac{dx}{dt} &= x(r - bx - \alpha y) - H_x \\ \frac{dy}{dt} &= y(-c + \beta x) - H_y, \end{aligned} \tag{3.1}$$

where r, $b = \frac{r}{K}$, α, c, β, H_x, H_y are positive constants. The constants H_x and H_y denote the rate of harvesting for the populations x and y respectively.

By setting $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ then we have the relations

$$x(r - bx - \alpha y) = H_x \tag{3.2}$$

$$y(-c + \beta x) = H_y. \tag{3.3}$$

From (3.2) we have $y = \frac{rx - bx^2 - H_x}{\alpha x}$ which follows that $r^2 - 4bH_x$ should be positive in order to get the equilibrium point in the positive quadrant. Hence we have to assume that $H_x < \frac{r^2}{4b}$. Since H_y is positive, then from (3.3) we should assume that $x > \frac{c}{\beta}$.

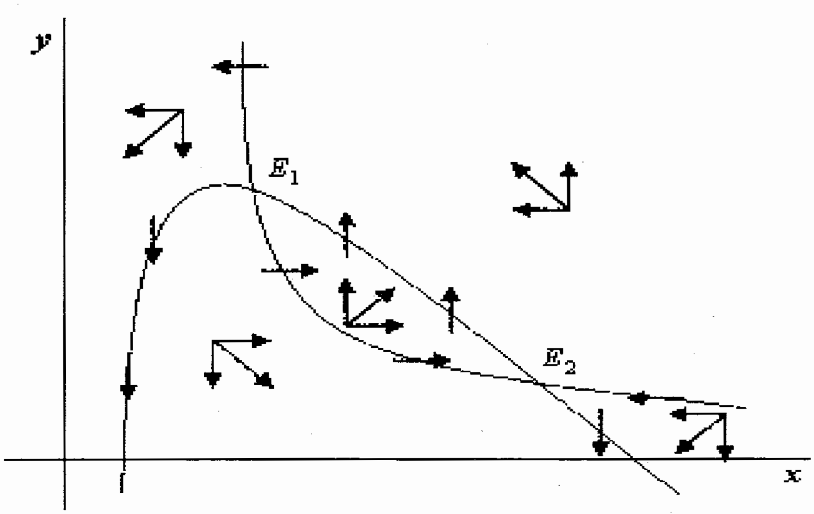

FIGURE 1. Phase plane and directions of the trajectories

From the phase plane, we know that it is possible to get two, one or no equilibrium points, Figure 1. There are two positive equilibrium points of the model when

$b\beta x^3 - (r\beta + bc)x^2 + (H_x\beta + H_y\alpha + rc)x - H_x c < 0$, for some positive $x > \frac{c}{\beta}$. Let the two equilibrium points be $E_1 = (x_1, y_1)$ and $E_2 = (x_2, y_2)$. The equilibrium point E_1 is possible to be asymptotically stable, while the equilibrium point E_2 is not stable, it is a saddle point.

To analyze the stability of the equilibrium point E_1 we linearize the model around the equilibrium point E_1 . The Jacobian matrix of the model is

$$J = \begin{pmatrix} r - 2bx - \alpha y & -\alpha x \\ \beta y & -c + \beta x \end{pmatrix}.$$

The characteristic equation of the Jacobian matrix at this point is

$$\lambda^2 - (P + S)\lambda + PS + QR = 0, \tag{3.4}$$

where

$$\begin{aligned} P &= r - 2bx_1 - \alpha y_1, \\ Q &= \alpha x_1, \\ R &= \beta y_1, \quad \text{and} \\ S &= -c + \beta x_1. \end{aligned}$$

Then the equilibrium point E_1 is asymptotically stable when $PS + QR > 0$ and $P + S < 0$.

Example 1. *Consider model (3.1) with parameters* $r = 1$, $b = 0.01$, $\alpha = 1$, $c = 0.3$, $\beta = 0.05$, $H_x = 0.01$, *and* $H_y = 0.02$. *The equilibrium points of the model in the positive quadrant are* $E_1 = (6.42819, 0.93416)$ *and* $E_2 = (99.56243, 0.00428)$. *The eigenvalues associated with the equilibrium point* E_1 *are* $-0.02066 \pm 0.54633i$ *and the eigenvalues associated with equilibrium point* E_2 *are* -0.99177 *and* 4.67437. *This reveals that the equilibrium point* E_1 *is asymptotically stable while the equilibrium point* E_2 *is a saddle point and unstable.*

Example 2. *Consider again model (3.1) with parameters* $r = 1$, $b = 0.01$, $\alpha = 1$, $c = 0.3$, $\beta = 0.05$, $H_x = 0.1$, *and* $H_y = 0.2$. *There are two equilibrium points of the model in the positive quadrant, they are,* $E_1 = (10.51819, 0.88531)$ *and* $E_2 = (95.42203, 0.04473)$. *The equilibrium point* E_1 *has eigenvalues* $-0.06512 \pm 0.66313i$ *and the equilibrium point* E_2 *has eigenvalues* -0.91354 *and* 4.43147. *This means that both equilibrium points are not stable.*

From Examples 1 and 2 we know that the equilibrium E_1 may be a stable or an unstable equilibrium point. It depends on the values of the parameters and the level of constant rate of harvesting. Apparently, the equilibrium point E_1 tends to the equilibrium point E^* when the harvesting function H_x and H_y approach zero. If the equilibrium point E^* for the non-harvesting model is asymptotically stable, then the eigenvalues of the Jacobian matrix of the linearized system have negative real parts. Since the eigenvalues are continuous in H_x and H_y, the equilibrium point E_1 is asymptotically stable for sufficiently small $H_x > 0$ and $H_y > 0$. On the other hand, when the equilibrium point E_1 is unstable, there exists an asymptotically stable limit cycle. Theory of perturbation of periodic solutions, Coddington and Levinson [5], shows that there is an asymptotically stable limit cycle for small $H_x > 0$ and $H_y > 0$. Thus, the qualitative behavior of the system for $H_x = 0$ and $H_y = 0$ carries over to small $H_x > 0$ and $H_y > 0$.

Now we consider the predator - prey population model with time delay and constant rate of harvesting. Both predator and prey populations are subjected to constant rate of harvesting. The model is

$$\begin{aligned}\frac{dx(t)}{dt} &= rx(t) - bx(t)x(t-\tau) - \alpha x(t)y(t) - H_x,\\ \frac{dy(t)}{dt} &= -cy(t) + \beta x(t)y(t) - H_y.\end{aligned} \tag{3.5}$$

A predator-prey model with time delays in the growth rate of the predator population and the prey harvested with constant rate has been analyzed by Martin and Ruan [14]. They showed that the time delays can induce instability, oscillations via Hopf bifurcation and switching stability.

To linearize the model about the equilibrium point E_1 of model (3.5), let $u(t) = x(t) - x_1$ and $v(t) = y(t) - y_1$. We then obtain the linearized model

$$\begin{aligned}\dot{u}(t) &= (r - bx_1 - \alpha y_1)u(t) - bx_1 u(t-\tau) - \alpha x_1 v(t)\\ \dot{v}(t) &= \beta y_1 u(t) + (-c + \beta x_1)v(t).\end{aligned} \tag{3.6}$$

From the linearized model we obtain the characteristic equation

$$\Delta(\lambda,\tau) = \lambda^2 + a_1\lambda e^{-\lambda\tau} - a_2\lambda - a_3 e^{-\lambda\tau} + a_4, \tag{3.7}$$

where

$$\begin{aligned}a_1 &= bx_1\\ a_2 &= r - c - bx_1 + \beta x_1 - \alpha y_1\\ a_3 &= -bcx_1 + b\beta x_1^2, \quad \text{and}\\ a_4 &= -rc + a\beta x_1 + bcx_1 + b\beta x_1^2 - \alpha c y_1.\end{aligned}$$

For $\tau = 0$, the characteristic equation (3.7) becomes $\lambda^2 + (a_1 - a_2)\lambda - a_3 - a_4 = 0$. This characteristic equation is the same with the characteristic equation (3.7). The eigenvalues of the characteristic equation are either real and negative or complex conjugate with negative real parts if and only if

$$a_1 - a_2 > 0 \quad \text{and} \quad -a_3 + a_4 > 0. \tag{3.8}$$

Hence, in the absence of time delay, the equilibrium point E_1 is locally asymptotically stable if and only if both conditions $a_1 - a_2 > 0$ and $-a_3 + a_4 > 0$ are satis£ed.

Now for $\tau \neq 0$, if $\lambda = i\omega$, $\omega > 0$, is a root for the characteristic equation (3.7), then we have

$$\omega^2 + a_1 i\omega\cos(\omega\tau) + a_1\omega\sin(\omega\tau) - a_2 i\omega - a_3\cos(\omega\tau) + a_3 i\sin(\omega\tau) + a_4 = 0.$$

Separating the real and imaginary parts, we get

$$\begin{aligned}-\omega^2 + a_4 + a_1\omega\sin(\omega\tau) - a_3\cos(\omega\tau) &= 0\\ -a_2\omega + a_1\omega\cos(\omega\tau) + a_3\sin(\omega\tau) &= 0,\end{aligned}$$

or equivalently

$$\begin{aligned}-\omega^2 + a_4 &= -a_1\omega\sin(\omega\tau) + a_3\cos(\omega\tau)\\ a_2\omega &= a_1\omega\cos(\omega\tau) + a_3\sin(\omega\tau).\end{aligned} \tag{3.9}$$

Squaring both sides gives

$$\omega^4 - 2a_4\omega^2 + a_4^2 = a_1^2\omega^2 \sin^2(\omega\tau) - 2a_1a_3\omega \sin(\omega\tau)\cos(\omega\tau) + a_3^2\cos^2(\omega\tau)$$
$$a_2^2\omega^2 = a_1^2\omega^2\cos^2(\omega\tau) + 2a_1a_3\omega\sin(\omega\tau)\cos(\omega\tau) + a_3^2\sin^2(\omega\tau).$$

Adding both equations and regrouping by powers of ω, we obtain the following fourth degree polynomial

$$\omega^4 - (a_1^2 + 2a_4 - a_2^2)\omega^2 + a_4^2 - a_3^2 = 0. \tag{3.10}$$

Then we obtain

$$\omega_\pm^2 = \frac{1}{2}\{(a_1^2 + 2a_4 - a_2^2) \pm \sqrt{(a_1^2 + 2a_4 - a_2^2)^2 - 4(a_4^2 - a_3^2)}\}. \tag{3.11}$$

From the equation (3.11), it follows that if

$$a_2^2 - 2a_4 - a_1^2 > 0 \quad \text{and} \quad a_4^2 - a_3^2 > 0, \tag{3.12}$$

then the equation (3.10) does not have any real solutions.

To £nd the necessary and suf£cient conditions for nonexistence of time delay induced instability, we now use the following theorem.

Theorem 3. *(Kar,* [12]*). A set of necessary and suf£cient conditions for an equilibrium point* (x_*, y_*) *to be asymptotically stable for all* $\tau \geq 0$ *is*

(1) *The real parts of all the roots of* $\Delta(\lambda, 0) = 0$ *are negative,*

(2) *For all real* ω *and* $\tau \geq 0$, $\Delta(i\omega, \tau) \neq 0$, *where* $i = \sqrt{-1}$.

Theorem 4. *If conditions (3.8), (3.12) and Theorem 3 are satis£ed, then the equilibrium point* E_1 *is locally asymptotically stable for all* $\tau \geq 0$.

Again, if

$$\begin{aligned} a_4^2 &- a_3^2 > 0, a_2^2 - 2a_4 - a_1^2 < 0, \quad \text{and} \\ (a_2^2 &- 2a_4 - a_1^2)^2 > 4(a_4^2 - a_3^2), \end{aligned} \tag{3.13}$$

hold, then there are two positive solutions of $\omega_\pm^2$. Substituting $\omega_\pm^2$ into equation (3.9) and solving for τ, we obtain

$$\tau_k^\pm = \frac{1}{\omega_\pm} \arctan\{\frac{\omega_\pm(a_1\omega_\pm^2 - a_1a_4 + a_2a_3)}{a_1a_2\omega_\pm^2 + a_3(a_4 - \omega_\pm^2)}\} + \frac{2k\pi}{\omega_\pm}, k = 0, 1, 2, \cdots. \tag{3.14}$$

Differentiating equation (3.7) with respect to τ, we obtain

$$(2\lambda - a_2 + a_1e^{-\lambda\tau} - \tau(a_1\lambda - a_3e^{-\lambda\tau}))\tfrac{d\lambda}{d\tau} = \lambda(a_1\lambda - a_3)e^{-\lambda\tau},$$

therefore

$$(\frac{d\lambda}{d\tau})^{-1} = \frac{2\lambda - a_2}{\lambda(a_1\lambda - a_3)e^{-\lambda\tau}} + \frac{a_1}{\lambda(a_1\lambda - a_3)} - \frac{\tau}{\lambda}.$$

From equation (3.7), we have $e^{-\lambda\tau} = \frac{-(\lambda^2 - a_2\lambda + a_4)}{(a_1\lambda - a_3)}$. Then we obtain

$$(\frac{d\lambda}{d\tau})^{-1} = \frac{2\lambda - a_2}{-\lambda(\lambda^2 - a_2\lambda - a_4)} + \frac{a_1}{\lambda(a_1\lambda - a_3)} - \frac{\tau}{\lambda}.$$

$$(\frac{d\lambda}{d\tau})^{-1} = \frac{2\lambda - a_2}{-\lambda(\lambda^2 - a_2\lambda - a_4)} + \frac{a_1}{\lambda(a_1\lambda - a_3)} - \frac{\tau}{\lambda}.$$

Thus,

$$\begin{aligned}
\text{sign}\frac{d(\text{Re}\lambda)}{d\tau}_{\lambda=i\omega} &= \text{sign}\{\text{Re}(\frac{d\lambda}{d\tau})^{-1}\}_{\lambda=i\omega} \\
&= \text{sign}\{\text{Re}(\frac{2\lambda - a_2}{-\lambda(\lambda^2 - a_2\lambda + a_4)})_{\lambda=i\omega} + \text{Re}(\frac{a_1}{\lambda(a_1\lambda - a_3)})_{\lambda=i\omega} \\
&+ \text{Re}(\frac{-\tau}{\lambda})_{\lambda=i\omega}\} \\
&= \text{sign}\{\text{Re}(\frac{2i\omega - a_2}{-i\omega(-\omega^2 - a_2 i\omega + a_4)}) + \text{Re}(\frac{a_1}{i\omega(a_1 i\omega - a_3)}) \\
&+ \text{Re}(\frac{-\tau}{i\omega})\} \\
&= \text{sign}\{\frac{a_2^2 + 2\omega^2 - 2a_4}{a_2^2\omega^2 + (-\omega^2 + a_4)^2} - \frac{a_1^2}{a_1^2\omega^2 + a_3^2}\}.
\end{aligned}$$

From equation (3.10), we know that

$$a_1^2\omega^2 + a_3^2 = \omega^4 + (a_2^2 - 2a_4)\omega^2 + a_4^2 = a_2^2\omega^2 + (-\omega^2 + a_4)^2,$$

then we obtain

$$\begin{aligned}
\text{sign}\{\frac{d(\text{Re}\lambda)}{d\tau}\}_{\lambda=i\omega} &= \text{sign}\{\frac{a_2^2 + 2\omega^2 - 2a_4}{a_2^2\omega^2 + (-\omega^2 + a_4)^2} - \frac{a_1^2}{a_2^2\omega^2 + (-\omega^2 + a_4)^2}\} \\
&= \text{sign}\{2\omega^2 - (a_1^2 + 2a_4 - a_2^2)\}. \qquad (3.15)
\end{aligned}$$

Theorem 5. *Let $\tau_k^\pm$ be de£ned by equation (3.14). If the conditions (3.7) and (3.13) are satis£ed, then the equilibrium point E_1 is stable when $\tau \in [0, \tau_0^+) \bigcup (\tau_0^-, \tau_1^+) \bigcup \cdots \bigcup (\tau_{m-1}^-, \tau_m^+)$ and unstable when $\tau \in [\tau_0^+, \tau_0^-) \bigcup (\tau_1^+, \tau_1^-) \bigcup \cdots \bigcup (\tau_{m-1}^+, \tau_{m-1}^-)$, for some positive integer m. Therefore there are bifurcations at the equilibrium point E_1 when $\tau = \tau_k^\pm$, $k = 0, 1, 2, \cdots$.*

Proof. Since the conditions (3.8) and (3.13) are satis£ed, then to prove the theorem we need only to verify the transversality conditions, see Cushing [6],

$$\frac{d(\text{Re}\lambda)}{d\tau}|_{\tau=\tau_k^+} > 0 \quad \text{and} \quad \frac{d(\text{Re}\lambda)}{d\tau}|_{\tau=\tau_k^-} < 0,$$

$$\frac{d(\text{Re}\lambda)}{d\tau}|_{\tau=i\tau_k^+} > 0 \text{ and } \frac{d(\text{Re}\lambda)}{d\tau}|_{\tau=i\tau_k^-} < 0.$$

From (3.15) and (3.11), it follows that

$$\begin{aligned}\text{sign}\{\frac{d(\text{Re}\lambda)}{d\tau}\}_{\lambda=i\omega_+} &= \text{sign}\{2\omega_+^2 - (a_1^2 + 2a_4 - a_2^2)\}\\ &= \text{sign}\{\sqrt{(a_1^2 + 2a_4 - a_2^2)^2 - 4(a_4^2 - a3^2)}\},\end{aligned}$$

therefore,

$$\frac{d(\text{Re}\lambda)}{d\tau}|_{\omega=\omega_+,\tau=\tau_k^+} > 0,$$

$$\frac{d(\text{Re}\lambda)}{d\tau}|_{\omega=\omega_+,\tau=\tau_k^+} > 0.$$

Again,

$$\begin{aligned}\text{sign}\{\frac{d(\text{Re}\lambda)}{d\tau}\}_{\lambda=i\omega_-} &= \text{sign}\{2\omega_-^2 - (a_1^2 + 2a_4 - a_2^2)\}\\ &= \text{sign}\{-\sqrt{(a_1^2 + 2a_4 - a_2^2)^2 - 4(a_4^2 - a3^2)}\},\end{aligned}$$

therefore,

$$\frac{d(\text{Re}\lambda)}{d\tau}|_{\omega=\omega_-,\tau=\tau_k^-} < 0.$$

Hence, the transversality conditions are satis£ed. This completes the proof. □

Example 6. *Consider model (3.5) with parameters* $r = 3.5$, $b = 0.04$, $\alpha = 1$, $c = 0.3$, $\beta = 0.05$, $H_x = 0.02$, *and* $H_y = 0.01$. *The equilibrium point of the model is* $E_1 = (6.06146, 3.25424)$. *For* $\tau = 0$, *the Jacobian matrix of the model associated with the equilibrium point has eigenvalues* $-0.11804 \pm 0.98570i$. *This means that the equilibrium point of the model without time delay is stable. The conditions (3.8) and (3.13) are satis£ed. Some trajectories of* $x(t)$ *and* $y(t)$ *with various time delays are given in Figures 2, 3, and 4.*

From Figures 2a and 2b with time delay $\tau = 1.2$, the equilibrium point $(6.06146, 3.25424)$ is stable. Figures 3a and 3b with time delay $\tau = 1.53$ show that the equilibrium point $(6.06146, 3.25424)$ is unstable. The £rst critical value of time delay is $\tau = \tau_0^+ = 1.37941$. When $\tau < 1.37941$, the equilibrium point $(6.06146, 3.25424)$ is asymptotically stable; when $\tau = 1.37941$ the equilibrium point $(6.06146, 3.25424)$ loses its stability; and when $\tau > 1.37941$ but less then the second critical value of time delay, the equilibrium point $(6.06146, 3.25424)$ becomes unstable and there is a bifurcating periodic solution, see Figure 4. Following Theorem 5 we have

$$\begin{aligned}\tau_0^+ &= 1.37941, & \tau_0^- &= 5.39314,\\ \tau_1^+ &= 6.98104, & \tau_1^- &= 12.53884,\\ \tau_2^+ &= 12.58266, & \tau_2^- &= 19.68453,\\ \tau_3^+ &= 18.68453, \quad \text{and} & \tau_3^- &= 26.83023.\end{aligned}$$

Then we have 2 stability switches from stability to instability and to stability.

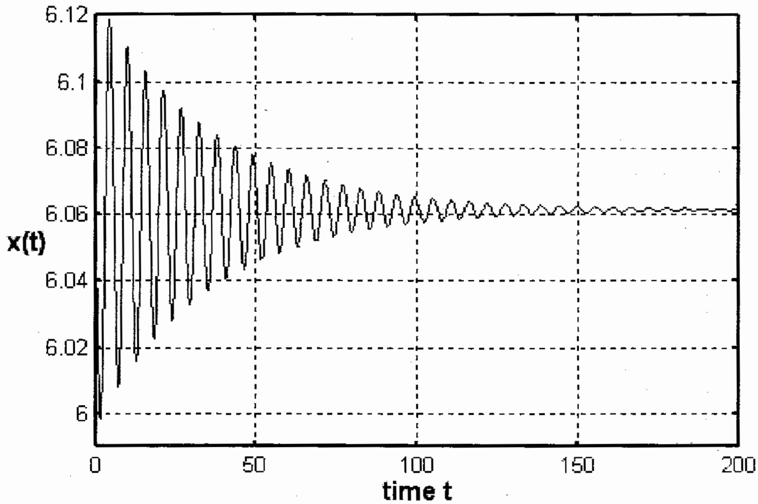

FIGURE 2a. Trajectory of prey with $x(0) = 6.0715$ and $\tau = 1.2$

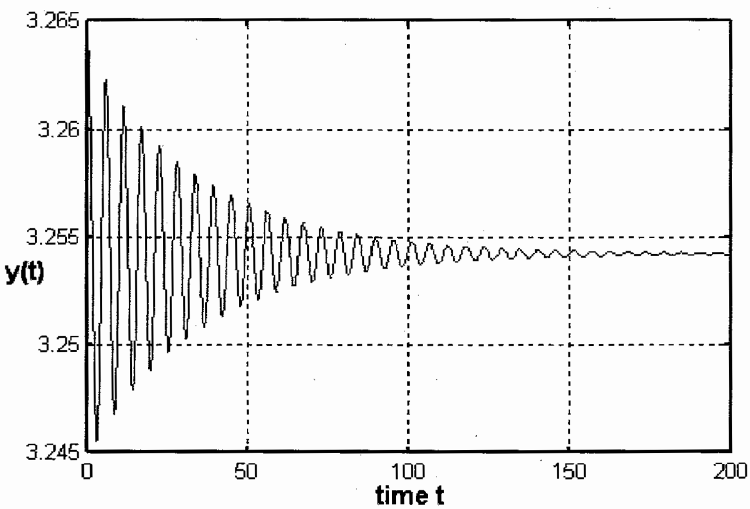

FIGURE 2b. Trajectory of predator with $y(0) = 3.2642$ and $\tau = 1.2$

4. Discussion

In the analysis of the positive equilibrium point of model (3.1), it is quite diff£cult to determine the value of the equilibrium points analytically. We just state that there exists either one, or two, or none positive equilibrium points by inspection the phase plane of the model. In the case of two positive equilibrium points occur, one of the equilibrium point is possibly stable and the other is a saddle point. In this paper, we just analyze the case of there exist two positive equilibrium points and focus on analyzing the effect of the time delay on the stable equilibrium point. Actually we may also try to analyze the effect of the time delay on the stability of the unstable equilibrium point.

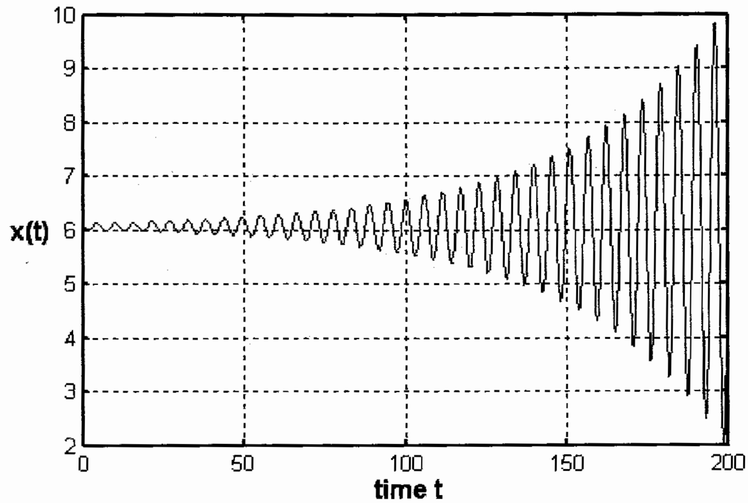

FIGURE 3a. Trajectory of prey with $x(0) = 6.0715$ and $\tau = 1.53$

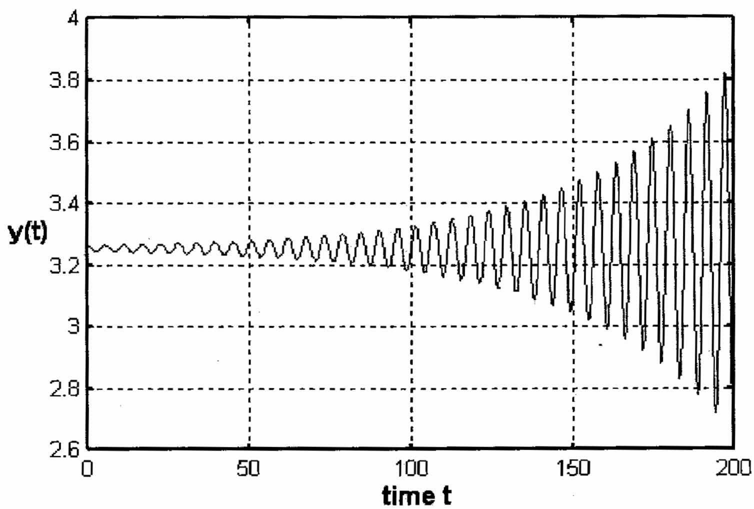

FIGURE 3b. Trajectory of predator with $y(0) = 3.2642$ and $\tau = 1.53$

There is still a lot of work to do in the predator-prey models with time delay and harvesting. For example, it would be interesting to consider time delay and harvesting in generalized Gause-type predator-prey model and in some another generalized predator-prey models as in Martin and Ruan [14]. It would also be interesting to study the Wangersky-Cunningham model with some delays in both the predator and prey model as in the Bartlett's model, see Bartlett [1] and Hasting [8].

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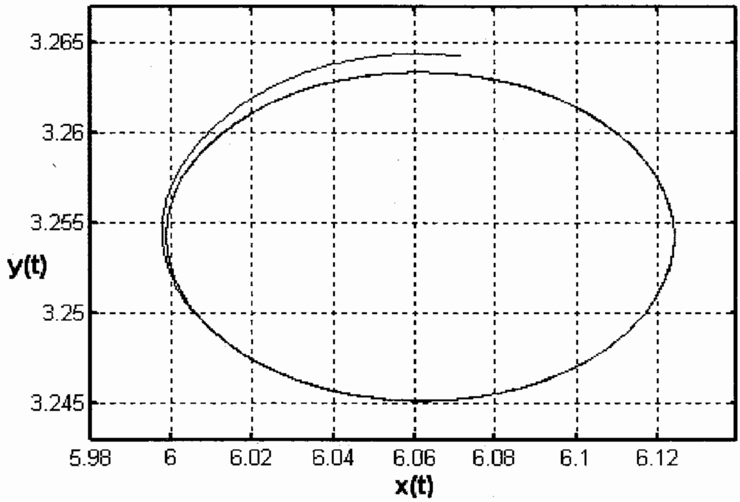

FIGURE 4. Trajectory of $(x(t), y(t))$, with $x(0) = 6.0715, y(0) = 3.2642$ and $\tau = 1.37941$

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On Products of Ordered Normed Spaces

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Abstract. In this paper certain order properties are investigated which the products of ordered linear spaces and ordered normed spaces inherit from their component spaces.

1. Introduction

F. Riesz, H. Freudenthal, L.V. Kantorovitch, Kakutani and others initiated the study of ordered linear spaces in the late 1930's. The theory developed into a mathematical discipline around 1950's. It is now one of the most important branches of functional analysis being effectively used to solve such problems, which are posed in more general setting. Several authors have studied regarding products of ordered linear spaces. Bonsall [5], Peressini [12], Jameson [9], Dalen [7], Cristescu [6], Martin [11], Karim [10] and Hong [8] have significant contributions in this area. The authors investigated [1] the inheritance of order properties from ordered product spaces to their component spaces through projections. In [2] relatively uniform convergence, order convergence etc., in product spaces of ordered linear spaces have been discussed. Order-completeness and order-separability in product spaces of ordered linear spaces were studied in [3].

A little work is done with reference to the order properties of ordered normed spaces. The first author attempted to investigate the mutual relationship of ordered product normed spaces and their component spaces in [4]. In section 3 of this article we mainly discuss how a base for the wedge W of a product ordered linear space can be investigated. Section 4 includes some order properties, which are closed under the formation of products of ordered normed spaces.

2. Preliminaries

A wedge is a non-empty subset W of a real linear space X such that

(W_1) $\quad W + W \subseteq W,$

(W_2) $\quad \alpha W \subseteq W$ for $\alpha \geq 0,$

The wedge W in X defines an ordering or preorder relation (a reflexive and transitive relation) " $\leq$ " on X by

$$x \leq y \Leftrightarrow y - x \in W$$

which is compatible with the linear structure of X , that is, " $\leq$ " satisfies the conditions:

(O_1) $x, y \in X, x \leq y \Rightarrow x + z \leq y + z$ for all $z \in X$

(O_2) $x, y \in X, x \leq y, \alpha \geq 0 \Rightarrow \alpha x \leq \alpha y$

A wedge W in X is said to be a cone if $W \cap (-W) = 0$ i.e., $x, -x \in W \Rightarrow x = 0$. A cone C in X defines a partial order relation " $\leq$ " on X. If the partial order (resp: ordering) on the real linear space X is due to a cone C (resp: a wedge W) then we call X an ordered (resp: a pre-ordered) linear space with cone C(resp: wedge W). The element x of an ordered linear space $(X, \leq)$ is said to be a positive element if $x \geq 0$ and the set $X_+ = \{x \in X : x \geq 0\}$is referred to as a positive cone.

3. Order Properties of Product Spaces of Ordered Linear Spaces

It is well known that if W_α (resp: C_α) is a wedge (resp: cone) in the real linear space X_α where $\alpha \in I$ then $W = \prod_{\alpha \in I} W_\alpha$ (resp: $C = \prod_{\alpha \in I} C_\alpha$) is a wedge (resp: cone) in the product linear space $X = \prod_{\alpha \in I} X_\alpha$. Thus the product linear space X is a preordered (resp: ordered) linear space for the ordering (resp: order) generated by the wedge W (resp: cone C).

Using the same order notation in each space, the ordering (resp: partial order) associated with the wedge W (resp: the cone C) is given by

$$(x_1, x_2, ..., x_n, ...) \leq (y_1, y_2, ..., y_n, ...) \Leftrightarrow x_\alpha \leq y_\alpha \quad forall \quad \alpha \in I.$$

Theorem 1. *Let $X_1, X_2, ..., X_n$ be preordered linear spaces with wedges $W_1, W_2, ..., W_n$ respectively. Let $X = \prod_{i=1}^n X_\alpha$ and $W = \prod_{i=1}^n W_\alpha$, then*

(1) $W - W = X \Leftrightarrow W_\alpha - W_\alpha = X_\alpha, \alpha = 1, 2, ..., n.$

(2) *$(e_1, e_2, ..., e_n)$ is an order unit in $X \Leftrightarrow e_\alpha$ is an order unit in X_α for every $\alpha = 1, 2, ..., n$.*

(3) *W is Archimedean $\Leftrightarrow$ each W_α $(\alpha = 1, 2, ..., n)$ is.*

Definition 2. Let X be a preordered linear space with wedge W. A base for the wedge W is a convex subset B such that for each x in $W - \{0\}$, there exists $\lambda > 0$ and $b \in B$ such that the representation $x = \lambda b$ is unique.

Let X and Y be preordered linear spaces with wedges W and W' respectively. Given the bases B of W and B' of W' , what the basis is of $W \times W'$. A natural suggestion would be to try $B_o = B \times B'$. Then given $(x, y) \in X \times Y$ there are unique $\lambda, \sigma > 0$ and $b \in B, b' \in B'$ such that $x = \lambda b$ and $y = \sigma b'$ so that $(x, y) = (\lambda b, \sigma b')$. Obviously, this construction does not give us a base. For example, consider R^2 with the usual positive cone. A base B is the line segment joining the points $(1, 0)$ and $(0, 1)$: a typical element has coordinates $(x, 1 - x)$ with $0 < x < 1$. A base B' for R is the point 1, so that $B \times B' = \{(x, 1 - x, 1) : 0 < x < 1\}$ which cannot be a base for a cone in R^3. On the other hand convex cover of the three points $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$ is a base for the usual positive cone of R^3. This leads us to :

Theorem 3. *Let X and Y be preordered linear spaces with wedges W and W' respectively and B , B' be the bases for W and W' respectively. If $0 \leq \lambda \leq 1$, then*

$$B_o = \{(\lambda b, (1-\lambda)b') : b \in B, b' \in B'\}$$

is a base for $W \times W'$.

Proof. Let $A = (x, y) \in W \times W'$ then $x \in W$ and $y \in W'$. By the definition of B and B' there are unique $k, h > 0$ and $b \in B, b' \in B'$ such that $x = kb$ and $y = hb'$. Put $\lambda = \frac{k}{k+h}$ so that $1 - \lambda = \frac{h}{k+h}$. Thus

$$b_o = (\lambda b, (1-\lambda)b') \in B_o \text{ and } (x, y) = (kb, hb').$$

Now

$$(x, y) = (k+h)(\tfrac{k}{k+h}b, \tfrac{h}{k+h}b') = (k+h)(\lambda b, (1-\lambda)b') = (k+h)b_o$$

By construction b_o and $k + h$ are uniquely determined from (x, y). Hence B_o is a base for $W \times W'$. □

Definition 4. Bonsall [5] defines a perfect subspace as a subspace E of an ordered linear space X with an order-unit e, which satisfies the following condition:

"given x in E and $\varepsilon > 0$, there exists y in E such that $y + \varepsilon e \geq x$ and $y + \varepsilon e \geq 0$"

Theorem 5. *Let* E_i $(i = 1, 2, ..., n)$ *be perfect subspace of ordered linear space* X_i *with order-unit* e_i. *If* $E = \prod_{i=1}^{n} E_i$ *and* $X = \prod_{i=1}^{n} X_i$, *then* E *is a perfect subspace of* X.

Proof. Since $e = (e_1, e_2, ..., e_n)$ is an order-unit in X, therefore by Theorem 1(2), the result follows. □

4. Order Properties of Product Spaces of Ordered Normed Spaces

If $X_i (i = 1, 2, ..., n)$ is a normed linear space then $X = \prod_{i=1}^{n} X_i$ is also a normed linear space with norm defined by

$$\|(x_1, x_2, ..., x_n)\| = \|x_1\| + \|x_2\| + ... + \|x_n\|$$

OR

$$\|(x_1, x_2, ..., x_n)\| = \max\{\|x_1\|, \|x_2\|, ..., \|x_n\|\} \text{ or } \|x_1\| \vee \|x_2\| \vee ... \vee \|x_n\|$$

The two norms are equivalent and the choice as to which of the two is to be used depends on the context.

In [4] the first author has studied inheritance of certain order properties from product ordered normed spaces to their component spaces. In this section some order properties have been studied which the product normed spaces inherit from their component spaces.

Definition 6. A preordered normed linear space X is said to have the property (R_1) if given $x, y \in X$, $\|x\| \leq \|y\|$ whenever $-y \leq x \leq y$.

Theorem 7. *Let* $X_1, X_2, ..., X_n$ *be preordered normed linear spaces with wedges* $W_1, W_2, ..., W_n$ *respectively. Let* $X = \prod_{\alpha=1}^{n} X_\alpha$ *and* $W = \prod_{\alpha=1}^{n} W_\alpha$, *then*

(1) X *has the property: "Given* $x \in X$ *with* $\|x\| \leq 1$, *there is* $y \geq x, -x$ *with* $\|y\| \leq \alpha$" *if each* X_i *has this property.*
(2) X *has property* (R_1) *if each* X_i *has property* (R_1);

Proof. (1) If $x = (x_1, x_2, ..., x_n) \in X$ with $\|x\| \leq 1$, then for every x_i where $i = 1, 2, 3, ..., n$, $\|x_i\| \leq \max\{\|x_1\|, \|x_2\|, ..., \|x_n\|\} = \|x\| \leq 1$. Therefore, by hypothesis there exists y_i in X_i with $y_i \geq x_i, -x_i$ and $\|y_i\| \leq \alpha$. Taking $y = (y_1, y_2, ..., y_n)$, we get
$y \geq x, -x$ and $\|y\| = \max\{\|y_1\|, \|y_2\|, ..., \|y_n\|\} \leq \alpha$.

(2) Let $x, y \in X$ be such that $-y \leq x \leq y$. If $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ then $-y_i \leq x_i \leq y_i$ where $i = 1, 2, 3, ..., n$. Since each X_i has property (R_1), therefore, $\|x_i\| \leq \|y_i\|$ for every $i = 1, 2, 3, , n$. Now, $\|x\| = \|x_1\| + \|x_2\| + ... + \|x_n\| \leq \|y_1\| + \|y_2\| + ... + \|y_n\| = \|y\|$.

□

Corollary 8. (1) *Part (1) implies that X has the property: "Given $x \in X$ with $\|x\| \leq 1$, there is $y \geq x, 0$ with $\|y\| \leq \alpha$" if each X_i has this property.*
(2) *Part (2) implies that norm on X is monotonic if norm on each X_i is monotonic;*

Definition 9. A subset D of a preordered linear space $(X, \leq)$ is said to be directed if for every piar of elements x, y from D there exist elements u, v in D such that $u \geq x, y$ and $v \leq x, y$.

Theorem 10. *Let X_i be a preordered normed linar space with wedge $W_i, i = 1, 2, ..., n$. Let $X = \prod_{i=1}^{n} X_i$ and $W = \prod_{i=1}^{n} W_i$, then*
(1) *the open(closed) unit ball in X is directed if open(closed) unit ball in each X_i is directed;*
(2) *norm is additive on W if norm is additive on each W_i.*

Proof. (1) We show that open unit ball in X is directed whenever the open unit ball in each of its component spaces is directed. The case for the closed unit ball is similar. Let $x, y \in X$ with $\|x\| < 1, \|y\| < 1$. If $x = (x_1, x_2, ..., x_n)$ then for every $i = 1, 2, 3, ..., n$, $\|x_i\| \leq \max\{\|x_1\|, \|x_2\|, ..., \|x_n\|\} = \|x\| < 1$. Similarly, if $y = (y_1, y_2, ..., y_n)$ then $\|y_i\| < 1$. Since unit ball in each X_i is directed therefore for each x_i, y_i of the unit ball in X_i there exist u_i and v_i in X_i with $\|u_i\| < 1$ and $\|v_i\| < 1$ such that $u_i \geq x_i, y_i$ and $v_i \leq x_i, y_i$. Taking $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$ we get $\|u\| < 1$ and $\|v\| < 1$ such that $u \geq x, y$ and $v \leq x, y$.

(2) Let $x, y \in W$. If $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ then $x_i, y_i \in W_i$ where $i = 1, 2, 3, ..., n$. Since norm on each W_i is additive, therefore, for every $i = 1, 2, 3, ..., n$, $\|x_i + y_i\| = \|x_i\| + \|y_i\|$.

Now

$$\begin{aligned} \|x + y\| &= \|(x_1 + y_1, x_2 + y_2, ..., x_n + y_n)\| \\ &= \|x_1 + y_1\| + \|x_2 + y_2\| + ... + \|x_n + y_n\| \\ &= \|x\| + \|y\| \end{aligned}$$

which shows that the norm is additive on W.

□

Definition 11. Let $(X, \|.\|)$ be an ordered normed space with cone C.

(1) The cone C is said to be α -normal if $\|u\|,\|v\| \leq 1$ and $u \leq x \leq v$ imply $\|x\| \leq \alpha$. In other words, the cone C is said to be α-normal if $u \leq x \leq v$ implies that $\|x\| \leq \alpha \ \max\{\|u\|,\|v\|\}$.
(2) The cone C is said to be α -generating if given $x \in X$ there are $u, v \in W$ such that $x = u - v$ with $\|u\| + \|v\| \leq \alpha\|x\|$.
(3) X is said to be (α, n)-generating if given $x_1, x_2, ..., x_n \in X$ with $\|x_j\| \leq 1$ $(j = 1, 2, 3, ..., n)$ we have $x \geq x_j$ such that $\|x\| \leq \alpha$.
(4) X is said to be (α, n)-additive if given $x_1, x_2, ..., x_n \in X$ we have

$$\sum_{j=1}^{n} \|x_j\| \leq \alpha \left\| \sum_{j=1}^{n} x_j \right\|$$

Theorem 12. *Let $X_1, X_2, ..., X_n$ be ordered normed linear spaces with cones $C_1, C_2, ..., C_n$ respectively. Let $X = \prod_{i=1}^{n} X_i$ and $C = \prod_{i=1}^{n} C_i$, then*
(1) *C is α -normal if each C_i is α -normal,*
(2) *C is α -generating if each C_i is α -generating*
(3) *X is (α, n)-generating if each X_i is (α, n)-generating,*
(4) *X is (α, n)-additive if each X_i is (α, n)-additive.*

Proof. (1) Let $x, y, z \in X$ be such that $y \leq x \leq z$. If $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n)$ and $z = (z_1, z_2, ..., z_n)$ then $y_i \leq x_i \leq z_i$ where $i = 1, 2, 3, ..., n$. Since each C_i is α -normal, so $\|x_i\| \leq \alpha \max\{\|y_i\|, \|z_i\|\}$. Now,

$$\begin{aligned} \|x\| &= \|x_1\| \vee \|x_2\| \vee ... \vee \|x_n\| \\ &\leq \alpha\ (\|y_1\| \vee \|z_1\|) \vee \alpha\ (\|y_2\| \vee \|z_2\|) \vee ... \vee \alpha\ (\|y_n\| \vee \|z_n\|) \\ &= \alpha\ [\ \max\ \{\|y_1\|, \|y_2\|, ..., \|y_n\|\} \vee\ \max\ \{\|z_1\|, \|z_2\|, ..., \|z_n\|\}] \\ &= \alpha\ \max\ \{\|y\|, \|z\|\} \end{aligned}$$

which shows that the cone C is α -normal.

(2) Let $x = (x_1, x_2, ..., x_n)$ be an element of X. Since each C_i is α -generating, therefore for x_i in $X_i (i = 1, 2, 3, ..., n)$ there are u_i, v_i in C_i such that $x_i = u_i - v_i$ and $\|u_i\| + \|v_i\| \leq \alpha\ \|x_i\|$. Taking $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$ then $u, v \in C$ for which $x = u - v$. Further,

$$\|u\| + \|v\| = \sum_{i=1}^{n} \|u_i\| + \sum_{i=1}^{n} \|v_i\| = \sum_{i=1}^{n} (\|u_i\| + \|v_i\|) \leq \alpha\ \sum_{i=1}^{n} \|x_i\| = \alpha\ \|x\|$$

which shows that the cone C is α -generating.

(3) Let $x_1, x_2, ..., x_n \in X$ with $\|x_j\| \leq 1$ where $j = 1, 2, 3, ..., n$. Then $x_j = (\varsigma_1^j, \varsigma_2^j, ..., \varsigma_n^j)$ where each $\varsigma_i^j (i = 1, 2, 3, ..., n)$ is an element of X_i. Since for each fixed $j = 1, 2, 3, ..., n$, $\|x_j\| = \max_{1 \leq i \leq n} \|\varsigma_i^j\| \leq 1$, therefore, $\|\varsigma_i^j\| \leq 1$ for every $i = 1, 2, 3, ..., n$. Also, since each X_i is (α, n)-generating therefore for each fixed $i = 1, 2, 3, ..., n$ there exists η_i in X_i such that $\eta_i \geq \varsigma_i^j$ for all $j = 1, 2, 3, ..., n$, and $\|\eta_i\| \leq \alpha$. Taking, $x = (\eta_1, \eta_2, ..., \eta_n)$ we have, $x \geq x_j$ for all $j = 1, 2, 3, ..., n$.
Furthermore,

$$\|x\| = \max_{1 \leq i \leq n} \|\eta_i\| \leq \alpha.$$

showing that X is (α, n)-generating.

(4) Let $x_1, x_2, ..., x_n \in X$. Then for each $j = 1, 2, 3, ..., n$, $x_j = (\varsigma_1^j, \varsigma_2^j, ..., \varsigma_n^j)$, where $\varsigma_i^j \in X_i (i = 1, 2, 3, ..., n)$. Since each X_i is (α, n)-additive, therefore, for each fixed $i = 1, 2, 3, ..., n$,

$$\sum_{j=1}^n \|\varsigma_i^j\| \leq \alpha \| \sum_{j=1}^n \varsigma_i^j \|$$

Now,

$$\sum_{j=1}^n \|x_j\| = \sum_{j=1}^n (\sum_{i=1}^n \left\|\varsigma_i^j\right\|) = \sum_{i=1}^n (\sum_{j=1}^n \left\|\varsigma_i^j\right\|) \leq \alpha \sum_{i=1}^n \left\|\sum_{j=1}^n \varsigma_i^j\right\| = \alpha \left\|\sum_{j=1}^n x_j\right\|$$

which shows that X is (α, n)-additive.

□

Definition 13. Let $(X, \|.\|)$ be a normed linear space;

(1) A wedge W in X is said to have the property (G) with constant α if, given x in X there exists y in X such that $-y \leq x \leq y$ and $\|y\| \leq \alpha\|x\|$.
(2) A wedge W in X is said to have the property (N) with constant α if, $-x \leq y \leq x$ implies that $\|y\| \leq \alpha\|x\|$.

Theorem 14. *Let X_i be a preordered normed space with wedge $W_i, i = 1, 2, ..., n$. Let $X = \prod_{i=1}^n X_i$ and $W = \prod_{i=1}^n W_i$, then*

(1) *the wedge W has the property (N) with some constant α if each W_i has the property (N) with constant α_i.*
(2) *the wedge W has the property (G) with some constant α if each W_i has the property (G) with constant α_i.*

Proof. (1) Let $x, y \in X$ be such that $-x \leq y \leq x$. If $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ where $x_i, y_i \in X_i$ then $-x_i \leq y_i \leq x_i$, for every $i = 1, 2, ..., n$. Since each X_i has property (N) with constant α_i, therefore $\|y_i\| \leq \alpha_i\|xi\|$. Now since,
$\|y_1\| + \|y_2\| + ... + \|y_n\| \leq \alpha_1\|x_1\| + \alpha_2\|x_2\| + ... + \alpha_n\|x_n\|$
therefore, $\|y\| \leq \alpha\|x\|$ where $\alpha = \max\{\alpha_1, \alpha_2, ..., \alpha_n\}$.

(2) Let $x \in X$ then $x = (x_1, x_2, ..., x_n)$ where $x_i \in X_i$ for every $i = 1, 2, ..., n$. Since each X_i has property (G) with constant α_i, therefore, there exist y_i in X_i such that $-y_i \leq x_i \leq y_i$, and $\|y_i\| \leq \alpha_i\|x_i\|$.

Taking $\alpha = \max\{\alpha_1, \alpha_2, ..., \alpha_n\}$, we obtain a point $y = (y_1, y_2, ..., y_n)$ in X such that , $-x \leq y \leq x$, and $\|y\| \leq \alpha\|x\|$.

□

Definition 15. Let X be a preordered linear space with wedge W. A subset D of X is said to be *decomposable* if for each u in D there exist u_1, u_2 in $D \cap W$ such that $u = \alpha_1 u_1 - \alpha_2 u_2$ for $\alpha_1, \alpha_2 \geq 0$ with $\alpha_1 + \alpha_2 = 1$. A wedge W in a normed linear space $(X, \|.\|)$ gives an open *decomposition* of X if given x in X there exist, $\alpha > 0$ and u_1, u_2 in W such that $x = u_1 - u_2$ and $\|u_1\|, \|u_2\| \leq \alpha\|x\|$.

A wedge W in a normed linear space $(X, \|.\|)$ gives a *bounded decomposition* of X if it gives an open decomposition of X [9].

Theorem 16. *If the wedge W_i in a preordered normed space X_i gives a bounded decomposition of X_i for every $i = 1, 2, ..., n$, then the wedge $W = \prod_{i=1}^{n} W_i$ in $X = \prod_{i=1}^{n} X_i$ gives a bounded decomposition of X.*

Proof. Let $x \in X$ then $x = (x_1, x_2, ..., x_n)$ where $x_i \in X_i$ for every $i = 1, 2, ..., n$. Since each W_i gives a bounded decomposition of X_i , therefore, there exist $\alpha > 0$ and $w_i^{(1)}, w_i^{(2)}$ in W_i such that $x_i = w_i^{(1)} - w_i^{(2)}$ and $\|w_i^{(1)}\|, \|w_i^{(2)}\| \leq \alpha_i \|x_i\|$. Let $w_1 = (w_1^{(1)}, w_2^{(1)}, ..., w_n^{(1)})$ and $w_2 = (w_1^{(2)}, w_2^{(2)}, ..., w_n^{(2)})$ then $w_1, w_2 \in W$ for which $x = w_1 - w_2$ and $\|w_1\|, \|w_2\| \leq \alpha \|x\|$ where $\alpha = \max\{\alpha_1, \alpha_2, ..., \alpha_n\}$. □

Conclusion

Order properties are of much importance when studied with reference to the context. For example, first duality theory in ordered linear spaces is due to α - normal and α-generating wedges whereas second duality theory, which is concerned with order-intervals of the form $[-x, x]$, is due to order property (G) and order property (N) of wedges. A rich theory of ordered linear spaces grows through these order properties. This article discusses various order properties of ordered normed spaces, which are closed under the formation of products. We also investigate how product of two base-normed spaces would be a base-normed space.

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Maximum Principles For Parabolic Systems

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Abstract. In this paper we introduce a strong maximum principle for some nonlinear parabolic systems with convex invariant regions. We also obtain a version of the Hopf boundary lemma for the systems.

AMS (MOS) Subject Classification Codes: 35B50, 35K40.
Key Words: Maximum Principles, Parabolic Systems.

1. Introduction

Consider the parabolic system

$$\frac{\partial u}{\partial t} - A(x,t,u)\sum_{i,j=1}^{n} a_{ij}(x,t,u)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i(x,t,u)\frac{\partial u}{\partial x_i} = f(x,t,u) \qquad (1.1)$$

on $D \times (0,T)$,where $u = \begin{bmatrix} u_1 \\ . \\ . \\ . \\ u_m \end{bmatrix}$, D is a domain in $\mathbb{R}^n$, $A(x,t,u)$, and $B_i\ (x,t,u)$ $(i = 1,2,...,n)$ are $m \times m$ matrix- valued functions on $D \times (0 \times T) \times \mathbb{R}^m$, $a_{ij}(x,t,u)$ $(i,j = 1,...,n)$ are real valued functions.

Under the hypothesis that the differential operator on the left- hand side of (1.1) is locally uniformly parabolic on $D \times (0,T)$, that (1.1) has a C^2 convex invariant region $S \subset \mathbb{R}^m$, and under some regularity conditions, we will show that, for (1.1) Weinberger's version of strong maximum principles holds, which says that if there exist a $(x^*,t^*) \in D \times (0,T)$ such that $u(x^*,t^*) \in \partial S$, then $u(D \times (0,t^*]) \subset \partial S$. Moreover, if in addition that D satisfies the interior sphere condition, we will prove that a version of the Hopf boundary lemma holds for (1.1).

The weak and strong maximum principles for the case that in (1.1), $A(x,t,u) \equiv I$ and B_i $(i = 1,2,...,n)$ are real valued functions are studied by [6], the boundary point lemma, however, was not mentioned in [6] (see the main theorem in [3]). Our basic method is the same as Weinberger's. The local defining functions of ∂S plays an important in [6], we prefer the distance function of ∂S, making the proofs more geometric.

An extension of the boundary lemma was found by W. Troy [11] for nonnegative solution of the elliptic system

$$\sum_{i,k=1}^{n} a_{jk}^{i}(x)\frac{\partial^2 u_i}{\partial_{x_j}\partial_{x_k}} + \sum_{j=1}^{n} b_j^i(x)\frac{\partial u_i}{\partial_{x_j}} + \sum_{j=1}^{n} C_{ij}(x)u_j = 0$$

on D, where $i = 1, ..., m$. $C_{ij}(x) \geq 0$ on D for $i \neq j, 1 \leq i, j \leq m$.

The weak maximum principles for (1.1) also has been studied by K. Chueh, C. Conley and J. Smoller [1]. Their results show that for a C^1 domain $S \subset \mathbb{R}^m$ to be an invariant region we need the following condition which we assume that it holds through this paper.

Condition 1. *S is convex and for any u in ∂S, the inward unit normal $v(u)$ at u is a left eigenvector of A and $B_i(i = 1, 2, ..., n)$, and $v(x).f(x,t,u) \geq 0$ for all (x,t) in $D \times (0,T)$.*

Weakly coupled semilinear parabolic systems in unbounded domains in $\mathbb{R}^2$ or $\mathbb{R}^3$ with polynomial nonlinearities are investigated in [2], and three conditions to insure the stability of the zero solution with respect to nonnegative H^2-perturbations are given.[4] formulate a criteria for validity of the maximum norm principle as a number of equivalent algebraic conditions describing the relation between the geometry of the unit sphere of the given norm and coefficients of the system under consideration.

In this paper we use the distance function of ∂S instead of choosing a general defining function as in [6] since it makes the proofs more geometric.

2. Preliminaries

All materials discussed in this section can be found in the Appendix of chapter 14 of [10], and they are included here for the reader's convenience.

First recall some classical definitions. Let S be a C^2 domain in $\mathbb{R}^m$ with $\partial S \neq \emptyset$. For any $u \in \partial S$, let $v(u)$ denote the unit inner normal to ∂S at u. For a fixed $u_0 \in \partial S$, construct a coordinate system $(u_1, ..., u_m)$ such that u_m- axis lies in the direction $v(u_0)$ and the origin is at u_0. Near u_0, ∂S can be expressed by $u_m = \varphi(u_1, ..., u_{m-1})$. Then the Gaussian curvature of ∂S at u_0 is $det[A^2\varphi(0)]$ and the principal curvatures of ∂S at u_0 are the eigenvalues $k_1, ..., k_{m-1}$ of the matrix $[A^2\varphi(0)]$. Now if we rotate the coordinate frame with respect to the u_m axis, we can let $u_1, .., u_m$ axes lie on eigenvector directions corresponding to $k_1, ..., k_{m-1}$, respectively. We call such a new coordinate system *a principal coordinate system at* u_0. In this system $[A^2\varphi(0)] = diag[k_1, ..., k_{m-1}]$.

For $u \in \mathbb{R}^m$, the distance function d is defined by $d(u) = dist(u, \partial S)$.

Lemma 2. *Let S be a C^k domain in $R^m.k \geq 2$ and $\partial S \neq \emptyset$. Then there exists an open (w.r.t the topology of $\overline{S}$) subset G of $\overline{S}$ such that $\partial D \subset G$, d in $C^2(G)$, and for any u in G, exists unique $y(u)$ in ∂S such that*

$$|u - y(u)| = d(u)(i.e\ u = y(u) + v(y(u))d(u)),$$
$$Ad(u) = v(y(u)), 1 - k_i(y(u))d(u) > 0, \ (i = 1, ..., m-1)$$

where $k_i(y(u))(i = 1, ...m-1)$ are principal curvatures of ∂S at $y(u)$. Moreover, for $u \in G$, at a principal coordinate system at $y(u)$,

$$[A^2d(u)] = diag[\frac{-k_1}{1-k_1 d}, ..., \frac{-k_{m-1}}{1-k_{m-1}d}, 0]$$

3. The Main Result And Its Proof

We assume that u is a solution of (1.1) and A, a_{ij} , and B_i are functions of (x,t) only due to the compositions.

Theorem 3. *Suppose that A, a_{ij}, and $B_i (1 \leq i,j \leq n)$ are locally bounded on $D \times (0,T), A_{m\times m}$ and $(a_{ij})_{n\times n}$ locally uniformly positive - definite on $D \times (0,T)$ and$f(x,t,u)$ is Lipschitz continuous in u locally uniformly with respect to (x,t) on $D \times (0,T)$. Assume also that there exist a C^2 domain S in R^m s.t condition (1) is satisfied. Then if $u(D\times(0,T)) \subset \overline{S}$ and there exists $(x^*,t^*) \in D\times(0,T)$ s.t $u^* = u(x^*,t^*) \in \partial S$, then $u(D \times (0,t^*]) \subset \partial S$. Furthermore , if there exist a $x_0 \in \partial D$ and $0 < t_0 < T$ s.t D satisfies the interior sphere condition at x_0 and u is continuous at (x_0,t_0) with $u(x_0,t_0) \in \partial S$, then either $u(D \times (0,t_0]) \subset \partial S$ or $v(u(x_0,t_0)).\partial u/\partial\eta < 0$.(if the directional derivative exists), where η is any outward pointing direction to $(\partial D \times (0,T)$ at (x_0,t_0),[3].*

Proof. Let us take a bounded open neighborhood $D_1 \subset D$ of x^* and $0 < t_1 < t^*$ s.t $u(D_1 \times [t_1,t^*]) \subset G$ where G is defined in Lemma (2). let $\mu(x,t,v)$ be the eigenvalue corresponding to the eigenvector v of $A(x,t)$ and $\lambda_i(x,t,v)$ be the eigenvalue of $B_i(x,t)$.Then on $D_1 \times [t_1,t^*]$

$$L = \frac{\partial}{\partial t} - \mu(x,t,v(y(u(x,t))) \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2}{\partial x_i\, \partial x_j} + \sum_{i=1}^{n} \lambda_i(x,t,v(y(x,t))))\frac{\partial}{\partial x_i}$$

is uniformly parabolic (for definitions of v and $y(u)$), [1].
let $\bar{d}(x,t) = d(u(x,t))$. then on $D_1 \times [t_1,t^*]$ we have

$$\begin{aligned}
L\bar{d} &= A_u d(u)\frac{\partial u}{\partial t} - \mu(x,t,v(y(u))) \\
&\quad \times \sum_{i,j=1}^{n} a_{ij}(x,t)\left(\sum_{\alpha,\beta=1}^{m}\frac{\partial^2 d(u)}{\partial u_\alpha \partial u_\beta}\frac{\partial u_\alpha}{\partial x_i}\frac{\partial u_\beta}{\partial x_j} + \sum_{\alpha=1}^{m}\frac{\partial d(u)}{\partial u_\alpha}.\frac{\partial^2 u_\alpha}{\partial x_i \partial x_j}\right) \\
&\quad + \sum_{i=1}^{n}\lambda_i(x,t,v(y(u)))\sum_{\alpha=1}^{m}\frac{\partial d(u)}{\partial u_\alpha}.\frac{\partial u_\alpha}{\partial x_i} \\
&= A_u d(u)\frac{\partial u}{\partial t} - I(x,t) - \mu(x,t,v(y(u)))A_u d(u)\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} \\
&\quad + \sum_{i=1}^{n}\lambda_i(x,t,v(y(u)))A_u d(u)\frac{\partial u}{\partial x_i} \\
&= A_u d(u)\frac{\partial u}{\partial t} - A_u d(u)A(x,t)\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + A_u d(u)\sum_{i=1}^{n} B_i\frac{\partial u}{\partial x_i} \\
&\quad - I(x,t) \\
&= A_u d(u) f(x,t,u) - I(x,t),
\end{aligned}$$

where I is defined by the second equality and in the third step we use the fact that $A_u d(u) = v(y(u))$ and condition (1).

Now by condition (1) again,

$$v(y(u))f(x,t,y(u)) \geq 0, \text{ i.e. } A_u d(y(u(x,t))).f(x,t,y(u(x,t))) \geq 0 \text{ on } A_1 \times [t_1, t^*].$$

Hence we have

$$Ld \geq A_u d(u(x,t))f(x,t,u(x,t)) - A_u d(y(u(x,t))).f(x,t,y(u(x,t))) - I(x,t)$$

$$= c(x,t).(u(x,t) - y(u(x,t))) - I(x,t),$$

where $c(x,t)$ is a vector function in $\mathbb{R}^m$ and is obtained by noticing $d \in C^2(G)$ and f is Lipschitz in $u.c(x,t)$ is bounded on $D_1 \times [t_1, t^*]$.

Since

$$u = y(u) + v(y(u))d(u),$$

we have

$$Ld \geq c(x,t)v(y(u(x,t)))d(u(x,t)) - I(x,t),$$

i.e.

$$L\bar{d} \geq c(x,t)d - I(x,t) \quad \text{On} \quad D_1 \times [t_1, t^*] \tag{3.2}$$

where c is bounded.

Next, we prove that $I \leq 0$ on $D_1 \times [t_1, t^*]$.

Fix $(x_0, t_0) \in D_1 \times [t_1, t^*]$.Since

$$\sum_{\alpha,\beta=1}^{m} \frac{\partial^2 d(u)}{\partial u_\alpha \partial u_\beta} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial u_\beta}{\partial x_j}$$

is invariant under any parallel translation and rotation of u coordinate system, we assume that we work in a principal coordinate system at $y(u(x_0,t_0)) \in \partial S$. Then by Lemma (2) we have

$$D_u^2 d(u(x_0,t_0)) = diag\left[\frac{-k_1}{1 - k_1 d(u(x_0,t_0))}, ..., \frac{-k_1}{1 - k_{m-1} d(u(x_0,t_0)}, 0\right]$$

where $k_1, ..., k_{m-1}$ are the principal curvatures of ∂S at $y(u(x_0,t_0))$. Thus

$$\frac{I}{\mu}(x_0,t_0) = \sum_{i,j=1}^{n} a_{ij}(x_0,t_0) \sum_{\alpha=1}^{m-1} \frac{-k_\alpha}{1 - k_\alpha d(u(x_0,t_0))} \frac{\partial u_\alpha}{\partial x_i}(x_0,t_0) \frac{\partial u_\alpha}{\partial x_j}(x_0,t_0),$$

i.e.

$$\frac{I}{\mu}(x_0 t_0) = \sum_{\alpha=1}^{m-1} \frac{-k_\alpha}{1 - k_\alpha d(u(x_0,t_0)} \sum_{i,j=1}^{n} a_{ij}(x_0,t_0) \frac{\partial u_\alpha}{\partial x_i}(x_0,t_0) \frac{\partial u_\alpha}{\partial x_i}(x_0,t_0). \tag{3.3}$$

Since S is convex, $k_\alpha \geq 0, 1 \leq \alpha \leq m-1$. Recall in the Lemma (2) that $1 - k_\alpha(y(u))d(u) \geqslant 0$ for $u \in G \quad (\alpha = 1.2, ...m-1)$, so

$$\frac{I}{\mu}(x_0,t_0) \leq 0 \quad on D_1 \times [t_1,t^*].$$

In view of (3.2), we have

$$L\overline{d} \geq c(x,t) \text{ on } D_1 \times [t_1,t^*].$$

By the classical strong maximum principle we have, $\overline{d} = 0$ on $D_1 \times [t_1,t^*]$, that is $u(D_1 \times [t_1,t^*] \subset \partial S$. Thus we have proved that $u^{-1}(\partial S)$ is relatively open in $D\times(0,t^*]$. Obviously $u^{-1}(\partial S)$ is relatively closed in$D\times(0,t^*]$. Hence $u(D\times(0,t^*]) \subset \partial S$.

To prove the remaining part of the theorem, we choose a bounded neighborhood D_2 of x_0 which is relatively open in $\overline{D}$ as well as a small $\delta > 0$ such that $u(D_2 \times (t_0-\delta,t_0+\delta) \subset G$. By the same way as above we have for some bounded c_0

$$L\overline{d} \geq c_0(x,t)\overline{d} \text{ on } D_2 \times (t_0-\delta,t_0+\delta).$$

Thus the classical boundary point lemma gives the result. □

4. Concluding Remarks

If the strict inequality in condition (1) holds for all $(x,t) \in D \times (0,t)$, then there is no $(x^*,t^*) \in D \times (0,T)$. In theorem (3), S can be the intersection of several C^2 domains S_j which satisfy condition (1). (In the case that S_j's meet at angles less than $\pi/2$, by this theorem proof, we just need S to satisfy condition (1). Combining (3.3) with $d \equiv 0$, we have $I \geq 0$. In view of (3.3) we have that $k_\alpha > 0$ for all $\alpha = 1,...,m-1$. Thus we can add to the theorem that if ∂S has positive Gaussian curvature everywhere, then u is independent of x when $0 < t \leq t^*$. Theorem (3) holds for elliptic systems corresponding to (1.1) with some modifications.Furthermore, it's also possible to extend the boundary point lemma for domains with corners, [3], [5].

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On the Midpoint Method for Solving Generalized Equations

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Abstract. We approximate a locally unique solution of a generalized equations in a Banach space setting using a new midpoint methods (see (1.2) and (3.10)). An existence–convergence theorem and a radius of convergence are given under Lipschitz and center–Lipschitz conditions on the first order Fréchet derivative and Lipschitz–like continuity property of set–valued mappings. We show that our method (1.2) is locally quadratically convergent using a fixed points theorem [10]. Motivated by optimization considerations [3], [4] related to the resolution on nonlinear equations, a smaller ratio and a larger radius of convergence are also provided. Our methods extend the midpoint method related to the resolution of nonlinear equations [7].

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Key Words: Banach Space, Newton's Method, Midpoint Method, Generalized Equation, Aubin Continuity, Lipschitz Condition, Radius of Convergence, Set–Valued Map.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of the generalized equation

$$0 \in F(x) + G(x), \tag{1.1}$$

where F is a continuous function defined in a neighborhood V of the solution x^* included in a Banach space X with values in itself, and G is a set–valued map from X to its subsets with closed graph.

Many problems in mathematical economics, variational inequalities and other fields can be formulated as in equation (1.1) [14]–[17].
We consider a new midpoint method for $x_0 \in V$ being the initial guess and all $k \geq 0$

$$\begin{cases} 0 \in F(x_k) + \nabla F(x_k)\ (y_k - x_k) + G(y_k) \\ 0 \in F(y_k) + \nabla F(\dfrac{x_k + y_k}{2})\ (x_{k+1} - y_k) + G(x_{k+1}), \end{cases} \tag{1.2}$$

where $\nabla F(x)$ is the first order Fréchet derivative of F at x.

For $G = \{0\}$, the mildpoint method was introduced in [1]–[5], [7] to solve nonlinear equations:

$$\begin{cases} y_k = x_k - (\nabla F(x_k))^{-1}\ F(x_k) \\ x_{k+1} = x_k - \left(\nabla F(\dfrac{x_k + y_k}{2})\right)^{-1}\ F(x_k). \end{cases} \tag{1.3}$$

In [7] the convergence of order three of iterative method (1.3) is studied under Kantorovich–type assumptions. A special Lipschitz–type condition on ∇F is used in [1] to obtain a Kantorovich–type convergence theorem. In [5] a midpoint two–step method is introduced to solve nonlinear equations under mild Newton–Kantorovich–type assumptions; the obtained results are extended the case in wich the underlying operator may be differentiable. Ezquerro et al. [11] presented a convergence result of method (1.3) using a new type of recurrence relations for this method. Hernández and Salanova [12] investigated a modified midpoint method by changing an evaluation of ∇F at $z_k = \dfrac{x_k + y_k}{2}$ in method (1.3) by an evaluation of operator F at the same point.

The purpose of this paper is to study the convergence analysis of method (1.2) under Lipschitz–type conditions on the first order Fréchet derivative and Lipschitz–like continuity of set–valued mappings.

The structure of this paper is the following. In section 2, we collect a number of basic definitions and recall a fixed points theorem for set–valued maps. In section 3, we show the existence and the quadratically convergence of the sequence defined by (1.2). Finally, we give some remarks on our method using some ideas related to nonlinear equations [3], [4].

2. Preliminaries and assumptions

In order to make the paper as self–contained as possible we reintroduce some results on fixed point theorem [3]–[10]. We let $\mathcal{Z}$ be a metric space equipped with the metric ρ. For $A \subset \mathcal{Z}$, we denote by dist $(x, A) = \inf\ \{\rho(x,y),\ y \in A\}$ the distance from a point x to A. The excess e from A to the set $C \subset \mathcal{Z}$ is given by $e(C, A) = \sup\ \{\text{dist}\,(x, A),\ x \in C\}$. Let $\Lambda : X \rightrightarrows Y$ be a set–valued map, we denote by $gph\, \Lambda = \{(x, y) \in X \times Y,\ y \in \Lambda(x)\}$ and $\Lambda^{-1}(y) = \{x \in X,\ y \in \Lambda(x)\}$ is the inverse of Λ. We call $B_r(x)$ the closed ball centered at x with radius r.

Definition 1. (see [6], [13], [16]) A set–valued Λ is said to be pseudo–Lipschitz around $(x_0, y_0) \in gph\, \Lambda$ with modulus M if there exist constants a and b such that

$$e(\Lambda(y') \cap B_a(y_0), \Lambda(y'')) \leq M \parallel y' - y'' \parallel, \quad \text{for all } y' \text{ and } y'' \text{ in } B_b(x_0). \tag{2.4}$$

We need the following fixed point theorems.

Lemma 2. (see [10]) *Let $(Z, \| \cdot \|)$ be a Banach space, let ϕ a set–valued map from Z into the closed subsets of Z, let $\eta_0 \in Z$ and let r and λ be such that $0 \leq \lambda < 1$ and*
(a) dist $(\eta_0, \phi(\eta_0)) \leq r(1-\lambda)$,
(b) $e(\phi(x_1) \cap B_r(\eta_0), \phi(x_2)) \leq \lambda \ \| x_1 - x_2 \|, \ \forall x_1, x_2 \in B_r(\eta_0)$,
then ϕ has a fixed-point in $B_r(\eta_0)$. That is, there exists $x \in B_r(\eta_0)$ such that $x \in \phi(x)$. If ϕ is single–valued, then x is the unique fixed point of ϕ in $B_r(\eta_0)$.

We suppose that, for every point x in a open convex neighborhood V of x^*, $\nabla F(x)$ exist. We will make the following assumptions:

($\mathcal{H}0$) The first order Fréchet derivative ∇F is L–Lipschitz on V. That is

$$\| \nabla F(x) - \nabla F(y) \| \leq L \ \| x - y \| \text{ for all } x, y \in V. \tag{2.5}$$

It follows from (2.5) that there exists $L_0 \in [0, L]$ such that

$$\| \nabla F(x) - \nabla F(x^*) \| \leq L_0 \ \| x - x^* \| \text{ for all } x \in V. \tag{2.6}$$

($\mathcal{H}1$) $[F(x^*) + \nabla F(x^*)(. - x^*) + G(.)]^{-1}$ is M–pseudo–Lipschitz around $(0, x^*)$.

Before stating the main result on this study, we need to introduce some notations. First, for $k \in I\!N$ and (y_k), (x_k) defined in (1.2), let us define the set–valued mappings Q, ψ_k, $\phi_k : X \rightrightarrows X$ by the following

$$Q(.) := F(x^*)+\nabla F(x^*)(.-x^*)+G(.); \ \psi_k(.) := Q^{-1}(Z_k(.)); \ \phi_k(.) := Q^{-1}(W_k(.)) \tag{2.7}$$

where Z_k and W_k are defined from X to X by

$$\begin{aligned} Z_k(x) &:= F(x^*) + \nabla F(x^*)(x - x^*) - F(y_k) - \nabla F(\frac{x_k + y_k}{2}) \ (x - y_k) \\ W_k(x) &:= F(x^*) + \nabla F(x^*)(x - x^*) - F(x_k) - \nabla F(x_k) \ (x - x_k) \end{aligned} \tag{2.8}$$

3. Local convergence analysis for method (1.2)

We show the main local convergence result for method (1.2):

Theorem 3. *We suppose that assumptions ($\mathcal{H}0$) and ($\mathcal{H}1$) are satisfied. For every constant $C > C_0 = \dfrac{3\,M\,L}{2}$, there exist $\delta > 0$ such that for every starting point x_0 in $B_\delta(x^*)$ (x_0 and x^* distinct), and a sequence (x_k) defined by (1.2) which satisfies*

$$\| x_{k+1} - x^* \| \leq C \, \| x_k - x^* \|^2 . \tag{3.9}$$

Remark 4. (a) Theorem 3 remains valid if one replaces the algorithm (1.2) by the following method

$$\left\{ \begin{array}{l} 0 \in F(x_k) + \nabla F(x_k) \ (y_k - x_k) + G(y_k) \\ 0 \in F(x_k) + \nabla F(\dfrac{x_k + y_k}{2}) \ (x_{k+1} - x_k) + G(x_{k+1}). \end{array} \right. \tag{3.10}$$

(b) The results of this paper seem also true for a general assumption: F is defined in a neghborhood V of the solution x^* included in a Banach space X with values in another Banach space Y, and G is a set–valued map from X to its subsets of Y with closed graph.

The proof of Theorem 3 is by induction on k. We need to give two results. In the first, we prove the existence of starting point y_0 for x_0 in V. In the second, we state a result which the starting point (x_0, y_0). Let us mention that y_0 and x_1 are a fixed points of ϕ_0 and ψ_0 respectively if and only if $0 \in F(x_0) + \nabla F(x_0)\ (y_0 - x_0) + G(y_0)$ and $0 \in F(y_0) + \nabla F(\dfrac{x_0 + y_0}{2})\ (x_1 - y_0) + G(x_1)$ respectively.

Proposition 5. *Under the assumptions of Theorem 3, there exists $\delta > 0$ such that for every starting point x_0 in $B_\delta(x^*)$ (x_0 and x^* distinct), the set–valued map ϕ_0 has a fixed point y_0 in $B_\delta(x^*)$, and satisfying*

$$\| y_0 - x^* \| \leq C \| x_0 - x^* \|^2 . \tag{3.11}$$

Proof. By hypothesis ($\mathcal{H}1$) there exist positive numbers M, a and b such that

$$e(Q^{-1}(y') \cap B_a(x^*), Q^{-1}(y'')) \leq M \| y' - y'' \|, \ \forall y', y'' \in B_b(0). \tag{3.12}$$

Fix $\delta > 0$ such that

$$\delta < \delta_0 = \min\left\{ a\ ,\ \sqrt{\frac{2\ b}{3\ L}}\ ,\ \frac{1}{C} \right\}. \tag{3.13}$$

The main idea of the proof of Proposition 5 is to show that both assertions (a) and (b) of Lemma 2 hold; where $\eta_0 := x^*$, ϕ is the function ϕ_0 defined in (2.7) and where r and λ are numbers to be set. According to the definition of the excess e, we have

$$\text{dist}\,(x^*, \phi_0(x^*)) \leq e\Big(Q^{-1}(0) \cap B_\delta(x^*), \phi_0(x^*) \Big). \tag{3.14}$$

Moreover, for all point x_0 in $B_\delta(x^*)$ (x_0 and x^* distinct) we have

$$\begin{aligned} \| W_0(x^*) \| &= \| F(x^*) - F(x_0) - \nabla F(x_0)\ (x^* - x_0) \| \\ &= \| \int_0^1 (\nabla F(x_0 + t(x^* - x_0)) - \nabla F(x_0))\ (x^* - x_0)\ dt \| . \end{aligned}$$

In view of assumption ($\mathcal{H}0$) we obtain

$$\| W_0(x^*) \| \ \leq \ \frac{L}{2}\ \| x^* - x_0 \|^2 . \tag{3.15}$$

Then (3.13) yields, $W_0(x^*) \in B_b(0)$.
Using (3.12) we have

$$\begin{aligned} e\Big(Q^{-1}(0) \cap B_\delta(x^*), \phi_0(x^*) \Big) &= e\Big(Q^{-1}(0) \cap B_\delta(x^*), Q^{-1}[W_0(x^*)] \Big) \\ &\leq \frac{M\,L}{2}\ \| x^* - x_0 \|^2 . \end{aligned} \tag{3.16}$$

By inequality (3.14), we get

$$\text{dist}\,(x^*, \phi_0(x^*)) \ \leq \ \frac{M\,L}{2}\ \| x^* - x_0 \|^2 . \tag{3.17}$$

Since $C > C_0$, there exists $\lambda \in [0, 1[$ such that $C(1 - \lambda) \geq C_0$ and

$$\text{dist}\,(x^*, \phi_0(x^*)) \leq C\ (1 - \lambda)\ \ \| x_0 - x^* \|^2 . \tag{3.18}$$

By setting $r := r_0 = C\ \ \| x_0 - x^* \|^2$ we can deduce from the inequality (3.18) that the assertion (a) in Lemma 2 is satisfied.
Now, we show that condition (b) of Lemma 2 is satisfied.

By (3.13) we have $r_0 \leq \delta \leq a$. Using $(\mathcal{H}0)$ we have for $x \in B_\delta(x^*)$ the following estimates

$$\begin{array}{rcl}\| W_0(x) \| & = & \| F(x^*) + \nabla F(x^*)\ (x - x^*) - F(x_0) - \nabla F(x_0)\ (x - x_0) \| \\ & \leq & \| F(x^*) - F(x_0) - \nabla F(x_0)\ (x^* - x_0) \| + \\ & & \| (\nabla F(x^*) - \nabla F(x_0))\ (x - x^*) \| \\ & \leq & \frac{L}{2} \| x^* - x_0 \|^2 + L_0\ \| x^* - x_0 \|\ \| x - x^* \| \\ & \leq & \frac{3L}{2}\ \delta^2. \end{array} \tag{3.19}$$

Then by (3.13) we deduce that for all $x \in B_\delta(x^*)$ we have $W_0(x) \in B_b(0)$. Then it follows that for all $x', x'' \in B_{r_0}(x^*)$, we have

$$e(\phi_0(x') \cap B_{r_0}(x^*), \phi_0(x'')) \leq e(\phi_0(x') \cap B_\delta(x^*), \phi_0(x'')),$$

which yields by (3.12) and $(\mathcal{H}0)$:

$$\begin{array}{rcl} e(\phi_0(x') \cap B_{r_0}(x^*), \phi_0(x'')) & \leq & M\ \| W_0(x') - W_0(x'') \| \\ & \leq & M\ \| (\nabla F(x_0) - \nabla F(x^*))\ (x'' - x') \| \\ & \leq & M\ L_0\ \delta\ \| x'' - x' \|. \end{array} \tag{3.20}$$

Without loss generality we may assume that $\delta < \dfrac{\lambda}{M\,L_0}$ and thus condition (b) of Lemma 2 is satisfied. Since both conditions of Lemma 2 are fulfilled, we can deduce the existence of a fixed point $y_0 \in B_{r_0}(x^*)$ for the map ϕ_0. This finishes the proof of Proposition 5. □

Proposition 6. *Under the assumptions of Theorem 3, there exist* $\delta > 0$ *such that for every starting point* x_0 *in* $B_\delta(x^*)$ *and* y_0 *given by Proposition 5 (*x_0 *and* x^* *distinct), and the set–valued map* ψ_0 *has a fixed point* x_1 *in* $B_\delta(x^*)$ *satisfying*

$$\| x_1 - x^* \| \leq C \| x_0 - x^* \|^2, \tag{3.21}$$

where the constant C *is given by Theorem 3.*

Proof. The proof of Proposition 6 is the same one as that of Proposition 5. The choice of δ is the same one given by (3.13). The inequality (3.14) is valid if we replace ϕ_0 by ψ_0. Moreover, for all point x_0 in $B_\delta(x^*)$ (x_0 and x^* distinct), we have

$$\begin{array}{rcl}\| Z_0(x^*) \| & = & \| F(x^*) - F(y_0) - \nabla F(\frac{x_0 + y_0}{2})\ (x^* - y_0) \| \\ & = & \| \displaystyle\int_0^1 (\nabla F(y_0 + t(x^* - y_0)) - \nabla F(\frac{x_0 + y_0}{2}))\ (x^* - y_0)\ dt \| \end{array}$$

In view of assumption $(\mathcal{H}0)$ and Proposition 5 we get

$$\begin{array}{rcl}\| Z_0(x^*) \| & \leq & L\ (\frac{1}{2} \| y_0 - x_0 \| + \frac{1}{2} \| y_0 - x^* \|)\ \| y_0 - x^* \| \\ & \leq & \frac{L}{2}\ (2 \| y_0 - x^* \| + \| x_0 - x^* \|)\ \ \| y_0 - x^* \| \\ & \leq & \frac{L\,C}{2}\ (2\,C\ \| x_0 - x^* \|^2 + \| x_0 - x^* \|)\ \ \| x^* - x_0 \|^2 \\ & \leq & \frac{L}{2}\ (2\,C^2\,\delta^2 + C\,\delta)\ \ \| x^* - x_0 \|^2 . \end{array} \tag{3.22}$$

By (3.13) and (3.22) we have

$$\| Z_0(x^*) \| \ \leq\ \frac{3\,L}{2}\ \| x_0 - x^* \|^2 . \tag{3.23}$$

Then (3.13) yields, $Z_0(x^*) \in B_b(0)$. Setting $r := r_0 = C \parallel x_0 - x^* \parallel^2$, we can deduce from the assertion (a) in Lemma 2 is satisfied.
By (3.13) we have $r_0 \le \delta \le a$, and moreover for $x \in B_\delta(x^*)$ we have

$$\begin{aligned} \parallel Z_0(x) \parallel &= \parallel F(x^*) + \nabla F(x^*)\,(x - x^*) - F(y_0) - \nabla F(\frac{x_0 + y_0}{2})\,(x - y_0) \parallel \\ &\le \parallel F(x^*) - F(y_0) - \nabla F(x^*)\,(x^* - y_0) \parallel + \\ &\quad \parallel (\nabla F(x^*) - \nabla F(\frac{x_0 + y_0}{2}))\,(x - y_0) \parallel . \end{aligned} \tag{3.24}$$

Using assumption $(\mathcal{H}0)$ we obtain

$$\parallel Z_0(x) \parallel \ \le \ \frac{5\,L_0}{2}\ \delta^2 \tag{3.25}$$

A slight change in the end of proof of Proposition 5 shows that the condition (b) of Lemma 2 is satisfied. The existence of a fixed point $x_1 \in B_{r_0}(x^*)$ for the map ψ_0 is ensured. This finishes the proof of Proposition 6. □

Proof of Theorem 3. Keeping $\eta_0 = x^*$ and setting $r := r_k = C \parallel x^* - x_k \parallel^2$, the application of Proposition 5 and Proposition 6 to the map ϕ_k and ψ_k respectively gives the existence of a fixed points y_k and x_{k+1} for ϕ_k and ψ_k respectively which is an elements of $B_{r_k}(x^*)$. This last fact implies the inequality (3.9), which is the desired conclusion. □

Remark 7. (a) It follows from the proof of Proposition 5 that constants C_0 and δ_0 can be replaced by the more precise

$$\overline{C_0} = \frac{M\,L}{2}, \tag{3.26}$$

and

$$\overline{\delta_0} = \min\left\{a\ ,\ \sqrt{\frac{2\,b}{L + 2\,L_0}}\ ,\ \frac{1}{C}\right\}, \tag{3.27}$$

respectively. Note that

$$\overline{C_0} \le C_0, \tag{3.28}$$

and

$$\delta_0 \le \overline{\delta_0}. \tag{3.29}$$

(b) The constant δ_0 in the proof of Proposition (6) can be given by :

$$\overline{\overline{\delta_0}} = \min\left\{a\ ,\ \sqrt{\frac{2\,b}{3\,L_0}}\ ,\ \frac{1}{C}\ ,\ 1\right\}. \tag{3.30}$$

Indeed by adding and substracting $\nabla F(x^*)\,(x^* - y_0)$ inside the first norm in the computation of $\parallel Z_0(x^*) \parallel$ we arrive at an estimate corresponding (3.23) :

$$\parallel Z_0(x^*) \parallel \ \le \ \frac{3\,L_0}{2}\ \parallel x_0 - x^* \parallel^2 . \tag{3.31}$$

Hence $\overline{\overline{\delta_0}}$ can be replace δ_0 in the proof of Proposition 6. This modification is usefull when $\overline{\overline{\delta_0}} > \delta_0$. These observations are important in computational mathematics since the allow a smaller ratio C and a larger radius of convergence [3], [4].

Remark 8. The sequence (y_n) given by algorithm (1.2) is also quadratically convergent to a solution x^* of (1.1) (see [9]). Note that the midpoint method for nonlinear equations was shown by us to be of order three (see [1]–[5], [7], [8]). However we had to introduce Lipschitz conditions on the second Fréchet derivative $\nabla^2 F$. Here we simply used hypotheses on ∇F only. In a future paper using the Ostrowski representation for F given in [8] we will recover the third order of convergence of method (1.2).

Application 9. (see [14])
Let K be a convex set in $I\!R^n$, P is a topological space and φ is a function from $P \times K$ to $I\!R^n$, the "perturbed" variational inequality problem consists of seeking k_0 in K such that

$$\text{For each } k \in K, \quad (\varphi(p, k_0); k - k_0) \geq 0 \tag{3.32}$$

where $(.;.)$ is the usual scalar product on $I\!R^n$ and p is fixed parameter in P. Let $\mathcal{I}_K$ be a convex indicator function of K and ∂ denotes the subdifferential operator. Then the problem (3.32) is equivalent to problem

$$0 \in \varphi(p, k_0) + \partial \mathcal{I}_K(k_0). \tag{3.33}$$

The problem (3.32) is equivalent to (3.33) which is a generalized equation in the form (1.1). Consequently, we can approximate the solution k_0 of (3.32) using our methods (1.2) and (3.10).

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Sum Intuitionistic Fuzzy Closure Spaces

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Abstract. We will prove the existence of final intuitionistic fuzzy topological spaces and final intuitionistic fuzzy closure spaces. From this fact, we can define intuitionistic quotient spaces of their spaces and sum of intuitionistic fuzzy closure spaces. In this paper, additivity of two kinds of intuitionistic fuzzy closure spaces are studied.

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1. Introduction

Šostak [19] introduced the fundamental concept of a fuzzy topological structure as an extension of both crisp topology and Chang fuzzy topology [5]. Later on he has developed the theory of fuzzy topological spaces in [20, 21]. In [16], Ramadan gave a similar definition, namely "smooth topological space". It has been developed in many direction [6,10-14].

As a generalization of fuzzy sets, the notion of intuitionistic fuzzy sets was introduced by Atanassov [2-4]. Recently, Çoker and his colleagues [8, 9] introduced the notion of intuitionistic fuzzy topological space using intuitionistic fuzzy sets. Samanta and Mondal [17, 18] introduced the notion of intuitionistic gradation of openness as a generalization of intuitionistic fuzzy topological spaces [9] and smooth topological spaces.

In this paper, we will prove the existence of final intuitionistic fuzzy topological spaces and final intuitionistic fuzzy closure spaces. From this fact, we will define intuitionistic quotient spaces of their spaces. Moreover, the additivity of two kinds of intuitionistic fuzzy closure spaces are studied.

Throughout this paper, let X be a nonempty set, $I = [0,1]$, $I_\circ = (0,1]$ and $I_1 = [0,1)$. For $\alpha \in I$, $\overline{\alpha}(x) = \alpha$ for all $x \in X$. The family of all fuzzy sets on X denoted by I^X. Notions and notations not described in this paper are standard and usual.

2. Intuitionistic Fuzzy Topological Spaces

Definition 1. [18] An intuitionistic gradation of openness (IGO, for short) on X is an ordered pair $(\tau, \tau*)$ of functions from I^X to I such that:

(IGO1): $\tau(\lambda) + \tau*(\lambda) \leq 1$, for each $\lambda \in I^X$,
(IGO2): $\tau(\underline{0}) = \tau(\underline{1}) = 1$, $\tau*(\underline{0}) = \tau*(\underline{1}) = 0$,
(IGO3): $\tau(\lambda_1 \wedge \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$ and $\tau*(\lambda_1 \wedge \lambda_2) \leq \tau*(\lambda_1) \vee \tau*(\lambda_2)$, for each $\lambda_i \in I^X$ and $i \in \{1, 2\}$,
(IGO4): $\tau(\bigvee_{i\in\Delta} \lambda_i) \geq \bigwedge_{i\in\Delta} \tau(\lambda_i)$ and $\tau*(\bigvee_{i\in\Delta} \lambda_i) \leq \bigvee_{i\in\Delta} \tau*(\lambda_i)$, for each family $\{\lambda_i \in I^X | i \in \Delta\}$.

The triplet $(X, \tau, \tau*)$ is called an intuitionistic fuzzy topological space (ifts, for short). τ and $\tau*$ may be interpreted as gradation of openness and gradation of nonopenness, respectively. Let $(\tau_1, \tau*_1)$ and $(\tau_2, \tau*_2)$ be IGO's on X. We say $(\tau_1, \tau*_1)$ is finer than $(\tau_2, \tau*_2)$ ($(\tau_2, \tau*_2)$ is coarser than $(\tau_1, \tau*_1)$) if $\tau_2(\lambda) \leq \tau_1(\lambda)$ and $\tau*_2(\lambda) \geq \tau*_1(\lambda)$ for all $\lambda \in I^X$.

Definition 2 ([18]). Let $(\tau, \tau*)$ be an IGO on X and the functions $\mathcal{F}, \mathcal{F}^* : I^X \to I$ defined by $\mathcal{F}(\mu) = \tau(\underline{1} - \mu)$ and $\mathcal{F}^*(\mu) = \tau * (\underline{1} - \mu)$ for all $\mu \in I^X$. Then $(\mathcal{F}, \mathcal{F}^*)$ is called an intuitionistic gradation of closedness (IGC, for short) on X.

Definition 3 ([15]). A function $\mathcal{C} : I^X \times I_\circ \times I_1 \to I^X$ is called an *intuitionistic fuzzy closure operator* if for each λ, $\mu \in I^X$, $r \in I_\circ$ and $s \in I_1$ with $r + s \leq 1$, the operator $\mathcal{C}$ satisfies the following conditions:

(C1) $\mathcal{C}(\underline{0}, r, s) = \underline{0}$.

(C2) $\lambda \leq \mathcal{C}(\lambda, r, s)$.

(C3) if $\lambda \leq \mu$, then $\mathcal{C}(\lambda, r, s) \leq \mathcal{C}(\mu, r, s)$.

(C4) $\mathcal{C}(\lambda, r, s) \vee \mathcal{C}(\mu, r, s) = \mathcal{C}(\lambda \vee \mu, r, s)$.

(C5) $\mathcal{C}(\lambda, r, s) \leq \mathcal{C}(\lambda, r_1, s_1)$ if $r \leq r_1$ and $s \geq s_1$ with $r_1 + s_1 \leq 1$.

The pair $(X, \mathcal{C})$ is called an intuitionistic fuzzy closure space. An intuitionistic fuzzy closure space $(X, \mathcal{C})$ is called *topological* if

(C6) $\mathcal{C}(\mathcal{C}(\lambda, r, s), r, s) = \mathcal{C}(\lambda, r, s)$, for each $\lambda \in I^X$ and $r \in I_\circ$, $s \in I_1$ with $r+s \leq 1$.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be intuitionistic intuitionistic fuzzy closure operators on X. We say that $\mathcal{C}_1$ is finer than $\mathcal{C}_2$ ($\mathcal{C}_2$ is coarser than $\mathcal{C}_1$) iff $\mathcal{C}_1(\lambda, r, s) \leq \mathcal{C}_2(\lambda, r, s)$, for each $\lambda \in I^X$ and $r \in I_\circ$, $s \in I_1$ with $r + s \leq 1$.

Theorem 4 ([15]). *Let $(X, \tau, \tau*)$ be an ifts. Then for each $r \in I_\circ$, $s \in I_1$, $\lambda \in I^X$ we define an operator $\mathcal{C}_{\tau,\tau*} : I^X \times I_\circ \times I_1 \to I^X$ as follows*

$$\mathcal{C}_{\tau,\tau*}(\lambda, r, s) = \bigwedge\{\mu \in I^X \mid \lambda \leq \mu, \tau(\underline{1} - \mu) \geq r, \tau*(\underline{1} - \mu) \leq s\}.$$

Then $\mathcal{C}_{\tau,\tau*}$ is an intuitionistic fuzzy closure operator.

Theorem 5 ([15]). *Let $(X, \mathcal{C})$ be an intuitionistic fuzzy closure space. Define the functions $\tau_\mathcal{C}, \tau*_\mathcal{C} : I^X \to I$ by*

$$\tau_\mathcal{C}(\lambda) = \bigvee\{r \in I_\circ \mid \mathcal{C}(\underline{1} - \lambda, r, s) = \underline{1} - \lambda\},$$

$$\tau*_\mathcal{C}(\lambda) = \bigwedge\{s \in I_1 \mid \mathcal{C}(\underline{1} - \lambda, r, s) = \underline{1} - \lambda\}.$$

Then:

(1) $(\tau_\mathcal{C}, \tau*_\mathcal{C})$ is an IGO on X.

(2) We have $\mathcal{C} = \mathcal{C}_{\tau_\mathcal{C},\tau*_\mathcal{C}}$ iff $(X, \mathcal{C})$ satisfies the following conditions:

(a) It is a topological intuitionistic fuzzy closure space.

(b) If $r_1 = \bigvee\{r \in I_\circ \mid \mathcal{C}(\lambda, r, s) = \lambda\}$ and $s_1 = \bigwedge\{s \in I_1 \mid \mathcal{C}(\lambda, r, s) = \lambda\}$, then $\mathcal{C}(\lambda, r_1, s_1) = \lambda$.

Definition 6 ([15]). Let $(X, \tau_1, \tau*_1)$ and $(Y, \tau_2, \tau*_2)$ be ifts's. A function $f : X \to Y$ is called an intuitionistic fuzzy continuous if $\tau_1(f^{-1}(\mu)) \geq \tau_2(\mu)$ and $\tau*_1(f^{-1}(\mu)) \leq \tau*_2(\mu)$ for all $\mu \in I^Y$. Equivalently, $\mathcal{F}_1(f^{-1}(\mu)) \geq \mathcal{F}_2(\mu)$ and $\mathcal{F}^*_1(f^{-1}(\mu)) \leq \mathcal{F}^*_2(\mu)$ for all $\mu \in I^Y$.

Definition 7 ([1]). Let $(X, \mathcal{C}_1)$ and $(Y, \mathcal{C}_2)$ be two intuitionistic fuzzy closure spaces. A function $f : (X, \mathcal{C}_1) \to (Y, \mathcal{C}_2)$ is said to be a $\mathcal{C}$-map if for all $\lambda \in I^X$, $r \in I_\circ$, $s \in I_1$ with $r + s \leq 1$,

$$f(\mathcal{C}_1(\lambda, r, s)) \leq \mathcal{C}_2(f(\lambda), r, s).$$

Theorem 8 ([1]). *Let $(X, \tau_1, \tau*_1)$ and $(Y, \tau_2, \tau*_2)$ be ifts's. A function $f : (X, \tau_1, \tau*_1) \to (Y, \tau_2, \tau*_2)$ is an intuitionistic fuzzy continuous iff $f : (X, \mathcal{C}_{\tau_1,\tau*_1}) \to (Y, \mathcal{C}_{\tau_2,\tau*_2})$ is a $\mathcal{C}$-map.*

Using Theorem 8, we can easily prove the following corollary:

Corollary 9 ([1]). *Let $(X, \mathcal{C}_1)$ and $(Y, \mathcal{C}_2)$ be intuitionistic fuzzy closure spaces. If $f : (X, \mathcal{C}_1) \to (Y, \mathcal{C}_2)$ is $\mathcal{C}$-map, then $f : (X, \mathcal{T}_{\mathcal{C}_1}, \mathcal{T}^*_{\mathcal{C}_1}) \to (Y, \mathcal{T}_{\mathcal{C}_2}, \mathcal{T}^*_{\mathcal{C}_2})$ is an intuitionistic fuzzy continuous map.*

Definition 10. An intuitionistic fuzzy topological property ξ is called an additive, if for any family of intuitionistic fuzzy topological spaces $\{(X_i, \tau_i, \tau*_i)\}_{i\in\Gamma}$ with property ξ, then the sum $(X, \tau, \tau*)$ also has property ξ

Definition 11. An intuitionistic fuzzy closure space $(X, \mathcal{C})$ is said to be:

(r, s)-fuzzy-T_0: If for all x, $y \in X$ such that $x \neq y$, there exists $\lambda \in I^X$ such that $\mathcal{C}(\lambda, r, s) = \lambda$ and $\lambda(x) \neq \lambda(y)$,where $r \in I_\circ$, $s \in S_1$.

(r, s)-fuzzy-T_1: If $\mathcal{C}(\chi_{\{x\}}, r, s) \leq \chi_{\{x\}}$, for each $x \in X$, where $r \in I_\circ$, $s \in S_1$ and $\chi_{\{x\}}$ is the characteristic function of x.

3. Final Intuitionistic Fuzzy Topological Spaces

Theorem 12. *Let Y be a set and $\{(X_i, \tau_i, \tau*_i)\}_{i\in\Gamma}$ be a family of intuitionistic fuzzy topological spaces. Let $f_i \colon X_i \to Y$ be a function for each $i \in \Gamma$. Define the functions $\tau, \tau* \colon I^Y \to I$ by*

$$\tau(\lambda) = \bigwedge_{i\in\Gamma} \tau_i(f_i^{-1}(\lambda)),\ \tau*(\lambda) = \bigvee_{i\in\Gamma} \tau*_i(f_i^{-1}(\lambda)).$$

Then:

(1) $(\tau, \tau*)$ is the finest intuitionistic fuzzy topology on Y for which each f_i is intuitionistic fuzzy continuous.

(2) $f \colon (Y, \tau, \tau*) \to (Z, \tau_Z, \tau*_Z)$ is an intuitionistic fuzzy continuous map iff each $fof_i \colon (X.\tau_i, \tau*_i) \to (Z, \tau_Z, \tau*_Z)$ is intuitionistic fuzzy continuous.

Proof. (1) (IGO1) and (IGO2) are trivial from the definition of τ, $\tau*$.

(IGO3)

$$\begin{aligned} \tau(\lambda\wedge\mu) &= \tau_i(f_i^{-1}(\lambda)\wedge f_i^{-1}(\mu)) \\ &\geq \tau_i(f_i^{-1}(\lambda)\wedge \tau_i(f_i^{-1}(\mu))) \\ &\geq \tau(\lambda)\wedge\tau(\mu). \end{aligned}$$

and

$$\begin{aligned} \tau*(\lambda\wedge\mu) &= \tau*_i(f_i^{-1}(\lambda)\wedge f_i^{-1}(\mu)) \\ &\leq \tau*_i(f_i^{-1}(\lambda)\vee \tau*_i(f_i^{-1}(\mu))) \\ &\leq \tau*(\lambda)\vee\tau*(\mu), \end{aligned}$$

which is a contradiction. Hence $\tau(\lambda\wedge\mu) \geq \tau(\lambda)\wedge\tau(\mu)$, $\tau*(\lambda\wedge\mu) \leq \tau*(\lambda)\vee\tau*(\mu)$.

(IGO4) We will try to prove that $\tau(\bigvee_{i\in\Gamma}\lambda_i) \geq \bigvee_{i\in\Gamma}\tau(\lambda_i)$, $\tau*(\bigvee_{i\in\Gamma}\lambda_i) \leq \bigvee_{i\in\Gamma}\tau*(\lambda_i)$. Suppose $\tau(\bigvee_{i\in\Gamma}\lambda_i) \not\geq \bigvee_{i\in\Gamma}\tau(\lambda_i)$ and $\tau*(\bigvee_{i\in\Gamma}\lambda_i) \not\leq \bigvee_{i\in\Gamma}\tau*(\lambda_i)$. From the definition of $(\tau, \tau*)$, there exists $i \in \Gamma$ such that

$$\tau(\bigvee_{i\in\Gamma}\lambda_i) \leq \tau_i(f^{-1}(\bigvee_{i\in\Gamma}\lambda_i)) < \bigvee_{i\in\Gamma}\tau(\lambda_i)$$

and

$$\tau*(\bigvee_{i\in\Gamma}\lambda_i) \geq \tau*_i(f^{-1}(\bigvee_{i\in\Gamma}\lambda_i)) > \bigvee_{i\in\Gamma}\tau*(\lambda_i).$$

But, we have

$$\begin{aligned} \tau_i(f^{-1}(\bigvee_{i\in\Gamma}\lambda_i)) &= \tau_i(\bigvee_{i\in\Gamma} f^{-1}(\lambda_i)) \\ &\geq \bigvee_{i\in\Gamma}\tau(f^{-1}(\lambda_i)) \\ &\geq \bigvee_{i\in\Gamma}\tau(\lambda_i). \end{aligned}$$

and

$$\begin{aligned}\tau*_i(f^{-1}(\bigvee_{i\in\Gamma}\lambda_i)) &= \tau*_i(\bigvee_{i\in\Gamma}f^{-1}(\lambda_i))\\ &\leq \bigvee_{i\in\Gamma}\tau*(f^{-1}(\lambda_i))\\ &\leq \bigvee_{i\in\Gamma}\tau*(\lambda_i),\end{aligned}$$

which is a contradiction. Hence $\tau(\bigvee_{i\in\Gamma}\lambda_i)\geq\bigvee_{i\in\Gamma}\tau(\lambda_i)$, $\tau*(\bigvee_{i\in\Gamma}\lambda_i)\leq\bigvee_{i\in\Gamma}\tau*(\lambda_i)$ for any family $\{\lambda_i\mid i\in\Gamma\}\subseteq I^Y$.

Secondly, since $\tau(\lambda)\leq\tau_i(f_i^{-1}(\lambda))$ and $\tau*(\lambda)\geq\tau*_i(f_i^{-1}(\lambda))$ for each $i\in\Gamma$, each f_i is intuitionistic fuzzy continuous map.

Finally, we will show that $(\tau,\tau*)$ is the finest intuitionistic fuzzy topology on Y for which each f_i is intuitionistic fuzzy continuous map. If $f_i\colon(X_i,\tau_i,\tau*_i)\to(Y,\tau^\star,\tau*^\star)$ is intuitionistic fuzzy continuous, we have $\tau^\star(\nu)\leq\tau_i^\star(f_i^{-1}(\nu))$ and $\tau*^\star(\nu)\geq\tau*_i(f_i^{-1}(\nu))$, for each $i\in\Gamma$, $\nu\in I^Y$. By using the definition of τ, $\tau*$, it follows $\tau^\star(\nu)\leq\tau(\nu)$ and $\tau*^\star(\nu)\geq\tau*(\nu)$ for all $\nu\in I^Y$.

(2) ($\Rightarrow$.) Trivial.

($\Leftarrow$.) Since $fof_i\colon(X,\tau_i,\tau*_i)\to(Z,\tau_Z,\tau*_Z)$ is an intuitionistic fuzzy continuous, we have for each $\mu\in I^Z$,

$$\tau_Z(\mu)\leq\tau_i((fof_i)^{-1}(\mu))=\tau_i(f_i^{-1}(f^{-1}(\mu))).$$

and

$$\tau*_Z(\mu)\geq\tau*_i((fof_i)^{-1}(\mu))=\tau*_i(f_i^{-1}(f^{-1}(\mu))).$$

By using the definition of τ, $\tau*$, it follows $\tau_Z(\mu)\leq\tau(f^{-1}(\mu))$ and $\tau*_Z(\mu)\geq\tau*(f^{-1}(\mu))$ for each $\mu\in I^Z$. Hence $f\colon(Y,\tau,\tau*)\to(Z,\tau_Z,\tau*_Z)$ is intuitionistic fuzzy continuous map.

□

Definition 13. The structure $(\tau,\tau*)$ defined in Theorem 12 is called the final intuitionistic fuzzy topology on Y associated with the families $\{(X_i,\tau_i,\tau*_i)\}_{i\in\Gamma}$ and $(f_i)_{i\in\Gamma}$.

Corollary 14. (Sum Intuitionistic fuzzy topological spaces)
Let $\{(X_i,\tau_i,\tau*_i)\}_{i\in\Gamma}$ be a family of intuitionistic fuzzy topological spaces, for different $i,j\in\Gamma$. X_i and X_J be disjoint, $X=\cup_{i\in\Gamma}X_i$. Let $id_i\colon X_i\to X$ be the identity map for which $i\in\Gamma$. Define functions $\tau,\tau*\colon I^Y\to I$ by

$$\tau(\lambda)=\bigwedge_{i\in\Gamma}\tau_i(id_i^{-1}(\lambda)),\ \tau*(\lambda)=\bigvee_{i\in\Gamma}\tau*_i(id_i^{-1}(\lambda)).$$

Then:

(1) $(\tau,\tau*)$ is the finest intuitionistic fuzzy topology on X for which each id_i is intuitionistic fuzzy continuous.

(2) $f\colon(X,\tau,\tau*)\to(Z,\tau_Z,\tau*_Z)$ is an intuitionistic fuzzy continuous map iff each $foid_i\colon(X,\tau_i,\tau*_i)\to(Z,\tau_Z,\tau*_Z)$ is intuitionistic fuzzy continuous map.

Corollary 15. *Let Y be a set and $(X, \tau, \tau*)$ be an intuitionistic fuzzy topological space. Let $f : X \to Y$ be a surjective function. Define the functions $\tau^f, \tau*^f : I^Y \to I$ by*

$$\tau^f(\lambda) = \tau(f^{-1}(\lambda)),\ \tau*^f(\lambda) = \tau*(f^{-1}(\lambda))$$

Then:

(1) $(\tau^f, \tau*^f)$ is the finest intuitionistic fuzzy topology on X which f is intuitionistic fuzzy continuous.

(2) $g : (Y, \tau^f, \tau*^f) \to (Z, \tau_Z, \tau*_Z)$ is an intuitionistic fuzzy continuous iff $gof : (X, \tau, \tau*) \to (Z, \tau_Z, \tau*_Z)$ is intuitionistic fuzzy continuous.

Definition 16. Let $(X, \tau, \tau*)$ be an intuitionistic fuzzy topological space and Y a set. Let $f : X \to Y$ be a surjective function. The final intuitionistic fuzzy topological spaces τ^f on Yassociated the $(X, \tau, \tau*)$ and f is called the quotient intuitionistic fuzzy topological space and the function f is called fuzzy quotient map.

Theorem 17. *Let $(X, \tau_1, \tau*_1)$ and $(Y, \tau_2, \tau*_2)$ be intuitionistic fuzzy topological spaces. Let $f : (X, \tau_1, \tau*_1) \to (Y, \tau_2, \tau*_2)$ is a surjective intuitionistic fuzzy continuous function .*

(1) If f is an intuitionistic open function, then f is intuitionistic fuzzy quotient function.

(2) If f is an intuitionistic closed function, then f is intuitionistic fuzzy quotient function.

Proof. we only show that $\tau_2 = \tau^f$. By using Corollary 15 , we have $\tau(\lambda) \le \tau^f(\lambda)$ and $\tau*_2(\lambda) \ge \tau*^f(\lambda)$ for all $\lambda \in I^Y$. Conversely, we have

$$\begin{aligned} \tau^f(\lambda) &= \tau_1(f^{-1}(\lambda)) \\ &\le \tau_2(f(f^{-1}(\lambda))) \\ &= \tau_2(\lambda). \end{aligned}$$

and

$$\begin{aligned} \tau*^f(\lambda) &= \tau*_1(f^{-1}(\lambda)) \\ &\ge \tau*_2(f(f^{-1}(\lambda))) \\ &= \tau*_2(\lambda). \end{aligned}$$

(2) Trivial. □

4. Final Intuitionistic Fuzzy Closure Spaces

Theorem 18. *Let Y be a set and $\{(X_i, \mathcal{C}_i)\}_{i\in\Gamma}$ be a family of intuitionistic fuzzy closure spaces. Let $f_i : X_i \to Y$ be a surjective function for each $i \in \Gamma$. Define the function $\mathcal{C} : I^Y \times I_\circ \times I_1 \to I^Y$ by*

$$\mathcal{C}_i(\lambda, r, s) = \bigvee_{i\in\Gamma} f_i(\mathcal{C}_i(f_i^{-1}(\lambda), r, s)).$$

Then:

(1) $\mathcal{C}$ is the finest intuitionistic fuzzy closure operator on Y for which each f_i is $\mathcal{C}$-map.

(2) $f\colon (Y,\mathcal{C}) \to (Z,\mathcal{C}_Z)$ *is a* $\mathcal{C}$*-map iff each* $fof_i\colon (X,\mathcal{C}_i) \to (Z,\mathcal{C}_Z)$ *is* $\mathcal{C}$*-map.*

Proof. (1) Firstly, we will show that $\mathcal{C}$ is intuitionistic fuzzy closure operator on Y. (C1), (C3) and (C4) are easily proved from the definition of $\mathcal{C}$. For (C2) we have

$$\begin{aligned}\mathcal{C}(\lambda,r,s) &= \bigvee_{i\in\Gamma} f_i(\mathcal{C}_i(f_i^{-1}(\lambda),r,s)) \\ &\geq f_i(\mathcal{C}_i(f_i^{-1}(\lambda),r,s)) \\ &\geq f_i(f_i^{-1}(\lambda)) = \lambda.\end{aligned}$$

Secondly, we have

$$\begin{aligned}\mathcal{C}(f_i(\lambda),r,s) &= \bigvee_{i\in\Gamma} f_i(\mathcal{C}_i(f_i^{-1}(\lambda),r,s)) \\ &\geq f_i(\mathcal{C}_i(f_i^{-1}(\lambda),r,s)) \\ &\geq f_i(\mathcal{C}_i(\lambda,r,s)).\end{aligned}$$

Hence $f_i\colon (X_i,\mathcal{C}_i) \to (Y,\mathcal{C})$ is $\mathcal{C}$-map.

Finally, we will show that $\mathcal{C}$ is the finest intuitionistic fuzzy closure operator on Y for which each f_i is $\mathcal{C}$-map. If $f_i\colon (X_i,\mathcal{C}_i) \to (Y,\mathcal{C}^*)$ is $\mathcal{C}$-map for each $i \in \Gamma$, then we have, for each $\lambda_i \in I^{X_i}$ and $r \in I_\circ$, $s \in I_1$, $f_i(\mathcal{C}_i(\lambda_i,r,s)) \leq \mathcal{C}^*(f_i(\lambda_i),r,s)$. It follows that

$$\begin{aligned}\mathcal{C}(\lambda,r,s) &= \bigvee_{i\in\Gamma} f_i(\mathcal{C}_i(f_i^{-1}(\lambda),r,s)) \\ &\leq \bigvee_{i\in\Gamma} \mathcal{C}^*(f_i(f_i^{-1}(\lambda),r,s)) \\ &= \mathcal{C}^*(\lambda,r,s).\end{aligned}$$

(2) ($\Rightarrow$). Trivial.

($\Leftarrow$). Let $fof_i\colon (X_i,\mathcal{C}_i) \to (Z,\mathcal{C}_Z)$ be $\mathcal{C}$-map, then we have

$$(fof_i)(\mathcal{C}_i(\lambda_i,r,s) \leq \mathcal{C}_Z(fof_i(\lambda_i),r,s)$$

It follows that

$$\begin{aligned}f(\mathcal{C}(\lambda,r,s)) &= f(\bigvee_{i\in\Gamma} f_i(\mathcal{C}_i(f_i^{-1}(\lambda),r,s)))) \\ &= \bigvee_{i\in\Gamma} f(f_i(\mathcal{C}_i(f_i^{-1}(\lambda),r,s))) \\ &\leq \bigvee_{i\in\Gamma} \mathcal{C}_Z(f(f_i(f_i^{-1}(\lambda))),r,s) \\ &= \mathcal{C}_Z(f(\lambda),r,s).\end{aligned}$$

□

From Theorem 18, we can state the following definition.

Definition 19. The structure $\mathcal{C}$ is called the final intuitionistic fuzzy closure operator on Y associated the families $\{(X_i, \mathcal{C}_i)\}$ and $(f_i)_{i\in\Gamma}$.

Corollary 20. (Sum intuitionistic fuzzy closure spaces) *Let $\{(X_i, \mathcal{C}_i)\}_{i\in\Gamma}$ be a family of intuitionistic fuzzy closure spaces, for different $i, j \in \Gamma$, X_i and X_j be disjoint, $X = \cup_{i\in\Gamma} X_i$. Let $id\colon X_i \to X$ be the identity map for each $i \in \Gamma$. Define the function $\mathcal{C}\colon I^X \times I_\circ \times I_1 \to I^X$ by*

$$\mathcal{C}(\lambda, r, s) = \bigvee_{i\in\Gamma} id_i(\mathcal{C}_i(id_i^{-1}(\lambda), r, s)).$$

Then:

(1) $\mathcal{C}$ is the finest intuitionistic fuzzy closure operator on X for which each id_i is $\mathcal{C}$-map.

(2) $f\colon (Y, \mathcal{C}) \to (Z, \mathcal{C}_Z)$ is a $\mathcal{C}$-map iff each $foid_i\colon (X, \mathcal{C}_i) \to (Z, \mathcal{C}_Z)$ is $\mathcal{C}$-map.

Definition 21. Let $(X, \mathcal{C})$ be an intuitionistic fuzzy closure space and Y a set. Let $f\colon X \to Y$ be a surjective function. Define the function $\mathcal{C}^f\colon I^Y \times I_\circ \times I_1 \to I^Y$ by

$$\mathcal{C}^f(\lambda, r, s) = f(\mathcal{C}(f^{-1}(\lambda), r, s)).$$

The $(Y, \mathcal{C}^f)$ induced by f is called the intuitionistic fuzzy quotient space of $(X, \mathcal{C})$ and the function f is called an intuitionistic fuzzy quotient map.

Theorem 22. *Let Y be a set and $\{(X_i, \tau_i, \tau*_i)\}_{i\in\Gamma}$ be a family of intuitionistic fuzzy topological spaces. let $f\colon X_i \to Y$ be surjective function for each $i \in \Gamma$ and $\{(X_i, \mathcal{C}_{\tau_i,\tau*_i})\}_{i\in\Gamma}$ a family of intuitionistic fuzzy closure spaces induced by $\{(X_i, \tau_i, \tau*_i)\}_{i\in\Gamma}$. Define the functions τ and $\tau*$ on Y as Theorem 18 and the function $\mathcal{C}\colon I^Y \times I_\circ \times I_1 \to I^Y$ by*

$$\mathcal{C}(\lambda, r, s) = \bigvee_{i\in\Gamma} f_i(\mathcal{C}_{\tau_i,\tau*_i}(f_i^{-1}(\lambda), r, s)).$$

Then:

(1) $\mathcal{C}$ is finer than $\mathcal{C}_{\tau,\tau*}$ induced by $(\tau, \tau*)$.

(2) $(\tau_{\mathcal{C}}, \tau*_{\mathcal{C}}) = (\tau, \tau*)$.

Proof. (1) Since $f_i\colon (X, \tau_i, \tau*_i) \to (Y, \tau, \tau*)$ is intuitionistic fuzzy continuous for each $i \in \Gamma$, by Theorem 8, $f_i\colon (X_i, \mathcal{C}_{\tau_i,\tau*_i}) \to (Y, \mathcal{C}_{\tau,\tau*})$ is a $\mathcal{C}$-map for each $i \in \Gamma$. From Theorem 18, $\mathcal{C}$ is finer than $\mathcal{C}_{\tau,\tau*}$.

(2) First, we will show that for each $i \in \Gamma$, $f_i\colon (X_i, \tau_i, \tau*_i) \to (Y, \tau_{\mathcal{C}}, \tau*_{\mathcal{C}})$ is intuitionistic fuzzy continuous. Suppose there exists $\lambda \in I^Y$ such that $\tau*_{\mathcal{C}}(\lambda) \nleq \tau*_i(f_i^{-1}(\lambda))$ and $\tau_{\mathcal{C}}(\lambda) \nleq \tau_i(f_i^{-1}(\lambda))$. Then there exists $r_o \in I_\circ$, $s_o \in I_1$ with $\mathcal{C}(\underline{1} - \lambda, r, s) = \underline{1} - \lambda$ such that $\tau*_{\mathcal{C}}(\lambda) \geq r_\circ > \tau*_i(f_i^{-1}(\lambda))$ and $\tau_{\mathcal{C}}(\lambda) \leq s_\circ < \tau_i(f_i^{-1}(\lambda))$. On the other hand, we have

$$\begin{aligned}\underline{1} - \lambda &= \mathcal{C}(\underline{1} - \lambda, r, s)\\ &= \bigvee_{i\in\Gamma} f_i(\mathcal{C}_{\tau_i,\tau*_i}(f_i^{-1}(\underline{1} - \lambda), r, s))\\ &\geq f_i^{-1}(\mathcal{C}_i(\underline{1} - f_i^{-1}(\lambda), r_\circ, s_\circ).\end{aligned}$$

It implies

$$\begin{aligned} f_i^{-1}(\lambda) &= f_i^{-1}(\underline{1}-\lambda) \\ &\geq f_i^{-1}(f_i(\mathcal{C}_{\tau,\tau*}(\underline{1}-f_i^{-1}(\lambda), r_\circ, s_\circ))) \\ &\geq \mathcal{C}_{\tau_i,\tau*_i}(f_i^{-1}(\lambda), r_\circ, s_\circ). \end{aligned}$$

But we have $\mathcal{C}_{\tau_i,\tau*_i}(f_i^{-1}(\lambda), r_\circ, s_\circ) = f_i^{-1}(\lambda)$. Since $\tau_{\mathcal{C}_{\tau_i}} = \tau_i$ and $\tau*_{\mathcal{C}_{\tau*_i}} = \tau*_i$, we have $\tau_i(f_i^{-1}(\lambda)) \geq r_\circ$ and $\tau*_i(f_i^{-1}(\lambda)) \leq s_\circ$, which is a contradiction. Hence $f_i\colon (X_i, \tau_i, \tau*_i) \to (Y, \tau_{\mathcal{C}}, \tau*_{\mathcal{C}})$ is intuitionistic fuzzy continuous.

Secondly, since $(\tau, \tau*)$ is the final intuitionistic fuzzy topology on Y, by Theorem 12, we have $\tau_{\mathcal{C}}(\lambda) \leq \tau(\lambda)$ and $\tau*_{\mathcal{C}}(\lambda) \geq \tau*(\lambda)$ for all $\lambda \in I^Y$. Conversely, since $\tau_{\mathcal{C}_{\tau,\tau*}} = \tau$ and $\tau*_{\mathcal{C}_{\tau,\tau*}} = \tau*$, we only show that $\tau_{\mathcal{C}_{\tau,\tau*}}(\lambda) \leq \tau_{\mathcal{C}}(\lambda)$ for all $\lambda \in I^Y$. Suppose there exists $\lambda \in I^Y$ such that $\tau_{\mathcal{C}_{\tau,\tau*}}(\lambda) \not\geq \tau_{\mathcal{C}}(\lambda)$ and $\tau*_{\mathcal{C}_{\tau*}}(\lambda) \not\leq \tau*_{\mathcal{C}}(\lambda)$. Then there exist $r_\circ \in I_\circ$, $s_\circ \in I_1$ with $\mathcal{C}_{\tau,\tau*}(\underline{1}-\lambda, r_\circ, s_\circ) = \underline{1}-\lambda$ such that $\tau_{\mathcal{C}_{\tau,\tau*}}(\lambda) \geq r_\circ > \tau_{\mathcal{C}}(\lambda)$ and $\tau*_{\mathcal{C}_{,\tau\tau*}}(\lambda) \leq s_\circ < \tau*_{\mathcal{C}}(\lambda)$. On the other hand, we have

$$\underline{1}-\lambda = \mathcal{C}_{\tau,\tau*}(\underline{1}-\lambda, r_\circ, s_\circ) \geq \mathcal{C}(\underline{1}-\lambda, r_\circ, s_\circ).$$

Hence $\mathcal{C}(\underline{1}-\lambda, r_\circ, s_\circ) = \underline{1}-\lambda$. So, $\tau_{\mathcal{C}}(\lambda) \geq r_\circ$ and $\tau*_{\mathcal{C}}(\lambda) \leq s_\circ$, which is a contradiction. □

Example 23. *Let $X = \{a, b\}$, $Y = \{x\}$ be sets. Define $\tau, \tau* : I^X \to I$ as follows:*

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{1}, \underline{0}; \\ \frac{1}{2}, & \text{if } \lambda = \underline{1} - a_{0.5} \text{ or } \underline{1} - b_{0.7}; \\ \frac{1}{2}, & \text{if } \lambda = \underline{1} - (a_{0.5} \vee b_{0.7}); \\ 0, & \text{otherwise.} \end{cases}$$

$$\tau*(\lambda) = \begin{cases} 0, & \text{if } \lambda = \underline{1}, \underline{0}; \\ \frac{1}{2}, & \text{if } \lambda = \underline{1} - a_{0.5} \text{ or } \underline{1} - b_{0.7}; \\ \frac{1}{2}, & \text{if } \lambda = \underline{1} - (a_{0.5} \vee b_{0.7}); \\ 1, & \text{otherwise.} \end{cases}$$

From Theorem 4, we obtain
$\mathcal{C}_{\tau,\tau*} : I^X \times I_\circ \times I_1 \to I^X$ as follows:

$$\mathcal{C}_{\tau,\tau*}(\lambda, r, s) = \begin{cases} \underline{0}, & \text{if } \lambda = \underline{0},\ r \in I_\circ,\ s \in I_1; \\ a_{0.5}, & \text{if } \underline{0} \neq \lambda \leq a_{0.5},\ 0 < r \leq \frac{1}{2},\ \frac{1}{2} < s \leq 1; \\ b_{0.7}, & \text{if } \underline{0} \neq \lambda \leq b_{0.7},\ 0 < r \leq \frac{1}{2},\ \frac{1}{2} < s \leq 1; \\ a_{0.5} \vee b_{0.7}, & \text{if } \lambda \leq a_{0.5} \vee b_{0.7},\ \lambda \not\leq a_{0.5},\ \lambda \not\leq b_{0.7}, \\ & 0 < r \leq \frac{1}{2},\ \frac{1}{2} < s \leq 1; \\ \underline{1}, & \text{otherwise.} \end{cases}$$

From Corollary 15, we have the quotient space $\tau^f, \tau*^f$ on Y of $(X, \tau, \tau*)$ as follows:

$$\tau^f(\nu) = \tau(f^{-1}(\nu)) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \underline{1}, \\ 0, & \text{otherwise.} \end{cases}$$

$$\tau*^f(\nu) = \tau*(f^{-1}(\nu)) = \begin{cases} 0, & \text{if } \nu = \underline{0}, \underline{1}, \\ 1, & \text{otherwise.} \end{cases}$$

From Theorem 4, we have

$$\mathcal{C}_{\tau^f,\tau*^f}(\nu,r,s)=\begin{cases}\underline{0}, & \textit{if } \nu=\underline{0},\ r\in I_\circ,\ s\in I_1,\\ \underline{1}, & \textit{otherwise.}\end{cases}$$

Since $\mathcal{C}^f(\nu,r,s)=f(\mathcal{C}_{\tau,\tau*}(f^{-1}(\nu),r,s))$ *from Theorem 22, we have*

$$\mathcal{C}^f(\nu,r,s)=\begin{cases}\underline{0}, & \textit{if } \nu=\underline{0},\ r\in I_\circ,\ s\in I_1,\\ x_{0.7}, & \textit{if } \underline{0}\neq\nu\leq x_{0.5},\ 0<r\leq\frac{1}{2},\ \frac{1}{2}<s\leq 1;\\ \underline{1}, & \textit{otherwise.}\end{cases}$$

Hence $\mathcal{C}_f$ *is finer than* $\mathcal{C}_{\tau^f,\tau*^f}$ *and* $\mathcal{C}_{\tau^f,\tau*^f}\neq\mathcal{C}^f$. *Moreover,* $\mathcal{C}^f$ *is topological from Theorem 4. Since*

$$x_{0.7}=\mathcal{C}^f(x_{0.5},\frac{1}{3},\frac{1}{3})\neq\mathcal{C}^f(\mathcal{C}^f(x_{0.5},\frac{1}{3},\frac{1}{3}),\frac{1}{3},\frac{1}{3})=1,$$

an intuitionistic fuzzy closure operator $\mathcal{C}^f$ *is not topological. From Theorem 5, we have*

$$\tau_{\mathcal{C}^f}(\nu)=\begin{cases}1, & \textit{if } \nu=\underline{0},\ \textit{or}\,\underline{1};\\ 0, & \textit{otherwise.}\end{cases},\qquad \tau_{\mathcal{C}^f}(\nu)=\begin{cases}0, & \textit{if } \nu=\underline{0},\ \textit{or}\,\underline{1};\\ 1, & \textit{otherwise..}\end{cases}$$

Hence $(\tau_{\mathcal{C}^f},\tau*_{\mathcal{C}^f})=(\tau^f,\tau*^f)$.

Theorem 24. *Let* $\{(X_i,\mathcal{C}_i)\}_{i\in\Gamma}$ *be a family of pairwise disjoint* (r,s)*-fuzzy-*T_0 *intuitionistic fuzzy closure spaces. Then their sum intuitionistic fuzzy closure space* $(X,\mathcal{C})$ *is also* (r,s)*-fuzzy-*T_0.

Proof. (1) $x,y\in X_i$, $i\in\Gamma$. Since $(X_i,\mathcal{C}_i)$ is $(r,s)-T_0$, there exists $\lambda\in I^{X_i}$ such that $\mathcal{C}_i(\lambda,r,s)=\lambda$ and $\lambda(x)\neq\lambda(y)$, since $\lambda\in I^X$. By using corollary 20, we have $\mathcal{C}(\lambda,r,s)=\lambda$.

(2) $x\in X_i$ and $y\in X_j$, $i,j\in\Gamma$, $i\neq j$. Let $\lambda\in I^{X_i}$, it can be easily checked that $\mathcal{C}(\lambda,r,s)=\lambda$ such that $\lambda(x)\neq\lambda(y)$, where $\mathcal{C}_i(\lambda,r,s)=\lambda$. Hence the sum $(X,\mathcal{C})$ is (r,s)-fuzzy-T_0. □

Theorem 25. *Let* $\{(X_i,\mathcal{C}_i)\}_{i\in\Gamma}$ *be a family of pairwise disjoint* (r,s)*-fuzzy-*T_1 *intuitionistic fuzzy closure spaces. Then their sum intuitionistic fuzzy closure space* $(X,\mathcal{C})$ *is also* (r,s)*-fuzzy-*T_1.

Proof. Let $x\in X=\cup X_i$, then $x\in X_{i_0}$ for some $i_o\in\Gamma$. But $(X_{i_0},\mathcal{C}_{i_0})$ is (r,s)-fuzzy-T_0, $\mathcal{C}_{i_0}(\chi_{\{x\}},r,s)\leq\chi_{\{x\}}$. Since $\chi_{\{x\}}\in I^X$. By using Corollary 20, $\mathcal{C}(\chi_{\{x\}},r,s)\leq\chi_{\{x\}}$. Hence the sum $(X,\mathcal{C})$ is (r,s)-fuzzy-T_1 space. □

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Simulation of Rotational Flows in Cylindrical Vessel with Rotating Single Stirrer

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Abstract. The purpose of this research is to investigate the influence of rotational speed and rotational direction of stirrer on the hydrodynamics and compare behavior against previously simulated numerical results in the dissolution vessel with fixed stirrer. The numerical simulation of two-dimensional incompressible complex flows of Newtonian fluid passed a stationary and rotating single stirrer within a cylindrical vessel is presented. The context is one, relevant to the food industry, of mixing fluid within a cylindrical vessel, where stirrer is located on the lid of the vessel eccentrically configured. Here, the motion is considered as driven by the rotation of the outer vessel wall, with various rotational speeds of vessel and stirrer. The numerical method adopted is based on a finite element semi-implicit time-stepping Taylor-Galerkin/pressure-correction scheme, posed in a cylindrical polar coordinate system. Numerical solutions are sought for Newtonian fluid. Variation with increasing speed of vessel, change in speed of stirrer and change in rotational direction of stirrer in mixer geometry are analysed, with respect to the flow structure and pressure drop.

Key Words: Numerical Simulation, Finite Element Method, Mixing Flows, Newtonian Fluids, Rotating Flow, Co-rotating Stirrer, Contra-rotating Stirrer.

1. Introduction

The rotational mixing in stirred vessel for the optimal design is of industrial importance, usually industrial problems are much harder to tackle, particularly in the field of chemical process applications, such as powder mixing processes [1], granular mixing, mixing of paper pulp in paper industry, mixing of dough in a food processing industry [2, 3] and many other industrial processes. In many mixing processes the complicating factors are the use of the fluids which exhibits very complex rheological behavior, the use of agitators with stirrer in fact that the agitator may be operated in the transitional regime and the direction of rotational speed of stirrer. The present problem is one of this form, expressed as the flow between an outer rotating cylindrical vessel wall and a single stationary and rotating cylindrical stirrer in both co-rotating and contra-rotating directions. Stirrer is located on the mixing vessel lid, and placed in an eccentric position with respect to the central cylindrical axis of the vessel. Under two-dimensional assumptions, the vessel essentially is considered to have infinite height. Elsewhere, the finite vessel problem in three-dimensions [3]-[7] has been analysed. In two-dimension, similar problem is also investigated with different number and shapes of stirrers [8, 9]. The motivation for this work is to advance fundamental technology modelling of the dough kneading with the ultimate aim to predict the optimal design of dough mixers themselves, hence, leading to efficient dough processing.

This problem has similarity to the classical journal bearing problem, associated with lubrication theory, involving a degree of eccentricity between outer and inner cylinders. The journal bearing problem has been solved for viscoelastic fluids employing finite element methods [10, 11] and spectral element methods [12]. Dris and Shaqfeh [10, 11] with finite elements, observed purely elastic flow instabilities in eccentric cylinder flow geometries. The velocity profiles vary as a function of eccentricity, azimuthal coordinate, and the ratio of cylindrical rotation rates. The local flow dynamics span over the entire range of flows from Taylor-Couette flow to Dean flow. The onset of flow instabilities has been shown to be the result of non-local effects in the flow [10]. Global effects drastically alter the hoop stresses in the base flow.

The present study adopts a semi-implicit Taylor-Galerkin/Pressure-Correction (TGPC) finite element time-marching scheme, which has been developed and refined over the last two decades. This scheme, initially conceived in sequential form, is appropriate for the simulation of incompressible Newtonian flows [14]-[17].

In Section 2, the complete problem is specified and the governing equations are described in Section 3. This is followed, in Section 4, by an outline of the TGCP numerical method employed for the simulations. Simulation results are presented in Section 5 and our conclusions are drawn in Section 6.

2. Problem Specification

The problem investigated here is two-dimensional mixing flows of Newtonian fluids, of relevance to the food industry such as occurs in dough kneading. Such flows are rotating, driven by the rotation of the outer containing cylindrical-shaped vessel. The stirrer is held in place by being attached to the lid of the vessel. In reality, within the industrial process, the lid of the vessel would rotate with stirrer attached. With a single stirrer, an eccentric configuration is adopted.

Initially, the problem is analysed for rotating flow between stationary stirrer in rotating cylindrical vessel, to validate the finite element discretisation in this cylindrical polar co-ordinate system to compare the numerical results against results obtained in previous investigations [7, 8]. Subsequently, two alternative rotational directions (Co-rotating and contra-rotating) of stirrer are investigated in a rotating cylindrical vessel. Throughout Newtonian fluid is considered.

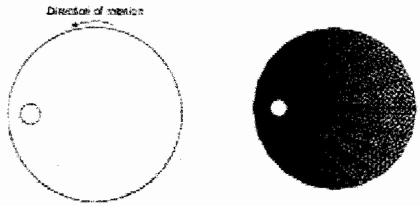


FIGURE 1. Eccentric rotating cylinder flow, with one stationary and rotating stirrer; Domain and finite element mesh

Domain and finite element mesh for the problem involved is displayed in Figure-1. In pervious investigations [7, 8], for mesh convergence studies, three meshes were generated, adopting a hierarchical mesh refinement technique. In this technique, each parent element of the coarser mesh is divided into four child elements. Between the solutions of any variable on two consecutive refined meshes, a discrepancy of order one percent tolerance was fixed. Due to enhancement in power of computation and based on pervious findings, the refined mesh M3 [7] is reasonably adoption for smooth and accurate solutions. The total number of elements, nodes and degrees-of-freedom are 3840, 7840 and 17680 respectively.

To provide a well-posed specification for each flow problem, it is necessary to prescribe appropriate initial and boundary conditions. Simulations commence from a quiescent initial state. Boundary conditions are taken as follows. For stationary stirrer the fluid may stick to the solid surfaces, so that the components of velocity vanish on the solid inner stirrer sections of the boundary ($v_r = 0$ and $v_\theta = 0$). For non-stationary stirrer, fixed constant velocity boundary conditions are applied. For co-rotating stirrer, vanishing radial velocity component ($v_r = 0$) is fixed and for azimuthal velocity component is fixed with three different non-dimensional speeds ($v_\theta = 0.5,\ 1$ and 2 unit). Similarly, for contra-rotating stirrer only azimuthal velocity component is changed and fixed in reverse direction (i.e., $v_\theta = -0.5,\ -1$ and -2 unit). On the outer rotating cylinder vessel a fixed constant velocity boundary condition is applied ($v_r = 0$ and $v_\theta = 1$ unit), and a pressure level is specified as zero for both co-rotating and contra-rotating stirrer on vessel wall. For stream function, outer cylinder is fixed zero and at inner stirrer is left unconstrained, being solutions on closed streamlines.

3. Governing System of Equations

The two-dimensional isothermal flow of incompressible Newtonian fluid can be modelled through a system comprising of the generalised momentum transport and conservation of mass equations. The coordinate reference frame is a two-dimensional cylindrical coordinate system taken over domain $\mathbf{\Omega}$. In the absence of body forces, the system of equations can be represented through the conservation of mass equation, as,

$$\nabla \cdot \mathbf{u} = 0, \tag{3.1}$$

the conservation of momentum transport equation, as,

$$\rho\frac{\partial \mathbf{u}}{\partial t} = \nabla \cdot \sigma - \rho \mathbf{u} \cdot \nabla \mathbf{u}, \tag{3.2}$$

where, $\mathbf{u}$ is the fluid velocity vector field, σ is the Cauchy stress tensor, ρ is the fluid density, t represents time and ∇ the spatial differential operator. The Cauchy stress tensor can be expressed in the form:

$$\sigma = -\mathbf{p}\delta + \mathbf{T}, \tag{3.3}$$

where p is the isotropic fluid pressure, δ is the Kronecker delta tensor, whilst $\mathbf{T}$ is the total stress tensor. For constant viscosity (μ) Newtonian fluids, the stress tensor $\mathbf{T}$ is given as

$$\mathbf{T} = 2\mu\mathbf{d}, \tag{3.4}$$

where the rate-of-strain tensor $\mathbf{d} = \frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^{\dagger}]$, and $\dagger$ represents the transpose operator.

Relevant non-dimensional Reynolds number is defined as:

$$Re = \frac{\rho V_c R}{\mu_c}, \tag{3.5}$$

The characteristic velocity V_c is taken to be the speed of the vessel, the characteristic length scale is the radius, R, of a stirrer and the characteristic viscosity μ_c is the zero shear-rate viscosity.

Appropriate scaling in each variable takes the form. At a characteristic rotational speed 50 rpm and zero shear viscosity of 105 Pa s, scaling yields dimensional variables $p = 2444p*$.

4. Numerical Method

As stated earlier, a time-marching finite element algorithm is employed in this investigation to compute numerical solutions through a semi-implicit Taylor-Galerkin /pressure-correction scheme [15], [21], [16]-[18], based on a fractional-step formulation. This involves discretisation, first in the temporal domain, adopting a Taylor series expansion in time and a pressure-correction operator-split, to built a second-order time-stepping scheme. Spatial discretisation is achieved via Galerkin approximation for the both momentum and stress constitutive equations. The finite element basis functions employed are quadratic (ϕ_j) for velocities, and linear (ψ_k) for pressure. Corresponding integrals are evaluated by a seven point Gauss quadrature rule.

Stage 1a:

$$\begin{aligned}[\frac{2\mathbf{M}}{\Delta t} + \frac{\mathbf{S}}{2Re}](\mathbf{V_r}^{n+\frac{1}{2}} - \mathbf{V_r}^{n}) &= {L_r}^{\dagger}P^n - \frac{\mu_c}{Re}\{S_{rr}V_r + S_{r\theta}V_\theta\}^n \\ &- \{\mathbf{N}(\mathbf{V})V_r - N_1(V_\theta)V_\theta\}^n\end{aligned}$$

$$\begin{aligned}[\frac{2\mathbf{M}}{\Delta t} + \frac{\mathbf{S}}{2Re}](\mathbf{V}_\theta^{n+\frac{1}{2}} - \mathbf{V}_\theta^{n}) &= {L_\theta}^{\dagger}P^n - \frac{\mu_c}{Re}\{S^{\dagger}_{r\theta}V_r + S_{\theta\theta}V_\theta\}^n \\ &- \{\mathbf{N}(\mathbf{V})V_\theta - N_1(V_\theta)V_r\}^n\end{aligned}$$

Stage 1b:

$$[\frac{2\mathbf{M}}{\Delta t}+\frac{\mathbf{S}}{2Re}](\mathbf{V_r}^{*}-\mathbf{V_r}^{n}) = L_r{}^{\dagger}P^n-\frac{\mu_c}{Re}\{S_{rr}V_r+S_{r\theta}V_\theta\}^n - \{\mathbf{N}(\mathbf{V})V_r-N_1(V_\theta)V_\theta\}^{n+\frac{1}{2}}$$

$$[\frac{2\mathbf{M}}{\Delta t}+\frac{\mathbf{S}}{2Re}](\mathbf{V}_\theta{}^{*}-\mathbf{V}_\theta{}^{n}) = L_\theta{}^{\dagger}P^n-\frac{\mu_c}{Re}\{S^{\dagger}_{r\theta}V_r+S_{\theta\theta}V_\theta\}^n - \{\mathbf{N}(\mathbf{V})V_\theta-N_1(V_\theta)V_r\}^{n+\frac{1}{2}}$$

Stage 2:

$$\theta\ \mathbf{KQ}^{n+1}=\frac{1}{\Delta t}\mathbf{LV}^{*},$$

Stage 3:

$$\frac{1}{\Delta t}\mathbf{M}(\mathbf{U}^{n+1}\ -\ \mathbf{U}^{*})=-\theta\mathbf{L}^{\dagger}\mathbf{Q}^{n+1},$$

where $\mathbf{V}^n$ are the nodal velocity vector at time t^n, respectively; $\mathbf{V}*$ is an intermediate non-divergence-free velocity vector, $\mathbf{V}^{n+1}$ is a divergence-free velocity vector at time step t^{n+1}. $\mathbf{P}^n$ is a pressure vector and $\mathbf{Q}^{n+1}=\mathbf{P}^{n+1}$ - $\mathbf{P}^n$ is a pressure difference vector. **M** is a mass matrix, N(V) is a convection matrix, **K** is a pressure stiffness matrix, **L** is a divergence/pressure gradient matrix and **S** is a momentum diffusion matrix. Utilising implied inner product notation < . > for domain integrals, the above system involves matrices of the form:

Mass matrix:

$$\mathbf{M}=\int_\Omega \phi_i\phi_j r d\Omega,$$

Non-linear advection matrices:

$$\mathbf{N}(\mathbf{V})=\int_\Omega \phi_i(\phi_l V_r^l\frac{\partial\phi_j}{\partial r}+\frac{\phi_l V_\theta^l}{\psi_k R_k}\frac{\partial\phi_j}{\partial\theta})rd\Omega,$$

and

$$\mathbf{N_1}(V_\theta)=\int_\Omega \phi_i\phi_l V_\theta^l\phi_j r d\Omega,\ \text{where } i,j,l=1,...,6$$

Pressure stiffness matrix:

$$\mathbf{K}_{km}=\int_\Omega \nabla\psi_k\nabla\psi_m r d\Omega,\ \text{where } k,m=1,2,3,$$

Pressure gradient matrix:

$$\mathbf{L}_{mi}=\int_\Omega \psi_m\nabla\phi_i r d\Omega$$

Momentum diffusion matrices:

$$\mathbf{S}=\begin{pmatrix} S_{rr} & S_{r\theta} \\ S^{\dagger}_{r\theta} & S_{\theta\theta}\end{pmatrix},$$

where † is transpose of the matrix and

$$\begin{aligned}
\mathbf{S}_{rr} &= \int_{\Omega}(2\frac{\partial\phi_i}{\partial r}\frac{\partial\phi_j}{\partial r}+\frac{2\phi_i\phi_j}{(\psi_kR_k)^2}+\frac{1}{(\psi_kR_k)^2}\frac{\partial\phi_i}{\partial\theta}\frac{\partial\phi_j}{\partial\theta})rd\Omega\\
\mathbf{S}_{r\theta} &= \int_{\Omega}(\frac{1}{\psi_kR_k}\frac{\partial\phi_i}{\partial\theta}\frac{\partial\phi_j}{\partial r}+\frac{2\phi_i}{(\psi_kR_k)^2}\frac{\partial\phi_j}{\partial\theta}-\frac{1}{(\psi_kR_k)^2}\frac{\partial\phi_i}{\partial\theta}\phi_j)rd\Omega,\\
\mathbf{S}_{\theta\theta} &= \int_{\Omega}(\frac{\partial\phi_i}{\partial r}\frac{\partial\phi_j}{\partial r}+\frac{\phi_i\phi_j}{(\psi_kR_k)^2}-\frac{1}{\psi_kR_k}\phi_i\frac{\partial\phi_j}{\partial r}-\frac{1}{\psi_kR_k}\frac{\partial\phi_i}{\partial r}\phi_j\\
&+ \frac{2}{(\psi_kR_k)^2}\frac{\partial\phi_i}{\partial\theta}\frac{\partial\phi_j}{\partial\theta})rd\Omega.
\end{aligned}$$

Repeated indices imply summation, taken over i, j and l for all velocity nodal points, and k, m for all vertex pressure nodal points on the triangular meshes. $\mathbf{F}^n$ is a forcing function vector due to body force and boundary conditions at time t_n (which vanishes here). To give the precise second-order form of the pressure-correction algorithm the Crank-Nicolson coefficient θ is taken as one half. Stage one and three are governed by augmented mass matrices and solved by a Jacobi iterative method that necessitates using only a small fixed number of mass iterations, typically three. At stage two, a Poisson equation emerges, with a matrix that is symmetric and positive definite. It possesses a banded structure, for which it is appropriate to employ a direct Choleski method. The bandwidth may be optimised by using an algorithm such as that of Sloan [23]. Here, n denotes the time step index. Velocity components at the half time step $n+\frac{1}{2}$ are computed in step 1a from data gathered at level n and in step 1b an intermediate non-solenoidal velocity field $\mathbf{V}*$ is computed at the full time step, using the solutions at the level n and $n+\frac{1}{2}$.

For pressure this leads naturally to a second step, where a Poisson equation is solved for the pressure-difference from a non-solenoidal velocity field $\mathbf{V}^*$ over the full time step. Solving for temporal pressure-difference has some specific advantages with respect to boundary conditions at the second step, see [15]. On a third and final step, a solenoidal velocity is captured at the end of the time-step cycle, computed from the pressure-difference field of step 2. For finite element approximation, the generalised weighting function w_i replaces ϕ_i, for the Galerkin formulation of momentum equation. In general, the time-step, Δt, is taken as 10^{-2}, so as to satisfy a local Courant Condition constraint [18]-[21] and a relative solution-increment time-step termination tolerance of 10^{-5} is enforced. The implicit splitting of pressure terms in the pressure correction leads to the factor θ, and a second-order scheme if taken as $\frac{1}{2}$. In addition, the Crank-Nicolson splitting of diffusion terms at stage-1, incorporates the implicit diffusion contribution to the left-hand-side of the equation.

5. Numerical Results

The numerical results are investigated from two distinct points of view: changing rotational speed and direction of stirrer. This leads to analysis with respect to increasing viscosity levels (decrease of Reynolds number) and comparison of flow structure and pressure variation across problem instances.

The predicted solutions are displayed for Newtonian fluid through contours plots of streamlines, and pressure isobars. Pressure isobar patterns are plotted with eleven contours, from the minimum to maximum value, over a fixed range. Streamlines are plotted in two regions: first from the vessel wall to the stirrer perimeter, seven contours are plotted ($\Psi = 0,\ 0.5,\ 1.0,\ 1.5,\ 2.0,\ 2.5$ and 2.95 units) and second from the stirrer to the centre of the recirculation, $3.05 \leq \Psi \leq \Psi_{max}$ at increments of 0.3 units. Comparative diagnostics may be derived accordingly.

Various increasing levels of zero-shear viscosities μ_c (characteristic) are considered, from which Reynolds number is computed, as defined above. For Reynolds numbers of $Re = 8.0$, $Re = 0.8$ and $Re = 0.08$, the corresponding zero shear viscosities are $\mu_c = 1.05$ Pa s, $\mu_c = 10.5$ Pa s and $\mu_c = 105.0$ Pa s. Of these levels, a range of material properties is covered from those for model fluids, to model dough, to actual dough, respectively.

5.1. **Flow Patterns and Pressure Differential for Stationary Stirrer with Increasing Inertia.** The effect of increasing Reynolds number upon streamline patterns on left and pressure differential on right isobars are represented in contour plots for stationary stirrer in figure-2. Computations are carried out at $Re = 0.08$, $Re = 0.8$ and $Re = 8$. At a low level of inertia, $Re = 0.08$, an intense recirculating region forms in the centre of the vessel, parallel to the stirrer and symmetrically intersecting the diameter that passes through the centres of the vessel and stirrer. Flow structure remains unaffected as Reynolds number rises to values of O(1); hence we suppress this data. However, upon increasing Reynolds number up to eight, so O(10), inertia takes hold and the recirculation region twist and shifts towards the upper-half plane, vortex intensity wanes and the vortex eye is pushed towards the vessel wall. The flow becomes asymmetric as a consequence of the shift in vortex core upwards. The diminishing trend in vortex intensity is tabulated in Table-1.

TABLE 1. Vortex intensity for Newtonian fluids: ($\mu_c = 105,\ 10.5$ and 1.05 Pas)

Problem	Speed	Re=0.08		Re=0.8		Re=8.0	
		Min.	*Max.*	*Min.*	*Max.*	*Min.*	*Max.*
Stationary Stirrer	Zero	0.00	5.091	0.00	5.087	0.00	4.852
Co-rotating stirrer	Double	0.00	10.259	0.00	10.294	0.00	11.482
	Same	0.00	7.295	0.00	7.299	0.00	7.299
	Half	0.00	7.295	0.00	7.295	0.00	7.295
Contra-rotating stirrer	Double	-2.884	2.865	-2.897	2.857	-3.913	2.171
	Same	0.00	3.734	0.00	3.731	0.00	3.557
	Half	0.00	4.340	0.00	4.340	0.00	4.160

Similar symmetry arguments apply across the geometry variants in pressure differential, at $Re = 0.08$, symmetric pressure isobars appear with equal magnitude in non-dimensional positive and negative extrema on the two sides (upper and lower) of the stirrer in the narrow-gap. As inertia increases from $Re = 0.8$ to $Re = 8$, asymmetric isobars are observed, with positive maximum on the top of the stirrer and negative minimum at the outer stirrer tip (near the narrow-gap), see also Table-2.

Asymmetrical flow structure is observed in all variables and across all instances as inertia increase from $Re = 0.08$ to $Re = 8.0$, recirculating flow-rate decrease by just five percent. In non-dimensional terms above $Re = 0.08$ (noting scale differences), there is increase in pressure-differential rise by as much as twenty-two percent, at $Re = 8.0$, whilst pressure differential increase on the lower part of the stirrer. For Newtonian fluid, the extrema of recirculating region along with vortex intensity and pressure differential, are tabulated for completeness in Tables (1 and 2) at all three Reynolds number values.

5.2. **Flow Patterns And Pressure Differential For Co-Rotating Stirrer With Increasing Inertia.** Equivalent field kinematic data for co-rotating stirrer with increasing Reynolds number from $Re = 0.08$ to $Re = 8.0$ is presented in figure-3, to make direct comparisons across all instances for Newtonian fluids, with particular reference to localised vortex intensity and pressure drops are tabulated in Tables (1 and 2).

In figure-3(i), for co-rotating case, stream lines are shown for decreasing speed of the stirrer (from left to right), double speed (left), same speed (centre) and half speed (right), only single vortex is formed, in contrast to the contra rotating case where three vortexes were formed, see figure 4(i). At $Re = 0.08$, doubling the speed of the stirrer the vortex is formed near to the stirrer and is much more circular and smooth in formation, but as the speed of the stirrer is reduced to half the vortex moves away from the stirrer towards the right and the centre of the vortex is circular on one side and on other side is suppressed, also showing an increase space between the centre of vortex and diameter of secondary streamline. Streamlines tend to increase in density at the edges of the stirrer. At $Re = 0.8$, the centre of the recirculating region is shifted towards the lower-half of the plane. The diameter of the vortex also increases and leaves no circulation of fluid in the centre of recirculating region. At $Re = 8.0$, the shape of the vortex centre is changed and further shifted towards the lower half of the plane. At the half speed of the stirrer and $Re = 8.0$, the shape of the recirculating region is changed and vortex centre amplifies in the size. Consequently, the fluid pushes towards vessel wall and create vacuum in the centre of the vessel.

For co-rotating instances, figure-3(ii) illustrates the pressure differential at all comparable parameter values. The pressure differentials are high and is about five times in negative extrema compare to stationary stirrer, at $Re = 8$ and small change is observed in positive maxima. Reducing the speed of stirrer from double to single speed, the pressure differentials is very low and remain in order of two for all inertial values. Subsequently, further reduction in the speed of stirrer to half virtually no change in the pressure differential is observed and remain unaltered for all Reynolds numbers values, see Table-2.

5.3. **Flow Patterns and Pressure Differential for Contra-Rotating Stirrer with Increasing Inertia.** Corresponding field kinematics data for contra-rotating stirrer situation with increasing Reynolds number at $Re = (0.08, 0.8 \text{ and } 8.0)$ the streamline contours and pressure differentials are presented in figure-4(i and ii) respectively. In figure-4(i), for contra-rotating case, streamlines are illustrated for decreasing speed of the stirrer from double (left) to half (right) against the speed of vessel the three vortices develops, two in the vicinity of stirrer, one in narrow gap and other in middle of the vessel, and third in the centre of vessel away from stirrer

TABLE 2. Pressure drop for Newtonian fluids: (μ_c = 105, 10.5 and 1.05 Pas)

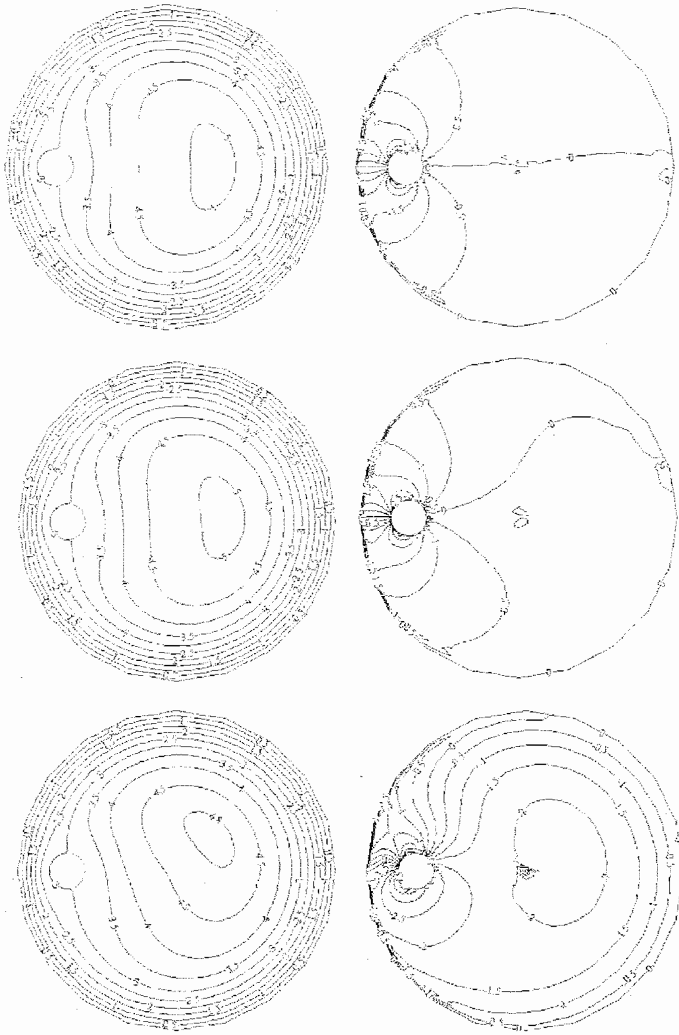
Problem	Speed	Re=0.08		Re=0.8		Re=8.0	
		Min.	*Max.*	*Min.*	*Max.*	*Min.*	*Max.*
Stationary Stirrer	Zero	-3.366	3.356	-3.421	3.325	-5.117	3.541
Co-rotating stirrer	Double	-5.234	4.916	-6.734	3.987	-26.117	4.924
	Same	-1.631	1.553	-1.984	1.209	-1.984	1.209
	Half	-1.984	1.209	-1.984	1.209	-1.984	1.209
Contra-rotating stirrer	Double	-11.578	11.380	-12.479	10.498	-28.358	6.906
	Same	-7.443	7.384	-7.719	7.127	-11.808	5.272
	Half	-5.394	5.372	-5.394	5.372	-8.218	4.849

close to vessel wall. In the narrow gap, where stirrer spins in counter direction of the vessel rotation, a small vortex appear with low vortex intensity, as the speed of stirrer decrease this vortex strength up to fifty percent high at low Re = 0.08. The augmentation in minima of vortex intensity is observed with increase in inertia, however, it suppress in maxima of vortex intensity. As inertia takes hold the second and third recirculation regions shifts the centres towards the upper-half plane of the vessel. For all Reynolds number values at double speed of stirrer, the central vortex rotate in counter direction against two other vortices. These recirculation regions have different rotational direction which is very important phenomena in homogenisation of the fluid.

For all three instances, comparable equilibrium influence apply across the geometry variants in pressure differential, at $Re = 0.08$ and double rotational speed of stirrer, symmetric pressure isobars appear with equal magnitude in non-dimensional positive and negative extrema on both sides (upper and lower) of the stirrer in the narrow-gap as shown in figure-4(ii). The associated values of pressure differentials are tabulated in Table-2. As inertia increase from $Re = 0.8$ to $Re = 8$, asymmetric isobars are observed, with positive maxima on the top of the stirrer and negative minima at the outer stirrer tip (near the narrow-gap), see also Table-3. For the contra-rotating instance, in contrast to co-rotating case, the pressure differentials are some what symmetrical in geometry at maxima and minima at twice the speed of stirrer and at half the speed of stirrer for both inertial values $Re = 0.08$ and $Re = 0.8$. However, upon increasing Reynolds number up to eight, thus O(10), inertia takes hold the pressure differentials are observed asymmetrical, increasing the speed of the stirrer to double increases the pressure differentials more than twice in negative minima and in contrast it decrease up to thirty five percent in positive maxima. Comparing against co-rotating case at same double speed of stirrer increase in minima is merely eight percent and increase in maxima is about thirty percent.

6. CONCLUSIONS

The use of a numerical flow simulator as a prediction tool for this industrial flow problem has been successfully demonstrated. We have been able to provide



(a) Re = 0.08

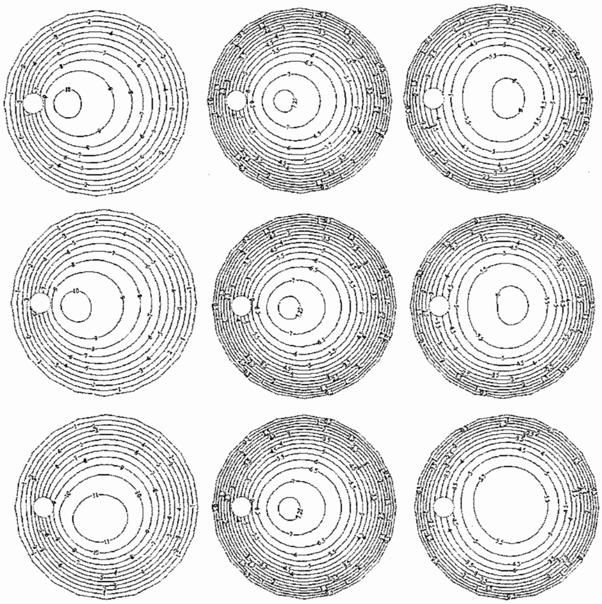
(b) Re = 0.8

(c) Re = 8.0

FIGURE 2. Streamline contours and pressure isobars of stationary stirrer with increasing Reynolds number.

physically realistic simulations for this complex mixing process using Newtonian fluid.

Addressing the rotation of the single stirrer case against stationary stirrer case in contra-rotating and co-rotating directions are being investigated with increasing inertia. For stationary stirrer case, it is clearly demonstrated that with increasing inertia fluid flow structure lose its symmetry and recirculating region move upwards in the direction of vessel motion and non-dimensional pressure differential increases. For co-rotating stirrer case, single recirculating region develops in the centre of the vessel and fluid suppressed towards vessel wall and leave big vacuum in the centre of the vessel. Whilst at twice the speed of stirrer pressure differentials are higher and lower at lower speed of stirrer. In contrast to these cases, contra-rotating case flow

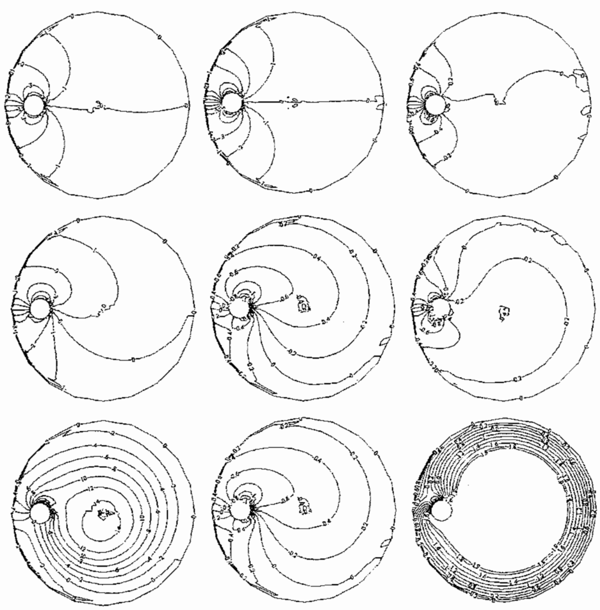


(a) Re = 0.08

(b) Re = 0.8

(c) Re = 8.0

FIGURE 3(i). Streamline contours for co-rotating stirrer case with decreasing speed of the stirrer from left to right (double ($V_\theta = 2$), same ($V_\theta = 1$) and half speed ($V_\theta = 0.5$)) against speed of vessel and increasing Reynolds number.

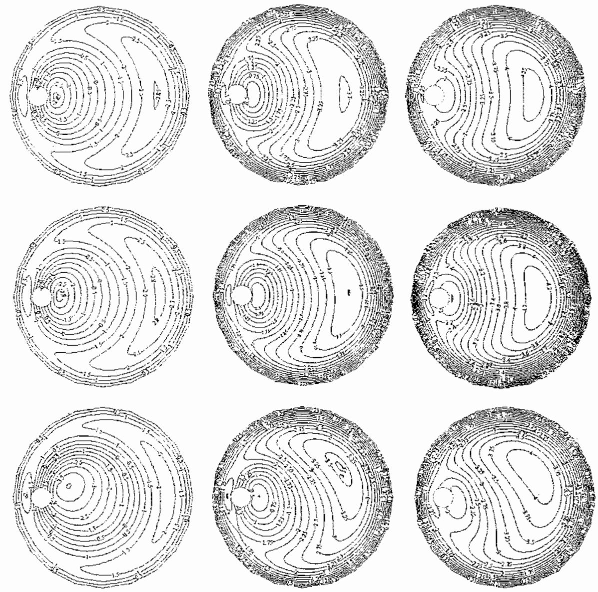


(a) Re = 0.08

(b) Re = 0.8

(c) Re = 8.0

FIGURE 3(ii). Pressure isobars for co-rotating stirrer case with decreasing speed of the stirrer from left to right (double ($V_\theta = 2$), same ($V_\theta = 1$) and half speed ($V_\theta = 0.5$)) against speed of vessel and increasing Reynolds number.

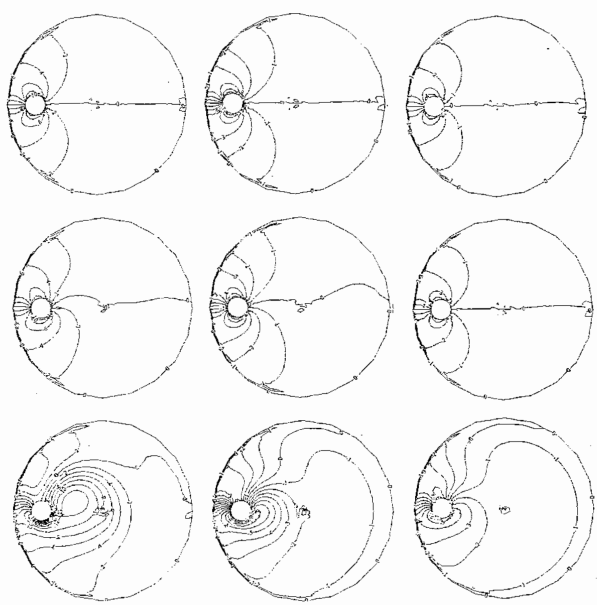


(a) Re = 0.08

(b) Re = 0.8

(c) Re = 8.0

FIGURE 4(i). Streamline contours for co-rotating stirrer case with decreasing speed of the stirrer from left to right (double ($V_\theta = 2$), same ($V_\theta = 1$) and half speed ($V_\theta = 0.5$)) against speed of vessel and increasing Reynolds number.



(a) Re = 0.08

(b) Re = 0.8

(c) Re = 8.0

FIGURE 4(ii). Pressure isobars for co-rotating stirrer case with decreasing speed of the stirrer from left to right (double ($V_\theta = 2$), same ($V_\theta = 1$) and half speed ($V_\theta = 0.5$)) against speed of vessel and increasing Reynolds number.

structure and pressure differential illustrates completely different picture. Instead of single vortex three recirculating regions have been developed with different position of vortex centres. The pressure differentials are generally higher, and similar balance in extrema is noted to those flows. However, the position, in those negative maxima exceeds to positive minima by about four times. Through the predictive capability generated, we shall be able to relate this to mixer design that will ultimately impact upon the processing of dough products.

Promising future directions of this work are investigation of rotation of two stirrers case in co-rotating, contra-rotating and mixed rotating directions, changing material properties using non-Newtonian fluids and introducing agitator in concentric configured stirrer.

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On Jordan k-Derivations of 2-Torsion Free Prime Γ_N-Rings

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Abstract. In this article, we define k-derivation and Jordan k-derivation of Γ-rings as well as different types of Γ-rings, and develop some important results relating to these concepts. In general, every Jordan k-derivation of a Γ-ring M is not a k-derivation of M. We prove that every Jordan k-derivation of a 2-torsion free prime Γ-ring (in the sense of Nobusawa) is a k-derivation.

AMS (MOS) Subject Classification Codes: Primary 16N60, Secondary 16W25, 16U80.

Key Words: Derivation, k-derivation, Jordan k-derivation, gamma ring, semiprime and prime gamma rings.

1. Introduction

Let M and Γ be two additive abelian groups. If there exists a mapping $(a, \alpha, b) \mapsto a\alpha b$ of $M \times \Gamma \times M \to M$ satisfying the following for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$:
(a) $(a + b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha + \beta)b = a\alpha b + a\beta b$, $a\alpha(b + c) = a\alpha b + a\alpha c$ and
(b) $(a\alpha b)\beta c = a\alpha(b\beta c)$,
then M is called a Γ-ring. This definition is due to Barnes [1].

If, in addition to the above, there exists a mapping $(\alpha, a, \beta) \mapsto \alpha a\beta$ of $\Gamma \times M \times \Gamma \to \Gamma$ satisfying the following for all $a, b \in M$ and $\alpha, \beta, \gamma \in \Gamma$:
(a*) $(\alpha + \beta)a\gamma = \alpha a\gamma + \beta a\gamma$, $\alpha(a + b)\beta = \alpha a\beta + \alpha b\beta$, $\alpha a(\beta + \gamma) = \alpha a\beta + \alpha a\gamma$,
(b*) $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$ and
(c*) $a\alpha b = 0$ for all $a, b \in M$ implies $\alpha = 0$,
then M is called a Γ-ring in the sense of Nobusawa[4], or simply, a Nobusawa Γ-ring and we say that M is a Γ_N-ring. Clearly, M is a Γ_N-ring always implies that Γ is an M-ring.

Let M be a Γ-ring. Then M is said to be 2-torsion free if $2a = 0$ implies $a = 0$ for all $a \in M$. Besides, M is called a prime Γ-ring if, for all $a, b \in M$, $a\Gamma M\Gamma b = 0$ implies either $a = 0$ or $b = 0$. And, M is called semiprime if $a\Gamma M\Gamma a = 0$ with $a \in M$ implies $a = 0$. Note that every prime Γ-ring is obviously semiprime.

The notions of derivation and Jordan derivation of a Γ-ring has been introduced by M. Sapanci and A. Nakajima in [5], whereas, the concept of k-derivation of a Γ-ring has been used and developed by H. Kandamar[3]. Afterwards, the concept of Jordan generalized derivation of a Γ-ring has been developed by Y. Ceven and M. A. Ozturk in [2].

Here we introduce the concept of Jordan k-derivation of a Γ-ring as follows and then we build up a relationship between the k-derivation and Jordan k-derivation of a Γ-ring in a concrete manner.

Let M be a Γ-ring and let $d : M \to M$ and $k : \Gamma \to \Gamma$ be two additive mappings. If $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$ holds for all $a, b \in M$ and $\alpha \in \Gamma$, then d is called a derivation of M. And, for all $a, b \in M$ and $\alpha \in \Gamma$, if $d(a\alpha b) = d(a)\alpha b + ak(\alpha)b + a\alpha d(b)$ is satisfied, then d is called a k-derivation of M. Finally, if $d(a\alpha a) = d(a)\alpha a + ak(\alpha)a + a\alpha d(a)$ holds for all $a \in M$ and $\alpha \in \Gamma$, then d is called a Jordan k-derivation of M.

From these definitions it is clear that every k-derivation of a Γ-ring M is a Jordan k-derivation of M. But, the converse statement is not true in general. Here we show that every Jordan k-derivation of a 2-torsion free prime Γ_N-ring M is a k-derivation of M. For this to happen we develop some important results as follows.

2. Main Results

Lemma 1. *Let M be a Γ_N-ring and let d be a Jordan k-derivation of M. Then for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, the following statements hold:*
(i) $d(a\alpha b + b\alpha a) = d(a)\alpha b + d(b)\alpha a + ak(\alpha)b + bk(\alpha)a + a\alpha d(b) + b\alpha d(a)$;
(ii) $d(a\alpha b\beta a + a\beta b\alpha a) = d(a)\alpha b\beta a + d(a)\beta b\alpha a + ak(\alpha)b\beta a + ak(\beta)b\alpha a$
$+ a\alpha d(b)\beta a + a\beta d(b)\alpha a + a\alpha bk(\beta)a + a\beta bk(\alpha)a + a\alpha b\beta d(a) + a\beta b\alpha d(a)$.
In particular, if M is 2-torsion free, then
(iii) $d(a\alpha b\alpha a) = d(a)\alpha b\alpha a + ak(\alpha)b\alpha a + a\alpha d(b)\alpha a + a\alpha bk(\alpha)a + a\alpha b\alpha d(a)$;
(iv) $d(a\alpha b\alpha c + c\alpha b\alpha a) = d(a)\alpha b\alpha c + d(c)\alpha b\alpha a + ak(\alpha)b\alpha c + ck(\alpha)b\alpha a$
$+ a\alpha d(b)\alpha c + c\alpha d(b)\alpha a + a\alpha bk(\alpha)c + c\alpha bk(\alpha)a + a\alpha b\alpha d(c) + c\alpha b\alpha d(a)$.
Especially, if M is 2-torsion free and if $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, then
(v) $d(a\alpha b\beta a) = d(a)\alpha b\beta a + ak(\alpha)b\beta a + a\alpha d(b)\beta a + a\alpha bk(\beta)a + a\alpha b\beta d(a)$;
(vi) $d(a\alpha b\beta c + c\alpha b\beta a) = d(a)\alpha b\beta c + d(c)\alpha b\beta a + ak(\alpha)b\beta c + ck(\alpha)b\beta a$
$+ a\alpha d(b)\beta c + c\alpha d(b)\beta a + a\alpha bk(\beta)c + c\alpha bk(\beta)a + a\alpha b\beta d(c) + c\alpha b\beta d(a)$.

Proof. Compute $d((a + b)\alpha(a + b))$ and cancel the like terms from both sides to obtain (i). Then replace $a\beta b + b\beta a$ for b in (i) to get (ii). Since M is 2-torsion free, (iii) is easily obtained by replacing α for β in (ii), and then (iv) is obtained by replacing $a + c$ for a in (iii). Again, since M is 2-torsion free and $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, (v) follows from (ii) and then finally, (vi) is obtained by replacing $a + c$ for a in (v). □

Lemma 2. *Let d be a Jordan k-derivation of a 2-torsion free Γ_N-ring M. Then for all $b \in M$ and $\beta \in \Gamma$, $k(\beta b\beta) = k(\beta)b\beta + \beta d(b)\beta + \beta bk(\beta)$.*

Proof. For all $a \in M$ and $\alpha \in \Gamma$, we have $d(a\alpha a) = d(a)\alpha a + ak(\alpha)a + a\alpha d(a)$. Let $b \in M$ and $\beta \in \Gamma$. Then putting $\beta b\beta$ for α, we get $d(a\beta b\beta a) = d(a)\beta b\beta a + ak(\beta b\beta)a + a\beta b\beta d(a)$. Expanding the LHS by Lemma (1)(iii), we obtain $a(k(\beta b\beta) - k(\beta)b\beta - \beta d(b)\beta - \beta bk(\beta))a = 0$. Hence, by applying the Nobusawa condition (c*) of the definition of Γ_N-ring, we get the proof. □

Lemma 3. *If d is a Jordan k_1-derivation as well as a Jordan k_2-derivation of a 2-torsion free Γ_N-ring M, then $k_1 = k_2$.*

Proof. Obvious. □

Remark 4. If d is a Jordan k-derivation of a 2-torsion free Γ_N-ring M, then k is uniquely determined.

Definition 5. Let M be a Γ-ring. Then for $a, b \in M$ and $\alpha \in \Gamma$, we define $[a, b]_\alpha = a\alpha b - b\alpha a$.

Lemma 6. *If M is a Γ-ring, then for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$,*
(i) $[a, b]_\alpha + [b, a]_\alpha = 0$;
(ii) $[a + b, c]_\alpha = [a, c]_\alpha + [b, c]_\alpha$;
(iii) $[a, b + c]_\alpha = [a, b]_\alpha + [a, c]_\alpha$;
(iv) $[a, b]_{\alpha+\beta} = [a, b]_\alpha + [a, b]_\beta$.

Proof. Obvious. □

Remark 7. Note that a Γ-ring M is commutative if and only if $[a, b]_\alpha = 0$ for all $a, b \in M$ and $\alpha \in \Gamma$.

Definition 8. Let d be a Jordan k-derivation of a Γ_N-ring M. Then for $a, b \in M$ and $\alpha \in \Gamma$, we define $F_\alpha(a, b) = d(a\alpha b) - d(a)\alpha b - ak(\alpha)b - a\alpha d(b)$.

Then we have, $F_\alpha(b, a) = d(b\alpha a) - d(b)\alpha a - bk(\alpha)a - b\alpha d(a)$.

Lemma 9. *l9 If d is a Jordan k-derivation of a Γ_N-ring M, then for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$,*
(i) $F_\alpha(a, b) + F_\alpha(b, a) = 0$;
(ii) $F_\alpha(a + b, c) = F_\alpha(a, c) + F_\alpha(b, c)$;
(iii) $F_\alpha(a, b + c) = F_\alpha(a, b) + F_\alpha(a, c)$;
(iv) $F_{\alpha+\beta}(a, b) = F_\alpha(a, b) + F_\beta(a, b)$.

Proof. Obvious. □

Remark 10. Note that d is a k-derivation of a Γ_N-ring M if and only if $F_\alpha(a, b) = 0$ for all $a, b \in M$ and $\alpha \in \Gamma$.

Lemma 11. *Let d be a Jordan k-derivation of a 2-torsion free Γ_N-ring M and suppose that $a, b \in M$ and $\alpha, \beta \in \Gamma$. Then*
(i) $F_\alpha(a, b)\alpha m\alpha[a, b]_\alpha + [a, b]_\alpha\alpha m\alpha F_\alpha(a, b) = 0$;
(ii) $F_\alpha(a, b)\beta m\beta[a, b]_\alpha + [a, b]_\alpha\beta m\beta F_\alpha(a, b) = 0$;
(iii) $F_\beta(a, b)\alpha m\alpha[a, b]_\beta + [a, b]_\beta\alpha m\alpha F_\beta(a, b) = 0$.

Proof. (i) Consider $G = d(a\alpha b\alpha m\alpha b\alpha a + b\alpha a\alpha m\alpha a\alpha b)$.
First, compute $G = d(a\alpha(b\alpha m\alpha b)\alpha a) + d(b\alpha(a\alpha m\alpha a)\alpha b)$ using Lemma 1 (iii) and then, $G = d((a\alpha b)\alpha m\alpha(b\alpha a) + (b\alpha a)\alpha m\alpha(a\alpha b))$ using Lemma 1 (iv). Since these two are equal, cancelling the similar terms from both sides of this equality and then rearranging them with the use of Lemma 2.5(i), we obtain the result of (i).

(ii) Considering $G = d(a\alpha b\beta m\beta b\alpha a + b\alpha a\beta m\beta a\alpha b)$ and proceeding in the same way as in the proof of (i) by the similar arguments, we get (ii).

(iii) Interchanging α and β in (ii), we obtain (iii). □

Lemma 12. *Let M be a 2-torsion free semiprime Γ_N-ring and suppose that $a, b \in M$. If $a\Gamma m\Gamma b + b\Gamma m\Gamma a = 0$ for all $m \in M$, then $a\Gamma m\Gamma b = b\Gamma m\Gamma a = 0$.*

Proof. Let m and m' be two arbitrary elements of M. Then by hypothesis, we have

$$\begin{aligned}(a\Gamma m\Gamma b)\Gamma m'\Gamma(a\Gamma m\Gamma b) &= -(b\Gamma m\Gamma a)\Gamma m'\Gamma(a\Gamma m\Gamma b)\\ = -(b\Gamma(m\Gamma a\Gamma m')\Gamma a)\Gamma m\Gamma b &= (a\Gamma(m\Gamma a\Gamma m')\Gamma b)\Gamma m\Gamma b\\ = a\Gamma m\Gamma(a\Gamma m'\Gamma b)\Gamma m\Gamma b &= -a\Gamma m\Gamma(b\Gamma m'\Gamma a)\Gamma m\Gamma b\\ &= -(a\Gamma m\Gamma b)\Gamma m'\Gamma(a\Gamma m\Gamma b).\end{aligned}$$

This implies, $2((a\Gamma m\Gamma b)\Gamma m'\Gamma(a\Gamma m\Gamma b)) = 0$.
Since M is 2-torsion free, $(a\Gamma m\Gamma b)\Gamma m'\Gamma(a\Gamma m\Gamma b) = 0$.
By the semiprimeness of M, $a\Gamma m\Gamma b = 0$ for all $m \in M$.
Hence we get, $a\Gamma m\Gamma b = b\Gamma m\Gamma a = 0$ for all $m \in M$. □

Corollary 13. *If M is a 2-torsion free semiprime Γ_N-ring, then for all $a, b \in M$ and $\alpha, \beta \in \Gamma$,*
(i) $F_\alpha(a,b)\alpha m\alpha[a,b]_\alpha = [a,b]_\alpha\alpha m\alpha F_\alpha(a,b) = 0$;
(ii) $F_\alpha(a,b)\beta m\beta[a,b]_\alpha = [a,b]_\alpha\beta m\beta F_\alpha(a,b) = 0$;
(iii) $F_\beta(a,b)\alpha m\alpha[a,b]_\beta = [a,b]_\beta\alpha m\alpha F_\beta(a,b) = 0$.

Proof. Using Lemma 12 in the result of Lemma 11, we obtain these results. □

Theorem 14. *Let M be a 2-torsion free semiprime Γ_N-ring. Then for all $a, b, c, d \in M$ and $\alpha, \beta, \gamma \in \Gamma$,*
(i) $F_\alpha(a,b)\alpha m\alpha[c,d]_\alpha = 0$;
(ii) $F_\alpha(a,b)\beta m\beta[c,d]_\alpha = 0$;
(iii) $F_\alpha(a,b)\alpha m\alpha[c,d]_\beta = 0$.

Proof. Replacing $a + c$ for a in Corollary 13 (i), we get

$$F_\alpha(a,b)\alpha m\alpha[c,b]_\alpha + F_\alpha(c,b)\alpha m\alpha[a,b]_\alpha = 0.$$

Therefore, we get

$$\begin{aligned}&F_\alpha(a,b)\alpha m\alpha[c,b]_\alpha\alpha m\alpha F_\alpha(a,b)\alpha m\alpha[c,b]_\alpha\\ &= -F_\alpha(a,b)\alpha m\alpha[c,b]_\alpha\alpha m\alpha F_\alpha(c,b)\alpha m\alpha[a,b]_\alpha = 0.\end{aligned}$$

Hence, by the semiprimeness of M, $F_\alpha(a,b)\alpha m\alpha[c,b]_\alpha = 0$.

Similarly, by replacing $b + d$ for b in this equality, we get

$$F_\alpha(a,b)\alpha m\alpha[c,d]_\alpha = 0.$$

Proceeding in the same way as before by the similar replacements in Corollary 13 (ii), we obtain (ii).

Finally, replacing $\alpha+\beta$ for α in (i), we get

$$F_\alpha(a,b)\alpha m\alpha[c,d]_\beta + F_\beta(a,b)\alpha m\alpha[c,d]_\alpha = 0.$$

Therefore, we have

$$\begin{aligned} &F_\alpha(a,b)\alpha m\alpha[c,d]_\beta\alpha m\alpha F_\alpha(a,b)\alpha m\alpha[c,d]_\beta \\ &= -F_\alpha(a,b)\alpha m\alpha[c,d]_\beta\alpha m\alpha F_\beta(a,b)\alpha m\alpha[c,d]_\alpha = 0. \end{aligned}$$

Hence, by the semiprimeness of M, we get $F_\alpha(a,b)\alpha m\alpha[c,d]_\beta = 0$. □

Theorem 15. *Every Jordan k-derivation of a 2-torsion free prime Γ_N-ring M is a k-derivation of M.*

Proof. Let d be a Jordan k-derivation of a 2-torsion free prime Γ_N-ring M. Since M is prime, we get from Theorem 14 (i) that either $F_\alpha(a,b)=0$ or, $[c,d]_\alpha = 0$ for all $a,b,c,d \in M$ and $\alpha \in \Gamma$.

If $[c,d]_\alpha \neq 0$ for all $c,d \in M$ and $\alpha \in \Gamma$. Then $F_\alpha(a,b) = 0$ for all $a,b \in M$ and $\alpha \in \Gamma$ and hence we get, d is a k-derivation of M.

But, if $[c,d]_\alpha = 0$ for all $c,d \in M$ and $\alpha \in \Gamma$, then M is commutative and, therefore, we have from Lemma 1 (i),

$$2d(a\alpha b) = 2d(a)\alpha b + 2ak(\alpha)b + 2a\alpha d(b).$$

Since M is 2-torsion free, we obtain that d is a k-derivation of M. □

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