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A Generalized System of Nonlinear Variational Inequalities in Hilbert Spaces

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Abstract. In this paper, we consider convergence of iterative-projection method for solutions of a generalized system for three different nonlinear relaxed co-coercive mappings in the framework of Hilbert spaces. Strong convergence theorems are established. Our results improve and extend the recent ones announced by many others.

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1. Introduction and Preliminaries

Variational inequalities are among the most interesting and intensively studied classes of mathematical problems and have wide applications in the fields of optimization and control, economics and transportation equilibrium and engineering sciences. Variational inequality problems have been generalized and extended in different directions using the novel and innovative techniques. Various kinds of iterative algorithms to solve the variational inequalities have been developed by many authors. There exists a vast literature [1]-[28] on the approximation solvability of nonlinear variational inequalities as well as nonlinear variational inclusions using projection type methods, resolvent operator type methods or averaging techniques. It is well known that variational inequalities are equivalent to fixed point problems. This alternative equivalent formulation is very important from the numerical analysis point of view and has played a significant part in several numerical methods for solving variational inequalities and complementarity problems. In particular, the solution of the variational inequalities can be computed using the iterative projection methods. For the convergence of the projection method, one may require that the operator is strongly monotone and Lipschitz continuous. Gabay [10] has shown that the convergence of a projection method can be proved for co-coercive operators. Note that co-coercivity is a weaker condition than strong monotonicity. Recently, Verma [22]-[25] introduced a new system of nonlinear strongly monotone variational inequalities and studied the approximate solvability of this system based on a system of projection methods. Projection methods have been applied widely to problems arising especially from complementarity, convex quadratic programming, and variational problems.

In this paper, we consider, based on the projection method, the approximate solvability of a system of nonlinear variational inequalities with different co-coercive mappings in the framework of Hilbert spaces. Solutions of the system of nonlinear relaxed co-coercive variational inequalities are also fixed points of asymptotically nonexpansisve mappings. Our results obtained in this paper generalize the results announced by Chang et al [3], Verma [22]-[24] and some others.

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$. Let C be a closed convex subset of H and let $T : C \to H$ be a nonlinear mapping. Let P_C be the projection of H onto the convex subset C. The classical variational inequality, denoted by $VI(C, T)$, is to find $u \in C$ such that

$$\langle Tu, v-u \rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

We see that the point $u \in C$ is a solution of the variational inequality (1.1) if and only if $u \in C$ is a fixed point of the mapping $P_C(I - \lambda T)$, where $\lambda > 0$ is a constant. That is,

$$u = P_C(u - \lambda T u). \tag{1.2}$$

One can easily see variational inequalities and fixed point problems are equivalent. This alternative equivalent formulation has played a significant role in the study of the variational inequalities and related optimization problems.

Let $T : C \to H$ be a mapping. Recall the following definitions.

(1) T is said to be monotone if

$$\langle Tu - Tv, u - v \rangle \geq 0, \quad \forall u, v \in C.$$

(2) T is said to be δ-strongly monotone if there exists a constant $\delta > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \delta \|x - y\|^2, \quad \forall x, y \in C.$$

This implies that

$$\|Tx - Ty\| \geq \delta \|x - y\|, \quad \forall x, y \in C,$$

that is, T is δ-expansive and, when $\delta = 1$, it is expansive.

(3) T is said to be γ-cocoercive if there exists a constant $\gamma > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \gamma \|Tx - Ty\|^2, \quad \forall x, y \in C.$$

(4) T is said to be relaxed γ-cocoercive if there exists a constant $\gamma > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq (-\gamma) \|Tx - Ty\|^2, \quad \forall x, y \in C.$$

(5) T is said to be relaxed (γ, δ)-cocoercive if there exist two constants $\gamma, \delta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq (-\gamma)\|Tx - Ty\|^2 + \delta\|x - y\|^2, \quad \forall x, y \in C.$$

Let $S : C \to C$ be a mapping. Recall the following definitions.

(6) S is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

(7) S is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|S^n x - S^n y\| \leq k_n\|x - y\|, \quad \forall x, y \in C,\ n \geq 0.$$

Next, we denote the fixed point of S by $F(S)$. If $x^* \in F(S) \cap VI(C, T)$, then we have the following

$$x^* = S^n x^* = P_C[x^* - \rho T x^*] = S^n P_C[x^* - \rho T x^*], \quad \forall n \geq 0.$$

This formulation is used to suggest the following iterative methods for finding a common element of the set of fixed points of asymptotically nonexpansive mappings and of the set of solutions to variational inequalities. Recently, three-step iterative method was studied by many authors to approximate solutions of variational inequalities and nonlinear operator equations. It has been shown that three-step schemes are numerically better than two-step and one step methods; see, for example, [15-17,27,28] and the reference therein.

Let $T_1, T_2, T_3 : C\times C\times C \to H$ be nonlinear mappings. Consider a system of nonlinear variational inequality problems (SNVI) as follows:

Find $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\langle sT_1(y^*, z^*, x^*) + x^* - y^*, x - x^*\rangle \geq 0, \quad \forall x \in C,\ s > 0, \tag{1.3}$$

$$\langle tT_2(z^*, x^*, y^*) + y^* - z^*, x - x^*\rangle \geq 0, \quad \forall x \in C,\ t > 0, \tag{1.4}$$

$$\langle rT_3(x^*, y^*, z^*) + z^* - x^*, x - x^*\rangle \geq 0, \quad \forall x \in C,\ r > 0. \tag{1.5}$$

One can easily see the SNVI problem (1.3)-(1.5) is equivalent to the following projection formulas:

$$\begin{aligned} x^* &= P_C[y^* - sT_1(y^*, z^*, x^*)], \quad s > 0,\\ y^* &= P_C[z^* - tT_2(z^*, x^*, y^*)], \quad t > 0,\\ z^* &= P_C[x^* - rT_3(x^*, y^*, z^*)], \quad r > 0, \end{aligned}$$

respectively, where P_C is the projection of H onto C.

Next, we consider some special classes of the SNVI problems (1.3)-(1.5) as follows:

(I) If C is a closed convex cone of H, then the SNVI problem (1.3)-(1.5) is equivalent to the following system (SNC) of nonlinear complementarity problems:

Find $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$T_1(y^*, z^*, x^*) \in C^*,\ T_2(z^*, x^*, y^*) \in C^*,\ T_3(x^*, y^*, z^*) \in C^*$$

$$\langle sT_1(y^*, z^*, x^*) + x^* - y^*, x^*\rangle = 0, \quad s > 0, \tag{1.6}$$

$$\langle tT_2(z^*, x^*, y^*) + y^* - z^*, x^*\rangle = 0, \quad t > 0, \tag{1.7}$$

$$\langle rT_3(x^*, y^*, z^*) + z^* - x^*, x^*\rangle = 0, \quad r > 0, \tag{1.8}$$

where C^* is the polar cone to C defined by

$$C^* = \{f \in H : \langle f, x\rangle \geq 0, \quad \forall x \in C\}. \tag{1.9}$$

(II) If $T_1 = T_2 = T_3$, then then the SVNI problem (1.3)-(1.5) reduces to the following SNVI problems:

Find $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\langle sT(y^*, z^*, x^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C,\, s > 0, \tag{1.10}$$

$$\langle tT(z^*, x^*, y^*) + y^* - z^*, x - x^* \rangle \geq 0, \quad \forall x \in C,\, t > 0, \tag{1.11}$$

$$\langle rT(x^*, y^*, z^*) + z^* - x^*, x - x^* \rangle \geq 0, \quad \forall x \in C,\, r > 0. \tag{1.12}$$

(III) If $T_1, T_2, T_3 : C \to H$ are univariate mappings, then the SVNI problem (1.3)-(1.5) reduces to the following SNVI problems:

Find $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\langle sT_1(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C,\, s > 0, \tag{1.13}$$

$$\langle tT_2(z^*) + y^* - z^*, x - x^* \rangle \geq 0, \quad \forall x \in C,\, t > 0, \tag{1.14}$$

$$\langle rT_3(x^*) + z^* - x^*, x - x^* \rangle \geq 0, \quad \forall x \in C,\, r > 0. \tag{1.15}$$

(IV) If $T_1 = T_2 = T_3 = T : C \to H$ is a univariate mapping, then the SVNI problem (1.3)-(1.5) reduces to the following SNVI problems:

Find $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\langle sT(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C,\, s > 0, \tag{1.16}$$

$$\langle tT(z^*) + y^* - z^*, x - x^* \rangle \geq 0, \quad \forall x \in C,\, t > 0, \tag{1.17}$$

$$\langle rT(x^*) + z^* - x^*, x - x^* \rangle \geq 0, \quad \forall x \in C,\, r > 0. \tag{1.18}$$

One can easily get the SNVI problem (1.16)-(1.18) is equivalent to the following projection formulas:

$$x^* = P_C[y^* - sT(y^*)], \quad s > 0, \tag{1.19}$$

$$y^* = P_C[z^* - tT(z^*)], \quad t > 0, \tag{1.20}$$

$$z^* = P_C[x^* - rT(x^*)], \quad r > 0. \tag{1.21}$$

Next, we introduce the following iterative methods for the above SNVI problems.

Algorithm 1. For any $x_0, y_0, z_0 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative process:

$$\begin{cases} z_n = S^n P_C[x_n - r_n T_3(x_n, y_n, z_n)], \\ y_n = S^n P_C[z_n - t_n T_2(z_n, x_n, y_n)], \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S^n P_C[y_n - s_n T_1(y_n, z_n, x_n)], \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ and S is an asymptotically nonexpansive mapping.

If $T_1 = T_2 = T_3 = T$ and $S = I$, the identity mapping, then Algorithm 1 is reduced to the following:

Algorithm 2. For any $x_0, y_0, z_0 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative process:

$$\begin{cases} z_n = P_C[x_n - r_n T(x_n, y_n, z_n)], \\ y_n = P_C[z_n - t_n T(z_n, x_n, y_n)], \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_C[y_n - s_n T(y_n, z_n, x_n)], \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ for all $n \geq 0$.

If $T_1, T_2, T_3 : C \to H$ are univariate mappings, then the Algorithm 1 is reduced to the following:

Algorithm 3. For any $x_0, y_0, z_0 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative process:

$$\begin{cases} z_n = S^n P_C[x_n - r_n T_3 x_n], \\ y_n = S^n P_C[z_n - t_n T_2 z_n], \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S^n P_C[y_n - s_n T_1(y_n)], \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ and S is an asymptotically nonexpansive mapping.

If $T_1 = T_2 = T_3 = T : C \to H$ is a univariate mapping and $S = I$, the identity mapping, then we have the following:

Algorithm 4. For any $x_0, y_0, z_0 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative process:

$$\begin{cases} z_n = P_C[x_n - r_n T x_n], \\ y_n = P_C[z_n - t_n T z_n], \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_C[y_n - s_n T(y_n)], \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

In order to prove our main results, we need the following lemmas and definitions.

Lemma 1. *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \lambda_n)a_n + b_n + c_n, \quad \forall n \geq n_0,$$

where n_0 is some nonnegative integer, $\{\lambda_n\}$ is a sequence in $(0, 1)$ with $\sum_{n=1}^{\infty} \lambda_n = \infty$, $b_n = o(\lambda_n)$ and $\sum_{n=0}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n = 0$.

Definition 2. A mapping $T : C \times C \times C \to H$ is said to be relaxed $((\gamma, \delta))$-cocoercive if there exist constants $(\gamma, \delta) > 0$ such that, for all $x, x' \in C$

$$\begin{aligned} &\langle T(x, y, z) - T(x', y', z'), x - x' \rangle \\ &\geq (-\gamma)\|T(x, y, z) - T(x', y', z')\|^2 + \delta\|x - x'\|^2, \quad \forall y, y', z, z' \in C. \end{aligned}$$

Definition 3. A mapping $T : C \times C \times C \to H$ is said to be μ-Lipschitz continuous in the first variable if there exists a constant $\mu > 0$ such that, for all $x, x' \in C$,

$$\|T(x, y, z) - T(x', y', z')\| \leq \mu\|x - x'\|, \quad \forall y, y', z, z' \in C.$$

2. Main Results

Theorem 4. *Let C be a closed convex subset of a real Hilbert space H. Let $T_i : C \times C \times C \to H$ be a relaxed (γ_i, δ_i)-cocoercive and μ_i-Lipschitz continuous mapping in the first variable for each $i = 1, 2, 3$ and $S : C \to C$ an asymptotically nonexpansive mapping with a fixed point. Suppose that $x^*, y^*, z^* \in F(S)$ and $(x^*, y^*, z^*) \in \Omega_1$, where Ω_1 denotes the set of solutions to the SNVI problems (1.3)-(1.5). Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be the sequences generated by Algorithm 1 and let $\{\alpha_n\}$ be a sequence in $[0, 1]$. Assume that the following conditions are satisfied.*

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(b) $k_n^3\theta_{1n}\theta_{2n}\theta_{3n} < 1$, *where*

$$\theta_{1n} = \sqrt{1 + s_n^2\mu_1^2 - 2s_n\delta_1 + 2s_n\gamma_1\mu_1^2}, \quad \theta_{2n} = \sqrt{1 + t_n^2\mu_2^2 - 2t_n\delta_2 + 2t_n\gamma_2\mu_2^2}$$

and

$$\theta_{3n} = \sqrt{1 + r_n^2\mu_3^2 - 2r_n\delta_3 + 2r_n\gamma_3\mu_3^2}.$$

Then the sequences $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge strongly to* x^*, y^* *and* z^*, *respectively.*

Proof. From the assumption, we have

$$\begin{cases} x^* = (1-\alpha_n)x^* + \alpha_n S^n P_C[y^* - s_n T_1(y^*, z^*, x^*)], \\ y^* = S^n P_C[z^* - t_n T_2(z^*, x^*, y^*)], \\ z^* = S^n P_C[x^* - r_n T_3(x^*, y^*, z^*)]. \end{cases}$$

It follows from Algorithm 1 that

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ &= \|(1-\alpha_n)x_n + \alpha_n S^n P_C[y_n - s_n T_1(y_n, z_n, x_n)] - x^*\| \\ &= \|(1-\alpha_n)x_n + \alpha_n S^n P_C[y_n - s_n T_1(y_n, z_n, x_n)] \\ &\quad - (1-\alpha_n)x^* + \alpha_n S^n P_C[y^* - s_n T_1(y^*, z^*, x^*)]\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n k_n \|y_n - y^* - s_n[T(y_n, z_n, x_n) - T(y^*, z^*, x^*)]\|. \end{aligned} \tag{2.1}$$

By the assumption that T_1 is relaxed (γ_1, δ_1)-cocoercive and μ_1-Lipschitz continuous in the first variable, we obtain

$$\begin{aligned} &\|y_n - y^* - s_n[T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)]\|^2 \\ &= \|y_n - y^*\|^2 - 2s_n\langle y_n - y^*, T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)\rangle \\ &\quad + s_n^2\|T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)\|^2 \\ &\le \|y_n - y^*\|^2 - 2s_n[-\gamma_1\|T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)\|^2 + \delta_1\|y_n - y^*\|^2] \\ &\quad + s_n^2\mu_1^2\|y_n - y^*\|^2 \\ &\le \|y_n - y^*\|^2 + 2s_n\gamma_1\mu_1^2\|y_n - y^*\|^2 - 2s_n\delta_1\|y_n - y^*\|^2 + s_n^2\mu_1^2\|y_n - y^*\|^2 \\ &= \theta_{1n}^2\|y_n - y^*\|^2, \end{aligned} \tag{2.2}$$

where $\theta_{1n} = \sqrt{1 + s_n^2\mu_1^2 - 2s_n\delta_1 + 2s_n\gamma_1\mu_1^2}$. Now, we estimate

$$\begin{aligned} \|y_n - y^*\| &= \|S^n P_C[z_n - t_n T_2(z_n, x_n, y_n)] - y^*\| \\ &= \|S^n P_C[z_n - t_n T_2(z_n, x_n, y_n)] - S^n P_C[z^* - t_n T_2(z^*, x^*, y^*)]\| \\ &\le k_n\|z_n - z^* - t_n[T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*)]\|. \end{aligned} \tag{2.3}$$

By the assumption that T_2 is relaxed (γ_2, δ_2)-cocoercive and μ_2-Lipschitz continuous in the first variable, we obtain

$$\begin{aligned} &\|z_n - z^* - t_n[T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*)]\|^2 \\ &= \|z_n - z^*\|^2 - 2t_n\langle z_n - z^*, T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*)\rangle \\ &\quad + t_n^2\|T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*)\|^2 \\ &\leq \|z_n - z^*\|^2 - 2t_n[-\gamma_2\|T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*)\|^2 + \delta_2\|z_n - z^*\|^2] \\ &\quad + t_n^2\mu_2^2\|z_n - z^*\|^2 \\ &\leq \|z_n - z^*\|^2 + 2t_n\gamma_2\mu_2^2\|z_n - z^*\|^2 - 2t_n\delta_2\|z_n - z^*\|^2 + t_n^2\mu_2^2\|z_n - z^*\|^2 \\ &\leq \theta_{2n}^2\|z_n - z^*\|^2, \end{aligned} \tag{2.4}$$

where $\theta_{2n} = \sqrt{1 + t_n^2\mu_2^2 - 2t_n\delta_2 + 2t_n\gamma_2\mu_2^2}$. On the other hand, we have

$$\begin{aligned} \|z_n - z^*\| &= \|S^n P_C[x_n - r_n T_3(x_n, y_n, z_n)] - z^*\| \\ &= \|S^n P_C[x_n - r_n T_3(x_n, y_n, z_n)] - S^n P_C[x^* - r_n T_3(x^*, y^*, z^*)]\| \\ &= k_n\|P_C[x_n - r_n T_3(x_n, y_n, z_n)] - P_C[x^* - r_n T_3(x^*, y^*, z^*)]\| \\ &\leq k_n\|x_n - x^* - r_n[T_3(x_n, y_n, z_n) - T_3(x^*, y^*, z^*)]\|. \end{aligned} \tag{2.5}$$

By the assumption that T_3 is relaxed (γ_3, δ_3)-cocoercive and μ_3-Lipschitz continuous in the first variable, we obtain

$$\begin{aligned} &\|x_n - x^* - r_n[T_3(x_n, y_n, z_n) - T_3(x^*, y^*, z^*)]\|^2 \\ &= \|x_n - x^*\|^2 - 2r_n\langle x_n - x^*, T_3(x_n, y_n, z_n) - T_3(x^*, y^*, z^*)\rangle \\ &\quad + r_n^2\|T_3(x_n, y_n, z_n) - T_3(x^*, y^*, z^*)\|^2 \\ &\leq \|x_n - x^*\|^2 - 2r_n[-\gamma_2\|T_3(x_n, y_n, z_n) - T_3(x^*, y^*, z^*)\|^2 + \delta_3\|x_n - x^*\|^2] \\ &\quad + r_n^2\mu_3^2\|x_n - x^*\|^2 \\ &\leq \theta_{3n}^2\|x_n - x^*\|^2, \end{aligned} \tag{2.6}$$

where $\theta_{3n} = \sqrt{1 + r_n^2\mu_3^2 - 2r_n\delta_3 + 2r_n\gamma_3\mu_3^2}$. Combining (2.2), (2.3), (2.4), (2.5) and (2.6), we see

$$\|y_n - y^* - s_n[T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)]\| \leq k_n^2\theta_{1n}\theta_{2n}\theta_{3n}\|x_n - x^*\|. \tag{2.7}$$

Substitute (2.7) into (2.1) yields that

$$\|x_{n+1} - x^*\| \leq [1 - \alpha_n(1 - k_n^3\theta_{1n}\theta_{2n}\theta_{3n})]\|x_n - x^*\|. \tag{2.8}$$

Applying Lemma 1 to (2.8), we can get the desired conclusion easily. This completes the proof.

□

Remark 5. Theorem 4 mainly improves the corresponding results of Chang, Lee and Chan [3] and also extends the results of Huang and Noor [11] to some extent.

As applications of Theorem 4, we have the following results immediately.

Corollary 6. *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \times C \times C \to H$ be a relaxed (γ, δ)-cocoercive and μ-Lipschitz continuous mapping in the first variable. Suppose that $(x^*, y^*, z^*) \in \Omega_2$, where Ω_2 denotes the set of solutions to the SNVI problems* (1.10)-(1.12). *Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be the sequences generated by*

Algorithm 2 and let $\{\alpha_n\}$ *be a sequence in* $[0,1]$. *Assume that the following conditions are satisfied.*

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(b) $\theta_{1n}\theta_{2n}\theta_{3n} < 1$, *where*

$$\theta_{1n} = \sqrt{1 + s_n^2\mu^2 - 2s_n\delta + 2s_n\gamma\mu^2}, \quad \theta_{2n} = \sqrt{1 + t_n^2\mu^2 - 2t_n\delta + 2t_n\gamma\mu^2}$$

and

$$\theta_{3n} = \sqrt{1 + r_n^2\mu^2 - 2r_n\delta + 2r_n\gamma\mu^2}.$$

Then the sequences $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge strongly to* x^*, y^* *and* z^*, *respectively.*

Corollary 7. *Let* C *be a closed convex subset of a real Hilbert space* H. *Let* $T_i : C \to H$ *be a relaxed* (γ_i, δ_i)*-cocoercive and* μ_i*-Lipschitz continuous mapping for each* $i = 1,2,3$ *and* $S : C \to C$ *an asymptotically nonexpansive mapping with a fixed point. Suppose that* $x^*, y^*, z^* \in F(S)$ *and* $(x^*, y^*, z^*) \in \Omega_3$, *where* Ω_3 *denotes the set of solutions to the SNVI problems* (1.13)-(1.15). *Let* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ *be the sequences generated by Algorithm 3 and let* $\{\alpha_n\}$ *be a sequence in* $[0,1]$. *Assume that the following conditions are satisfied.*

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(b) $k_n^3\theta_{1n}\theta_{2n}\theta_{3n} < 1$, *where*

$$\theta_{1n} = \sqrt{1 + s_n^2\mu_1^2 - 2s_n\delta_1 + 2s_n\gamma_1\mu_1^2}, \quad \theta_{2n} = \sqrt{1 + t_n^2\mu_2^2 - 2t_n\delta_2 + 2t_n\gamma_2\mu_2^2}$$

and

$$\theta_{3n} = \sqrt{1 + r_n^2\mu_3^2 - 2r_n\delta_3 + 2r_n\gamma_3\mu_3^2}.$$

Then the sequences $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge strongly to* x^*, y^* *and* z^*, *respectively.*

Corollary 8. *Let* C *be a closed convex subset of a real Hilbert space* H. *Let* $T : C \to H$ *be a relaxed* (γ, δ)*-cocoercive and* μ*-Lipschitz continuous mapping. Suppose that* $(x^*, y^*, z^*) \in \Omega_4$, *where* Ω_4 *denotes the set of solutions to the SNVI problems* (1.16)-(1.18). *Let* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ *be the sequences generated by Algorithm 4 and let* $\{\alpha_n\}$ *be a sequence in* $[0,1]$. *Assume that the following conditions are satisfied.*

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(b) $\theta_{1n}\theta_{2n}\theta_{3n} < 1$, *where*

$$\theta_{1n} = \sqrt{1 + s_n^2\mu^2 - 2s_n\delta + 2s_n\gamma\mu^2}, \quad \theta_{2n} = \sqrt{1 + t_n^2\mu^2 - 2t_n\delta + 2t_n\gamma\mu^2}$$

and

$$\theta_{3n} = \sqrt{1 + r_n^2\mu^2 - 2r_n\delta + 2r_n\gamma\mu^2}.$$

Then the sequences $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge strongly to* x^*, y^* *and* z^*, *respectively.*

References

[1] R.P. Agarwal, Y.J. Cho, N.J. Huang, *Sensitivity analysis for strongly nonlinear quasi-variational inclusions*, Applied Mathematics Letters, **13** (2000), 19-24.

[2] D.P. Bertsekas, J. Tsitsiklis, Parallel and Distributed Computation: Numerical Methods, Prentice Hall, Englewood Cliffs, New Jersey, 1989.

[3] S.S. Chang, H.W.J. Lee, C.K. Chan, *Generalized system for relaxed cocoercive variational inequalities in Hilbert spaces*, Applied Mathematics Letters, **20** (2007), 329-334.

[4] Y.J. Cho, J. Li, N.J. Huang, *Solvability of implicit complementarity problems*, Mathematical and Computer Modelling, **45** (2007), 1001-1009.

[5] Y.J. Cho, X. Qin, M. Shang, Y. Su, *Generalized nonlinear variational inclusions involving* (A, η)*-monotone mappings in Hilbert spaces*, Fixed Point Theory and Applications **2007** (2007), 29653.

[6] Y.J. Cho, H.Y. Lan, *Generalized nonlinear random* (A, η)*-accretive equations with random relaxed cocoercive mappings in Banach spaces*, Computers Mathematics with Applications, **55** (2008), 2173-2182.

[7] X.P. Ding, *Iterative algorithms for solving generalized nonlinear mixed variational inequalities*, Journal of Computational and Applied Mathematics, **172** (2004), 399-408.

[8] X.P. Ding, *Algorithms of solutions for completely generalized mixed implicit quasi-variational inclusions*, Applied Mathematics and Computation, **148** (2004), 47-66.

[9] F. Giannessi, A. Maugeri, Variational Inequalities and Network Equilibrium Problems, Plenum Press, New York, 1995.

[10] D. Gabay, Applications of the Method of Multipliers to Variational Inequalities, Augmented Lagrangian Methods, Edited by M. Fortin and R. Glowinski, North-Holland, Amsterdam, Holland, 1983, pp. 299-331.

[11] Z. Huang, M.A. Noor, *An explicit projection method for a system of nonlinear variational inequalities with different* (γ, r)*-cocoercive mappings*, Applied Mathematics and Computation, **190** (2007), 356-361.

[12] H.Y. Lan, Y.J. Cho, R.U. Verma, *Nonlinear relaxed cocoercive variational inclusions involving* $(A,)$*-accretive mappings in Banach spaces*, Computers Mathematics with Applications, **51** (2006), 1529-1538.

[13] N.H. Nie, Z. Liu, K.H. Kim, S.M. Kang, *A system of nonlinear variational inequalities involving strong monotone and pseudocontractive mappings*, Advances in Nonlinear Variational Inequalities, **6** (2003), 91-99.

[14] M.A. Noor, *Some developments in general variational inequalities*, Applied Mathematics and Computation, **152** (2004), 199-277.

[15] M.A. Noor, *New approximation schemes for general variational inequalities*, Journal of Mathematical Analysis and Applications, **251** (2000), 217-229.

[16] M.A. Noor, *General variational inequalities and nonexpansive mappings*, Journal of Mathematical Analysis and Applications, **331** (2007), 810-822.

[17] M.A. Noor, Z. Huang, *Three-step methods for nonexpansive mappings and variational inequalities*, Applied Mathematics and Computation, **187** (2007), 680-685.

[18] X. Qin, M. Shang, Y. Su, *Generalized variational inequalities involving relaxed monotone mappings in Hilbert spaces*, PanAmerican Mathematical Journal, **17** (2007), 81-88.

[19] X. Qin, M.A. Noor, *General Wiener-Hopf equation technique for nonexpansive mappings and general variational inequalities in Hilbert spaces*, Applied Mathematics and Computation, **201** (2008), 716-722.

[20] X. Qin, S.M. Kang, M. Shang, *Generalized system for relaxed cocoercive variational inequalities in Hilbert spaces*, Applicable Analysis, **87** (2008), 421-430.

[21] X. Qin, M. Shang, H. Zhou, *Strong convergence of a general iterative method for variational inequality problems and fixed point problems in Hilbert spaces*, Applied Mathematics and Computation, **200** (2008), 242-253.

[22] R.U. Verma, *Generalized system for relaxed cocoercive variational inequalities and its projection methods*, Journal of Optimizatoin Theory and Applications, **121** (2004), 203-210.

[23] R.U. Verma, *Generalized class of partial relaxed monotonicity and its connections*, Advances in Nonlinear Variational Inequalities, **7** (2004), 155-164.

[24] R.U. Verma, *General convergence analysis for two-step projection methods and application to variational problems*, Applied Mathematics Letters, **18** (2005), 1286-1292.

[25] R.U. Verma, *Projection methods and a new system of cocoercive variational inequality problems*, Internattion Journal of Differential Equattions and Applications, **6** (2002), 359-367.

[26] N.H. Xiu, J.Z. Zhang, *Local convergence analysis of projection type algorithms: Unified approach*,Journal of Optimizatoin Theory and Applications, **115** (2002), 211-230.

[27] Y. Yao, M.A. Noor, *Convergence of three-step iterations for asymptotically nonexpansive mappings*, Applied Mathematics and Computation, **187** (2007), 883-892.

[28] Y. Yao, M.A. Noor, R. Chen, Y.C. Liou, *Strong convergence of three-step relaxed hybrid steepest-descent methods for variational inequalities*, Applied Mathematics and Computation, **201** (2008), 175-183.

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On the Convergence of a Modified Newton Method for Solving Equations

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Abstract. We approximate a solution of a nonlinear operator equation in a Banach space setting, where the differentiability of the operator involved is not assumed using a modified Newton method considered also in [1], [2], [6], [11]–[15]. We provide new sufficient convergence conditions, which are weaker than before [1], [2], [6], [11]–[15], under the same computational cost. Numerical examples where our results apply, where earlier ones fail are also provided in this study.

AMS (MOS) Subject Classification Codes: 65K10, 65G99, 65J99, 49M15, 47J20
Key Words: Modified Newton Method, Newton Method, Banach Space, Fréchet Derivative, Majorizing Sequence, Newton–Kantorovich Hypothesis.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where, F is a continuous operator defined on a subset $\mathcal{D}_F$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time–invariant system is driven by the equation $\dot{x} = Q(x)$, for some suitable operator Q, where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative–when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

We use the modified Newton method:

$$x_{n+1} = x_n - G'(x_n)^{-1}\ F(x_n) \quad (n \geq 0), \quad (x_0 \in \mathcal{D}). \tag{1.2}$$

to generate a sequence approximating $x^\star$. Operator $G'(x) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ the space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$, denote the Fréchet–derivative of operator G [2], [5], [11]. If $G'(x) = F'(x)$ $(x \in \mathcal{D})$, we obtain Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}\ F(x_n) \quad (n \geq 0), \quad (x_0 \in \mathcal{D}), \tag{1.3}$$

[3]–[5], [8]–[11].
The semilocal convergence of the modified Newton method (1.2) has been given by several authors under Lipschitz–type conditions (see [1]–[15], and the references there).
Here, motivated by optimization considerations, we introduce the needed center–Lipschitz conditions (see (2.14)) to find upper bounds on the norms $\| \ G'(x)^{-1}\ G'(x_0)\ \|$ instead of the less precise Lipschitz conditions (see (2.4)). In turn out that this way, we obtain a new semilocal convergence analysis with the following advantages over the corresponding ones in [1], [2], [6], [11]–[15]: larger convergence domain, finer estimates on the distances $\| \ x_{n+1} - x_n \ \|$, $\| \ x_n - x^\star \ \|$ $(n \geq 0)$, and an at least as precise information on the location of the solution $x^\star$.
Numerical examples where our results apply to solve nonlinear equations (1.1) are provided, in cases earlier ones cannot [6]–[15].

2. Semilocal convergence using increasing majorizing sequences

We state a semilocal convergence result for the modified Newton method (1.2).

Theorem 1. *Let $F\ :\ \mathcal{D}_F \subset \mathcal{X} \longrightarrow \mathcal{Y}$ be continuous, and $G\ :\ D_G \subset \mathcal{X} \longrightarrow Y$, be satisfying the Fréchet differentiability on a disk $\mathcal{D} \subset \mathcal{D}_F \cap \mathcal{D}_G$.*
Assume there exist $x_0 \in \mathcal{D}$, and constants $\eta \geq 0$, $M \geq 0$, $K > 0$, such that for $x, y \in \mathcal{D}_0$:

$$G'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X}); \tag{2.1}$$

$$\| \ G'(x_0)^{-1}\ F(x_0)\ \| \leq \eta; \tag{2.2}$$

$$\| \ G'(x_0)^{-1}\ ((F-G)(x) - (F-G)(y))\ \| \leq M\ \ \| \ x - y\ \|; \tag{2.3}$$

$$\| \ G'(x_0)^{-1}\ (G'(x) - G'(y))\ \| \leq K\ \ \| \ x - y\ \|; \tag{2.4}$$

$$2\ K\ \eta \leq (1-M)^2; \tag{2.5}$$

and

$$\overline{U}(x_0, r_0^\star) = \{x \in \mathcal{X}\ : \| \ x - x_0\ \| \leq r_0^\star\} \subseteq \mathcal{D}, \tag{2.6}$$

where

$$r_0^\star \geq r^\star = \frac{1-M}{K}\left(1 - \sqrt{1 - \frac{2\ K\ \eta}{(1-M)^2}}\right). \tag{2.7}$$

Then, sequence $\{x_n\}$ $(n \geq 0)$, generated by modified Newton method (1.2) is well defined for all $n \geq 0$, remains in $\overline{U}(x_0, r_0^\star)$, and converges to a solution $x^\star$ of equation $F(x) = 0$. Moreover, the following estimates hold for all $n \geq 0$:

$$\| \ x_{n+1} - x_n\ \| \leq r_{n+1} - r_n \tag{2.8}$$

and

$$\| \ x_n - x^\star\ \| \leq r^\star - r_n, \tag{2.9}$$

where, scalar sequence $\{r_n\}$ is given by:

$$r_0 = 0,$$

$$r_{n+1} = r_n - \frac{\Big(K\ (r_n - r_{n-1}) + 2\ M\Big)\ (r_n - r_{n-1})}{2\ (1 - K\ r_n)} = r_n - \frac{g(r_n)}{1 - K\ r_n} \quad (n \geq 1) \tag{2.10}$$

and

$$g(r) = \frac{K}{2}\ r^2 - (1 - M)\ r + \eta. \tag{2.11}$$

Proof. Proof was provided in non–affine invariant form in ([12], p. 189). Here, we state that the same proof can be provided in affine invariant form, if we use $G'(x_0)^{-1}\ G(x)$, $G'(x_0)^{-1}\ F(x)$ instead of $G(x)$, $F(x)$ respectively used in [12].
That completes the proof of Theorem 1. □

Remark 2. Theorem 1 improves corresponding Theorem 5.3 in [12], since our results are provided in affine invariant form. The advantages of affine versus non–affine convergence results have been explained in detail in [8] (see also [5]).
It turns out that the results in Theorem 1 can be improved even further. Indeed, let us define scalar sequences $\{q_n\}$, $\{s_n\}$ $(n \geq 0)$ for some $L > 0$ by:

$$q_0 = 0, \qquad q_1 = \eta,$$

$$q_{n+2} = q_{n+1} - \frac{\Big(K\ (q_n - q_{n-1}) + 2\ M\Big)\ (q_n - q_{n-1})}{2\ (1 - L\ q_n)} \quad (n \geq 1), \tag{2.12}$$

and

$$s_0 = 0,$$

$$s_{n+1} = s_n - \frac{g(s_n)}{1 - L\ s_n} \quad (n \geq 0). \tag{2.13}$$

In view of (2.4), there exists $L > 0$ such that

$$\parallel G'(x_0)^{-1}\ (G'(x) - G'(x_0))\ \parallel \leq L\ \parallel x - x_0 \parallel \qquad \text{for all} \quad x \in \mathcal{D}. \tag{2.14}$$

Note that in general

$$L \leq K \tag{2.15}$$

holds, and $\frac{K}{L}$ can be arbitrarily large [3]–[5].

We can show the semilocal convergence theorem for the modified Newton's method (1.2).

Theorem 3. (1) *If hypothesis (2.5) holds, then following hold for all $n \geq 0$:*

$$0 \leq q_n \leq s_n \leq r_n, \tag{2.16}$$

$$0 \leq q_{n+1} - q_n \leq s_{n+1} - s_n \leq r_{n+1} - r_n, \tag{2.17}$$

$$0 \leq q^\star - q_n \leq s^\star - s_n \leq r^\star - r_n, \tag{2.18}$$

and

$$q^\star \leq s^\star \leq r^\star, \tag{2.19}$$

where

$$q^\star = \lim_{n \to \infty} q_n, \qquad s^\star = \lim_{n \to \infty} s_n. \tag{2.20}$$

and $r^\star$ *is given in (2.7).*

(2) *Under the hypotheses of Theorem 1, sequence* $\{x_n\}$ $(n \geq 0)$*, generated by modified Newton method (1.2) is well defined, remains in* $\overline{U}(x_0, q^\star)$ *for all* $n \geq 0$*, and converges to a solution* $x^\star \in \overline{U}(x_0, q^\star)$ *of equation* $F(x) = 0$.
Moreover, the following estimates hold for all $n \geq 0$*:*

$$\| x_{n+1} - x_n \| \leq q_{n+1} - q_n \tag{2.21}$$

and

$$\| x_n - x^\star \| \leq q^\star - q_n. \tag{2.22}$$

Proof. (1) The proof of this part follows using induction on n, (2.10)–(2.13), (2.15), and standard majorization techniques [2], [5], [11].

(2) Let $x \in \overline{U}(x_0, q^\star)$. Using (2.7), (2.14), and (2.19), we obtain

$$\| G'(x_0)^{-1} \, (G'(x) - G'(x_0)) \| \leq L \, \| x - x_0 \| \leq L \, r^\star \leq K \, t^\star < 1. \tag{2.23}$$

It follows from (2.23), and the Banach lemma on invertible operators [5], [11], that $G'(x)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and

$$\| G'(x)^{-1} \, G'(x_0) \| \leq \frac{1}{1 - L \, \| x - x_0 \|}. \tag{2.24}$$

Using (1.2) for $n = 0$, (2.2), and (2.6), we get $\| x_1 - x_0 \| \leq \eta \leq r^\star$. That is $x_1 \in \overline{U}(x_0, r^\star)$, and (2.21) holds for $n = 0$. Let us assume that $x_k \in \overline{U}(x_0, r^\star)$ for all $k \leq n$. Then, using (1.2), (2.3), (2.4), (2.10), (2.12), (2.15), (2.24), and the identity

$$\frac{\frac{K}{2} \, r_k^2 - (1 - M) \, r_k + \eta}{1 - K \, r_k} = \frac{\frac{K}{2} \, (r_k - r_{k-1})^2 + M \, (r_k - r_{k-1})}{1 - K \, r_k} \tag{2.25}$$

we obtain in turn

$$\begin{aligned}
&\| x_{k+1} - x_k \| = \| -(G'(x_k)^{-1} \, G'(x_0)) \, (G'(x_0)^{-1} \, F(x_k)) \| \\
&\leq \frac{1}{1 - L \, q_k} \Bigg(\| G'(x_0)^{-1} \, (G(x_k) - G(x_{k-1}) - G'(x) \, (x_k - x_{k-1})) \| + \\
&\| G'(x_0)^{-1} \, (F(x_k) - G(x_k) - (F(x_{k-1}) - G(x_{k-1}))) \| \\
&\leq \frac{1}{1 - L \, q_k} \left(\frac{K}{2} \, \| x_k - x_{k-1} \| + M \right) \, \| x_k - x_{k-1} \| \\
&\leq \frac{1}{1 - L \, q_k} \left(\frac{K}{2} \, (q_k - q_{k-1}) + M \right) \, (q_k - q_{k-1}) = q_{k+1} - q_k,
\end{aligned} \tag{2.26}$$

which shows (2.21) for all $n \geq 0$.

Using part 1. of Theorem 3, and (2.26), we deduce sequence $\{x_n\}$ is Cauchy in a Banach space $\mathcal{X}$, and such it converges to some $x^\star \in \overline{U}(x_0, q^\star)$ (since $\overline{U}(x_0, q^\star)$ is a closed set). By letting $k \longrightarrow \infty$ in (2.26), we obtain $F(x^\star) = 0$. Estimate (2.22) is obtained from (2.21) by using standard majorization techniques [5].

That completes the proof of Theorem 3. □

Remark 4. It follows from Theorem 3 that finer majorizing sequences than $\{r_n\}$ can be obtained, and under the same computational cost, since in practice, the evaluation of Lipschitz constant K requires the evaluation of center–Lipschitz constant L. One is then wondering if the Newton–Kantorovich–type hypothesis (2.5) can be weakened, since finer sequence $\{q_n\}$ may be converging under weaker hypotheses. In [3]–[5], we provided sufficient convergence conditions for more general majorizing sequences that $\{q_n\}$. One such condition which can be weaker than (2.5) is given in [4], [5]:

$$\left(\frac{K}{2}+\frac{2\ L}{2-\delta}\right)\ \delta\ \eta+2\ M\leq\delta \qquad \text{for some}\ \ \delta\in[0,2). \tag{2.27}$$

In the next section, we provide sufficient convergence conditions other that (2.5), and (2.27) using decreasing instead of increasing majorizing sequences.

3. Semilocal convergence using decreasing majorizing sequences

We need the following result on majorizing sequences for modified Newton method (1.2)

Lemma 5. *Let $\eta\geq 0$, $K>0$, $M\geq 0$, and $L>0$ be given constants. Set $t_0=\frac{1}{L}$. Define functions Δ, A, C on $[0,+\infty)^2$, and B on $[0,+\infty)$ by*

$$\Delta(t,\gamma)=(K\ t+M)^2-(K-2\ \gamma\ L)\ (K\ t+2\ M)\ t,$$

$$A(t,\gamma)=2\ \left(K\ t+M+\sqrt{\Delta(t,\gamma)}\right),$$

$$B(t)=2\ (K\ t+2\ M),$$

and

$$C(t,\gamma)=\frac{B(t)}{A(t,\gamma)}.$$

Assume any of the following hold:

function

$$f_t(x)=1-x-C(t,x) \tag{3.1}$$

has a non–negative zero $\gamma_0=\gamma(t_0)$ at $t=t_0$, such that:

$$\beta=2\ L\ \eta\leq\gamma_0\leq\frac{K}{2\ L}; \tag{3.2}$$

or

$$K+M-L\geq 0, \tag{3.3}$$

$$f_{t_0}(\beta)\geq 0, \tag{3.4}$$

and hypothesis (3.2) holds;
or
function f_t has a non–negative zero γ_0^1 at $t=t_1$, such that:

$$\gamma_0^1\leq\frac{K}{2\ L} \tag{3.5}$$

and

$$\beta<\beta_0, \tag{3.6}$$

where

$$\beta_0 = 4\left\{M+2+\sqrt{(M+2)^2+4\,(\frac{K}{2\,L}-1)}\right\}^{-1}; \tag{3.7}$$

or

$$f_{t_1}(\beta_1) \le 0, \qquad \beta_1 = \min\{\beta_0,\ \frac{K}{2\,L}\}, \tag{3.8}$$

$$f_{t_1}(\beta) \ge 0, \tag{3.9}$$

(3.5), and (3.6) hold.
Note that the existence of γ_0 (or γ_0^1) follows from the intermediate value theorem applied to function f_{t_0} (or f_{t_1}) on the interval $\left[\beta, \frac{K}{2\,L}\right]$ (or $[\beta, \beta_1]$) respectively.
Then, scalar sequence $\{t_n\}$ $(n \ge 0)$ generated by

$$t_1 = t_0 - \eta,$$

$$t_{n+1} = t_n - \frac{\Big(K\ (t_{n-1}-t_n) + 2\ M\Big)\ (t_{n-1}-t_n)}{2\ L\ t_n} \quad (n \ge 1), \tag{3.10}$$

is well defined, decreasing and converges to some $t^\star \in [0, t_0]$.

Proof. If $\eta = 0$, then $t_n = t_0 = t^\star$ $(n \ge 0)$. Let us assume $\eta \ne 0$. Function Δ is a quadratic polynomial with leading coefficient $2\ K\ \gamma_0\ L$ (for $\gamma = \gamma_0$), and whose sign of the discriminant is the same with: $2\ \gamma_0\ L\ (2\ \gamma_0\ L - K)$. It then follows by (2.2) that functions A and C are well defined on $[0, \infty) \times [0, \gamma_0]$. It also follows by definition of γ_0 that $\gamma_0 \in (0, 1)$. Set:

$$\frac{t_{n+1}}{t_n} = 1 - \gamma_n,$$

where,

$$\gamma_n = \gamma(t_n) = \frac{\Big(K\ (t_{n-1}-t_n) + 2\ M\Big)\ (t_{n-1}-t_n)}{2\ L\ t_n^2} \quad (n \ge 1).$$

We shall show: $t_k \ge (1-\gamma_0)\ t_{k-1}$, which if $t_{k-1} > 0$, implies $0 < t_k < t_{k-1}$. But, $t_k \ge (1-\gamma_0)\ t_{k-1}$ holds if $1 - \gamma_k \ge 1 - \gamma_0$ or $\gamma_k \le \gamma_0$ or

$$(K - 2\ L\ \gamma_0)\ t_k^2 - 2\ (K\ t_{k-1} + M)\ t_k + (K\ t_{k-1} + 2\ M)\ t_{k-1} \le 0,$$

or $t_k \ge C(t_{k-1}, \gamma_0)\ t_{k-1}$. Using (3.10), and the definition of t_0, we get $t_1 \ge (1-\gamma_0)\ t_0$. That is we have:
$t_1 \ge (1-\gamma_0)\ t_0 \Longrightarrow t_1 \ge C(t_0, \gamma_0)\ t_0 \Longleftrightarrow t_2 \ge (1-\gamma_0)\ t_1$.
Similarly, we show this implication holds in general, i.e.,
$t_{k-1} \ge (1-\gamma_0)\ t_{k-2} \Longrightarrow t_{k-1} \ge C(t_0, \gamma_0)\ t_{k-2} \Longleftrightarrow t_k \ge (1-\gamma_0)\ t_{k-1}$ $(k \ge 1)$.
If the alternative conditions (3.5)–(3.9) hold, then $t_2 > 0$, and $t_3 \ge (1-\gamma_0^1)\ t_2$, and the induction follows by analogy.
The induction is completed. Hence, sequence $\{t_n\}$ $(n \ge 0)$ is decreasing positive, and as such it converges to some $t^\star \in [0, t_0]$.
That completes the proof of Lemma 5. □

We can show the following semilocal convergence theorem for modified Newton method (1.2):

Theorem 6. *Let $F : \mathcal{D}_F \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be continuous, and $G : \mathcal{D}_G \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be satisfying the Fréchet differentiability on a disk $\mathcal{D} \subset \mathcal{D}_F \cap \mathcal{D}_G$. Assume there exist $x_0 \in \mathcal{D}_0 \subseteq \mathcal{D}$, and constants $\eta \geq 0$, $K > 0$, $M \geq 0$, $L > 0$, such that for $x, y \in \mathcal{D}$:*

$$G'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X});$$

$$\| G'(x_0)^{-1} F(x_0) \| \leq \eta;$$

$$\| G'(x_0)^{-1} ((F-G)(x) - (F-G)(y)) \| \leq M \| x - x \|;$$

$$\| G'(x_0)^{-1} (G'(x) - G'(y)) \| \leq K \| x - y \|;$$

$$\| G'(x_0)^{-1} (G'(x) - G'(x_0)) \| \leq L \| x - x_0 \|;$$

$$\overline{U}(x_0, t_0 - t^\star) \subseteq \mathcal{D} \qquad (\text{or } \overline{U}(x_0, \frac{1}{L}) \subseteq \mathcal{D}),$$

and hypotheses of Lemma 5 hold.
Then, sequence $\{x_n\}$ $(n \geq 0)$, generated by modified Newton method (1.2) is well defined, remains in $\overline{U}(x_0, t_0 - t^\star)$ for all $n \geq 0$, and converges to a solution $x^\star \in \overline{U}(x_0, t_0 - t^\star)$ of equation $F(x) = 0$.
Moreover, the following estimates hold for all $n \geq 0$:

$$\| x_{n+1} - x_n \| \leq t_n - t_{n+1} \tag{3.11}$$

and

$$\| x_n - x^\star \| \leq t_n - t^\star. \tag{3.12}$$

Furthemore, if

$$K\ (t_0 - t^\star) + M < L\ t^\star, \tag{3.13}$$

or

$$\frac{K}{2\ L} + M < 1, \tag{3.14}$$

$x^\star$ is the unique solution of equation $F(x) = 0$ in $\overline{U}(x_0, t_0 - t^\star)$.

Proof. As in (2.26) for $x_k \in \overline{U}(x_0, t^\star)$, we arrive at:

$$\| x_{k+1} - x_k \| \leq \frac{1}{1 - L\ (t_0 - t_k)} \left(\frac{K}{2}\ (t_{k-1} - t_k) + M \right) (t_{k-1} - t_k) = t_k - t_{k+1}, \tag{3.15}$$

which implies (3.11). Estimate (3.12) follows from (3.11) by using standard majorizing techniques [5], [11].
It is show uniqueness part. Let $y^\star \in \overline{U}(x_0, t_0 - t^\star)$ be a solution of equation $F(x) = 0$. Using (1.2), we obtain the identity

$$\begin{aligned} x_{k+1} - y^\star &= x_k - G'(x_k)^{-1}\ F(x_k) - y^\star \\ &= -(G'(x_k)^{-1}\ G'(x_0)) \Big((G'(x_k)\ (x_k - y^\star) - (G(x_k) - G(y^\star))) + \\ &\quad (F(x_k) - G(x_k) - (F(y^\star) - G(y^\star))) \Big), \end{aligned} \tag{3.16}$$

which as in (2.26) leads to:

$$\begin{aligned}
&\| x_{k+1} - y^\star \| \\
&\leq \frac{1}{1 - L \ \| x_k - y^\star \|} \left(\frac{K}{2} \ \| x_k - y^\star \| + M \right) \ \| x_k - y^\star \| \\
&\leq \frac{1}{1 - L \ (t_0 - t^\star)} \left(\frac{K}{2} \ (t_0 - t^\star) + M \right) \ \| x_k - y^\star \| \\
&\leq \frac{1}{L \ t^\star} \left(\frac{K}{2} \ (t_0 - t^\star) + M \right) \ \| x_k - y^\star \| \\
&\leq \| x_k - y^\star \| \qquad \text{(by (3.13) or (3.14))},
\end{aligned}$$

which shows $\lim_{k \to \infty} x_k = y^\star$. But, we already showed $\lim_{k \to \infty} x_k = x^\star$. Hence, we conlude $x^\star = y^\star$.

That completes the proof of Theorem 6. □

4. Special cases and applications

A direct comparison between Theorems 1 and 6 is not possible, since the former uses an increasing majorizing sequence and the latter a decreasing one. However, we can compare the sufficient convergence condition (2.5) with the corresponding ones in Lemma 5, at least in some intersting special cases.

(1) Case $F(x) = G(x)$, $(x \in \mathcal{D})$. (Newton's method). Condition (2.5) reduces to the famous Newton–Kantorovich hypothesis [5], [8], [11]:

$$h_K = 2 \ K \ \eta \leq 1, \tag{4.1}$$

since $M = 0$, where as condition (3.2) becomes:

$$h_A = 2 \ \overline{K} \ \eta \leq 1, \tag{4.2}$$

where

$$\overline{K} = \frac{1}{8} \left(K + 4 \ L + \sqrt{K^2 + 8 \ L \ K} \right), \tag{4.3}$$

since,

$$\gamma_0 = \frac{\sqrt{K^2 + 8 \ L \ K} - K}{\sqrt{K^2 + 8 \ L \ K} + K}. \tag{4.4}$$

Note that

$$K \leq \overline{K} \tag{4.5}$$

hold in general, and $\dfrac{\overline{K}}{K}$ can be arbitrarily large [3]–[5]. In case $L < K$, then strict inequality holds (4.5).

It follows from (4.1), (4.2), and (4.5) that

$$h_K \leq 1 \Longrightarrow h_A \leq 1, \tag{4.6}$$

but not necessarily vice verca unles, if $K = L$ (see also Example1).

(2) Case $F(x) \neq G(x)$ $(x \in \mathcal{D})$. Then we can only compare Theorem 1 with Theorem 6 using numerical examples.

Example1. Let $\mathcal{X} = \mathcal{Y} = \boldsymbol{R}$, $x_0 = 1$, $\mathcal{D} = [\alpha, 2-\alpha]$, $\alpha \in [0, \frac{1}{2})$, and define function F and G on $\mathcal{D}$ by

$$F(x) = G(x) + \epsilon \ |x-1| \qquad \text{and} \qquad G(x) = x^3 - \alpha, \tag{4.7}$$

where, ϵ is a given real number.
Using (2.1)–(2.4), we obtain:

$$\eta = \frac{1}{3}\ (1-\alpha), \quad L = 3-\alpha, \quad K = 2\ (2-\alpha), \quad \text{and} \quad M = |\epsilon|.$$

Note that function F is continuous but not differentiable on D, since a $F'(1)$ does not exist.
Hypothesis (4.1) is violated, since

$$4\ (2-\alpha)\left(\frac{1}{3}\ (1-\alpha) + |\epsilon|\right) > 1 \qquad \text{for all} \quad \epsilon, \quad \text{and} \quad \alpha \in [0, \frac{1}{2}). \tag{4.8}$$

That is there is no guarantee that sequence $\{x_n\}$, starting at $x_0 = 1$ converges under the hypotheses of Theorem 1.
However, our Theorem 6 can apply to solve equation $F(x) = 0$.

Let us consider two cases:

(1) <u>Case $\epsilon = 0$.</u> Then condition (4.2) holds for $\alpha \in [.450339002, \frac{1}{2}]$, which is the same range, given by us in [4] using a different approach.

(2) <u>Case $\epsilon \neq 0$.</u> Choose e.g.: $\epsilon = .1$, and $\alpha = .49$. Then, we get:

$$\eta = .17, \qquad L = 2.51, \qquad K = 3.02, \qquad M = .1, \qquad t_0 = .398406374,$$
$$t_1 = .22840637, \qquad \gamma_0 = .410812, \qquad \gamma_0^1 = .369936,$$

$$\beta = .8534, \qquad \frac{K}{2\ L} = .601593625, \qquad \text{and} \quad \beta_0 = 1.058703597.$$

Hence hypotheses (3.5) and (3.6) of Lemma 5 are satisfied. That is the conclusions of Theorem 6 apply to solve equation $F(x) = 0$.

We complete this study with an example involving a nonlinear integral equation of Chandrasekhar–type [1], [2], [5], [7], [11]. For simplicity, we choose $F(x) = G(x)$ $(x \in \mathcal{D})$.
Example2. Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0,1]$ be the space of real–valued continuous functions defined on the interval $[0,1]$ with norm

$$\| x \| = \max_{0 \le s \le 1} |x(s)|.$$

Let $\theta \in [0,1]$ be a given parameter. Consider the "cubic" integral equation

$$u(s) = u^3(s) + \lambda\, u(s) \int_0^1 q(s,t)\, u(t)\, dt + y(s) - \theta. \tag{4.9}$$

Here the kernel $q(s,t)$ is a continuous function of two variables defined on $[0,1] \times [0,1]$; the parameter λ is a real number called the "albedo" for scattering; $y(s)$ is a given continuous function defined on $[0,1]$ and $x(s)$ is the unknown function sought in $\mathcal{C}[0,1]$. Equations of the form (4.9) arise in the theory of radiative transfer, neutron transport, and the kinetic theory of gasses [1], [2], [4], [7].
For simplicity, we choose $u_0(s) = y(s) = 1$, and $q(s,t) = \dfrac{s}{s+t}$, for all $s \in [0,1]$, and

$t \in [0,1]$, with $s+t \neq 0$. If we let $\mathcal{D} = U(u_0, 1-\theta)$, $g = 0$ and define the operator F on $\mathcal{D}$ by

$$F(x)(s) = x^3(s) + \lambda\, x(s) \int_0^1 q(s,t)\, x(t)\, dt + y(s) - \theta, \tag{4.10}$$

for all $s \in [0,1]$, then every zero of F satisfies equation (4.9). We have the estimates

$$\max_{0 \leq s \leq 1} \left| \int \frac{s}{s+t}\, dt \right| = \ln 2.$$

Therfore, if we set $b_0 = \| F'(u_0)^{-1} \|$, then it follows from (2.1)–(2.4) that:

$$\eta = q\ (|\lambda|\ \ln 2 + 1 - \theta), \quad K = 2\ q\ (|\lambda|\ \ln 2 + 3\ (2-\theta)),$$
$$L = q\ (2\ |\lambda|\ \ln 2 + 3\ (3-\theta)).$$

It follows from Theorem 6 that if condition (4.2) holds, then problem (4.9) has a unique solution near u_0. This condition is weaker than the conditions given before using the Newton–Kantorovich hypothesis (4.1).
Note also that $L < K$ for all $\theta \in [0,1]$.

Conclusion

We provide a semilocal convergence analysis for a modified Newton method considered also in [4], [5], [12], [13], [15], in order to approximate a locally unique solution of an equation in a Banach space.
Using a combination of Lipschitz and center–Lipschitz conditions, insted of only Lipschitz conditions used in the works above, we provide an analysis with the following advantages: larger convergence domain and weaker sufficient convergence conditions. Note that these advantages are obtained under the same computational cost, since in practice the computation of the Lipschitz constant K requires the computation of L.
Numerical examples further validating the results are also provided.

References

[1] I.K. Argyros, Quadratic equations and applications to Chandrasekhar's and related equations, Bull. Austral. Math., 32 (1985), 275–292.

[2] I.K. Argyros, *Polynomial operator equations in abstract spaces and applications*, CRC Press LLC, Boca Raton, Florida, USA, 1998.

[3] I.K. Argyros, *On the Newton–Kantorovich hypothesis for solving equations*, J. Comput. Appl. Math., **11** (2004), 103–110.

[4] I.K. Argyros, *A unifying local–semilocal convergence analysis and applications for two–point Newton–like methods in Banach space*, J. Math. Anal. and Appl., **298** (2004), 374–397.

[5] I.K. Argyros, *Computational theory of iterative methods*, Studies in Computational Mathematics, **15**, Elsevier, 2007, New York, U.S.A.

[6] E. Catinas, *On some iterative methods for solving nonlinear equations*, Revue d'Anal. Numér. Théor. Approx., **23** (1994), 47–53.

[7] S. Chandrasekhar, *Radiative transfer*, Dover Publ., New York, 1960, USA.

[8] P. Deuflhard, G. Heindl, *Affine invariant convergence theorems for Newton's method, and extensions to related methods*, SIAM J. Numer. Anal., **16** (1979), 1–10.

[9] J.M. Gutierrez, *A new semilocal convergence theorem for Newton's method*, J. Comput. Appl. Math., **79** (1997), 131–145.

[10] M.A. Hernánadez, M.J. Rubio, J.A. Ezquerro, *Secant–like methods for solving integral equations of the Hammerstein type*, J. Comput. Appl. Math., **115** (2000), 245–254.

[11] L.V. Kantorovich, G.P. Akilov, *Functional analysis in normed spaces*, Pergamon Press, New York, 1964.

[12] H.Ch. Lai, P.Y. Wu, *Errors bounds of Newton type process on Banach spaces*, Numer. Math., **39** (1982), 175–193.

[13] D.F. Nguen, P.P. Zabrejko, *The majorant method in the theory of Newton–Kantorovich approximations, and the Pták error estimates*, Numer. Funct. Anal. Opt., **9(5–6)**, (1987), 671–684.

[14] F.A. Potra, *Sharp error bounds for a class of Newton–like methods*, Libertas Mathematica, **5** (1985), 71–84.

[15] P.D. Proinov, *A generalization of the Banach contraction principle with high order of convergence of successive approximations*, Nonlinear Anal., **67** (2007), 2361–2369.

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On the Local Convergence of the Gauss–Newton Method

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Abstract. The local convergence of the Gauss–Newton method is studied under a combination of the radius and center–Lipschitz average functions [3], [7], [8]. Using more precise estimates and under the same or less computational cost, we provide an analysis of this method with the following advantages over the corresponding results in [8]: larger convergence ball, and finer error estimates on the distances involved.

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1. INTRODUCTION

In this study we are concerned with the problem of approximating a solution $x^\star$ of equation

$$F'(x)^T\ F(x) = 0, \tag{1.1}$$

where F is a Fréchet–differentiable operator defined on $\mathcal{X} = \mathbb{R}^n$, with values on $\mathcal{Y} = \mathbb{R}^m$ $(m \geq n)$.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time–invariant system is driven by the equation $\dot{x} = T(x)$, for some suitable operator T, where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative–when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

We are seeking least–square solutions of (1.1). That is we solve the minimization problem:

$$\min_{x \in \mathcal{X}} \frac{1}{2} F(x)^T F(x). \tag{1.2}$$

We use the famous Gauss–Newton method

$$x_{k+1} = x_k - (F'(x_k)^T F'(x_k))^{-1} F'(x_k)^T F(x_k) \quad (x_0 \in \mathcal{X}), \quad (k \geq 0) \tag{1.3}$$

to generate a sequence approximating a solution $x^\star$ of (1.2).

There is an extensive literature on the local as well as the semilocal convergence analysis of Newton–type methods under various conditions in the more general setting when $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces [1]–[11].

In particular, in the case of Gauss–Newton method (1.3), Li et al. provided a local convergence analysis in [8] using the concept of the generalized Lipschitz condition with L average (inaugurated by Wang in [10]), which unified the Kantorovich–domain–type [1]–[3], [6] approach with the Smale–point–estimate–type approach [3], [9], [10].

Recently, we have successfully used in [1]–[3] a combination of Lipschitz and center–Lipschitz conditions (instead of only Lipschitz conditions as in to provide a finer local and semilocal convergence analysis for Newton–type methods, when F is an isomorphism. The main idea is derived from the observation that more precise upper bounds on the norms $\| F'(x)^{-1} F'(x^\star) \|$ can be obtained if the needed center–Lipschitz condition is used:

$$\begin{aligned} &\| F'(x^\star)^{-1} (F'(x) - F'(x^\star)) \| \leq \ell_0 \| x - x^\star \|, \\ &\text{for all } x \in U(x^\star, r_0) = \{x \in \mathcal{X} : \| x - x^\star \| \leq r_0\} \subseteq \mathcal{X}, \ r_0 > 0, \ \ell_0 > 0 \end{aligned} \tag{1.4}$$

instead of the commonly used Lipschitz condition ([4]–[11]):

$$\| F'(x^\star)^{-1} (F'(x) - F'(y)) \| \leq \ell \| x - y \|, \quad \text{for all } x, y \in U(x^\star, r_0), \ \ell > 0. \tag{1.5}$$

If condition (1.5) holds, then, it follows that there exists $\ell_0 \in [0, \ell]$, such that (1.4) is satisfied, and $\dfrac{\ell}{\ell_0}$ can be arbitrarily large [3].

It turn out that these ideas can be used to study the local convergence of the Gauss–Newton method (1.3). In particular, we provide a local convergence analysis with the following advantages over the work by Li et al. [8]:

(1) Larger convergence ball. Enlarging the convergence ball is very important in computational mathematics because it allows for a wider choice of initial guesses in the case of the local convergence of the Gauss–Newton method (1.3).
(2) Finer estimates on the distances involved, which implies that fewer iterations are needed to achieve a desired error tolerance.
(3) An at least as precise information is provided on the uniqueness of the solution.

The above improvements are also obtained under the same computational cost since the computation of the radius Lipschitz condition with L average (see (2.1)) requires that of the center–Lipschitz with L_0 average (see (2.2)).

2. Preliminaries

We need to introduce the concept of Lipschitz condition inaugurated in [10]:

Definition 1. The operator $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ satisfies the radius Lipschitz condition with L average on $U(x^\star, r_0)$ if

$$\| F(x) - F(x_\tau) \| \le \int_{\tau\, s(x)}^{s(x)} L(t)\, dt, \quad \text{for all } x \in U(x^\star, r_0),\ 0 \le \tau \le 1, \tag{2.1}$$

where, L is a positive non–decreasing function on $[0, r_0]$, $s(x) = \| x - x^\star \|$, and $x_\tau = x^\star + \tau\,(x - x^\star)$.

Definition 2. The operator $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ satisfies the center–Lipschitz condition with L_0 average on $U(x^\star, r_0)$ if

$$\| F(x) - F(x^\star) \| \le \int_0^{s(x)} L_0(t)\, dt, \quad \text{for all } x \in U(x^\star, r_0), \tag{2.2}$$

where, L_0 is a positive non–decreasing function on $[0, r_0]$.

Note that in [7], [8], [10], the same function L is used in Definitions 1 and 2. However,

$$L_0(t) \le L(t) \quad t \in [0, r_0) \tag{2.3}$$

holds in general and $\dfrac{L}{L_0}$ can be arbitrarily large [1]–[3].

We provide an example where strict inequality holds in (2.3).

Example1. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}^2$, be equipped with the Euclidean norm, $x^\star = 0$, and define function F on $U(0,1)$ by

$$F(z) = (e^x - 1, e^y - 1)^T, \quad z = (x, y)^T. \tag{2.4}$$

Then, using (2.1), and (2.2), we obtain:

$$L(t) = \ell = \sqrt{2}\, e, \quad \text{and} \quad L_0(t) = \ell_0 = \sqrt{2}\,(e-1) \quad \text{for all } t \in [0,1]. \tag{2.5}$$

It follows from (2.5) that

$$L_0(t) < L(t) \quad \text{for all } t \in [0,1]. \tag{2.6}$$

As noted in the introduction using our (2.2) instead of (2.1), which was employed in [8] for the computation of the norms

$$\| (F'(x)^T\, F'(x))^{-1}\, F'(x) \|\ \| F'(x) - F'(x^\star) \|, \quad x \in U(x^\star, r_0),$$

leads to the advantages, as stated at the end of the introduction of this paper.

Let $\mathbb{R}^{m\times n}$ be the set of all $m \times n$ matrices, and A^+ be the generalized inverse of $A \in \mathbb{R}^{m\times n}$. Then, when $m \ge n$, and A is of full rank, we have $A^+ = (A^T\, A)^{-1}\, A^T$.

We need the lemmas:

Lemma 3. [7] *Let $A, E \in \mathbb{R}^{m\times n}$. Assume $B = A + E$, and $\| A^+ \| \| E \| < 1$. Then, the following hold:*

$$\text{rank } (B) \geq \text{rank } (A).$$

If rank $(A) = n$, $m \geq n$, then we have rank $(B) = n$.

Lemma 4. [7] *Let $A, E \in \mathbb{R}^{m\times n}$. Assume $B = A + E$, and $\| A^+ \| \| E \| < 1$. Then, the following hold:*

$$\| B^+ \| \leq \frac{\| A^+ \|}{1 - \| A^+ \| \| E \|},$$

provided that rank $(B) =$ rank (A).

Moreover,

$$\| B^+ - A^+ \| \leq \frac{\sqrt{2} \ \| A^+ \|^2 \ \| E \|}{1 - \| A^+ \| \| E \|}$$

provided that rank $(A) =$ rank $(B) = \min \{m, n\}$.

Lemma 5. [7], [8] *Let M be a positive non–decreasing function on $[0, r_0]$. Then, for each $a \geq 0$, the functions*

$$f_a(t) = \frac{1}{t^{1+a}} \int_0^t u^a \ M(u) \ du$$

and

$$g(t) = \frac{1}{t^2} \int_0^t (2t - u) \ M(u) \ du$$

are non–decreasing on $[0, r_0]$.

3. Local convergence analysis of method (1.3)

We shall show the main local convergence result for the Gauss–Newton method (1.3) using a combination of the radius Lipschitz condition with L average, and the center–Lipschitz condition with L_0 average on $U(x^\star, r_0)$:

Theorem 6. *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be continuously Fréchet–differentiable on $U(x^\star, r_0)$, where $x^\star$ is a solution of (1.2) and $r_0 > 0$. Set*

$$b = \| (F'(x^\star)^T \ F'(x^\star))^{-1} \ F'(x^\star)^T \|, \quad \text{and} \quad c = \| F(x^\star) \| .$$

Moreover, assume operator $F'(x^\star)$ is of full rank and F' satisfies the radius Lipschitz condition with L average and the center–Lipschitz condition with L_0 average on $U(x^\star, r_0)$.

Furthemore, assume function h_0 has a minimal zero r on $[0, r_0]$, which also satisfies:

$$b \int_0^r L_0(t) \ dt < 1, \tag{3.1}$$

where,

$$h_0(p) = \left(\int_0^p L(t) \ t \ dt + (\sqrt{2} \ b \ c + p) \int_0^p L_0(t) \ dt \right) b - p. \tag{3.2}$$

Then, sequence $\{x_k\}$ $(k \geq 0)$ generated by the Gauss–Newton method (1.3) is well defined, remains in $U(x^\star, r)$ for all $k \geq 0$, and converges to $x^\star$, provided that $x_0 \in U(x^\star, r)$ with $x_0 \neq x^\star$.

Moreover, the following estimates hold for all $k \geq 1$:

$$\begin{aligned} \| x_k - x^\star \| &\leq \alpha \; \| x_{k-1} - x^\star \|^2 + \beta \; \| x_{k-1} - x^\star \| \\ &\leq q^k \; \| x_0 - x^\star \|, \end{aligned} \tag{3.3}$$

where,

$$\alpha = \frac{b \displaystyle\int_0^{s(x_0)} L(t)\; t\; dt}{\left(1 - b \displaystyle\int_0^{s(x_0)} L_0(t)\; dt\right) s(x_0)^2}, \tag{3.4}$$

$$\beta = \frac{\sqrt{2}\, b^2\; c \displaystyle\int_0^{s(x_0)} L(t)\; dt}{\left(1 - b \displaystyle\int_0^{s(x_0)} L_0(t)\; dt\right) s(x_0)}, \tag{3.5}$$

and

$$0 < q_0 = \alpha\; s(x_0) + \beta < 1. \tag{3.6}$$

Proof. Using (2.2), (3.1), and the choice of r, we obtain in turn:

$$\begin{aligned} \| (F'(x^\star)^T\; F'(x^\star))^{-1}\; F'(x^\star)^T \| \; \| F'(x) - F'(x^\star) \| &\leq b \int_0^{s(x_0)} L_0(t)\; dt \\ &< b \int_0^r L_0(t)\; dt < 1. \end{aligned} \tag{3.7}$$

In view of Lemmas 3, 4 respectively, and (3.7), $F'(x)$ is full rank and satisfies

$$\| (F'(x)^T\; F'(x))^{-1}\; F'(x)^T \| \leq \frac{b}{1 - b \displaystyle\int_0^{s(x_0)} L_0(t)\; dt}, \tag{3.8}$$

and

$$\begin{aligned} &\| (F'(x)^T\; F'(x))^{-1}\; F'(x)^T - (F'(x^\star)^T\; F'(x^\star))^{-1}\; F'(x^\star)^T \| \\ &\leq \frac{\sqrt{2}\; b \displaystyle\int_0^{s(x_0)} L_0(t)\; dt}{1 - b \displaystyle\int_0^{s(x_0)} L_0(t)\; dt} \end{aligned} \tag{3.9}$$

for all $x \in U(x^\star, r)$.

Moreover, in view of Lemma 5 for $a = 0$ and $a = 1$, we get in turn

$$\begin{aligned} \frac{\displaystyle\int_0^{s(x_0)} M(t)\; t\; dt}{s(x_0)} &= s(x_0)\; \frac{\displaystyle\int_0^{s(x_0)} M(t)\; t\; dt}{s(x_0)^2} \\ &\leq r\; \frac{\displaystyle\int_0^r M(t)\; t\; dt}{r^2} = \frac{\displaystyle\int_0^r M(t)\; t\; dt}{r}, \end{aligned} \tag{3.10}$$

and

$$\frac{\displaystyle\int_0^{s(x_0)} M(t)\ dt}{s(x_0)} \leq \frac{\displaystyle\int_0^{r} M(t)\ dt}{r}. \tag{3.11}$$

Using (3.1), (3.6), (3.10), (3.11), and the choice of r, for $M = L_0$ or L, we obtain:

$$\begin{aligned} 0 < q &= \frac{1}{s(x_0)} \left(\frac{b \displaystyle\int_0^{s(x_0)} L(t)\ t\ dt}{1 - b \displaystyle\int_0^{s(x_0)} L_0(t)\ dt} + \frac{\sqrt{2}\ b^2\ c \displaystyle\int_0^{s(x_0)} L(t)\ dt}{1 - b \displaystyle\int_0^{s(x_0)} L_0(t)\ dt} \right) \\ &\leq \frac{1}{r} \left(\frac{b \displaystyle\int_0^{r} L(t)\ t\ dt}{1 - b \displaystyle\int_0^{r} L_0(t)\ dt} + \frac{\sqrt{2}\ b^2\ c \displaystyle\int_0^{r} L(t)\ dt}{1 - b \displaystyle\int_0^{r} L_0(t)\ dt} \right) < 1. \end{aligned} \tag{3.12}$$

Using Gauss–Newton method (1.3), we obtain the identity

$$\begin{aligned} x_k - x^\star &= x_{k-1} - x^\star - (F'(x_{k-1})^T\ F'(x_{k-1}))^{-1}\ F'(x_{k-1})^T\ F(x_{k-1}) \\ &= (F'(x_{k-1})^T\ F'(x_{k-1}))^{-1}\ F'(x_{k-1})^T\ (F'(x_{k-1})\ (x_{k-1} - x^\star) - \\ &\quad F(x_{k-1}) + F(x^\star)) + (F'(x^\star)^T\ F'(x^\star))^{-1}\ F'(x^\star)^T\ F(x^\star) - \\ &\quad (F'(x_{k-1})^T\ F'(x_{k-1}))^{-1}\ F'(x_{k-1})^T\ F(x^\star). \end{aligned} \tag{3.13}$$

In particular for $k = 1$ in (3.12), since $x_0 \in U(x^\star, r)$, we obtain in turn using (2.1), (2.2), (3.8), and (3.9):

$$\begin{aligned} &\| x_1 - x^\star \| \\ &\leq \| (F'(x_0)^T\ F'(x_0))^{-1}\ F'(x_0)^T \| \times \\ &\| \int_0^1 (F'(x_0) - F'(x_0 + \tau\ (x^\star - x_0))\ (x_0 - x^\star)\ d\tau \| + \\ &\| (F'(x^\star)^T\ F'(x^\star))^{-1}\ F'(x^\star)^T - (F'(x_0)^T\ F'(x_0))^{-1}\ F'(x_0)^T \|\ \| F(x^\star) \| \end{aligned}$$

$$\begin{aligned} &\leq \frac{b}{1 - b \displaystyle\int_0^{s(x_0)} L_0(t)\ dt} \int_0^1 \int_{\tau\ s(x_0)}^{s(x_0)} L(t)\ dt\ s(x_0)\ d\tau + \frac{\sqrt{2}\ b^2\ c \displaystyle\int_0^{s(x_0)} L_0(t)\ dt}{1 - b \displaystyle\int_0^{s(x_0)} L_0(t)\ dt} \\ &= \frac{b}{1 - b \displaystyle\int_0^{s(x_0)} L_0(t)\ dt} \left(\int_0^{s(x_0)} L(t)\ t\ dt + \sqrt{2}\ b\ c \int_0^{s(x_0)} L_0(t)\ dt \right). \end{aligned} \tag{3.14}$$

It follows from (3.6) and (3.14) that:

$$\| x_1 - x^\star \| \leq q_0\ \| x_0 - x^\star \|, \tag{3.15}$$

which implies $x_1 \in U(x^\star, r)$.

Hence, (3.3) holds for $k = 0$.

Assume $x_k \in U(x^\star, r)$, by exchanging x_0, x_1 with x_k, x_{k+1}, we obtain:

$$\begin{aligned}
&\| x_{k+1} - x^\star \| \\
&\leq \frac{b \displaystyle\int_0^{s(x_k)} L(t)\ t\ dt}{1 - b \displaystyle\int_0^{s(x_k)} L_0(t)\ dt} + \frac{\sqrt{2}\ b^2\ c \displaystyle\int_0^{s(x_k)} L_0(t)\ dt}{1 - b \displaystyle\int_0^{s(x_k)} L_0(t)\ dt} \\
&\leq \frac{b \displaystyle\int_0^{s(x_k)} L(t)\ t\ dt\ s(x_k)^2}{s(x_k)^2 \left(1 - b \displaystyle\int_0^{s(x_0)} L_0(t)\ dt\right)} + \frac{\sqrt{2}\ b^2\ c \displaystyle\int_0^{s(x_k)} L_0(t)\ dt\ s(x_k)}{s(x_k) \left(1 - b \displaystyle\int_0^{s(x_k)} L_0(t)\ dt\right)} \\
&\leq q_0\ \| x_k - x^\star \| \leq q_0^{k+1}\ \| x_0 - x^\star \| < r
\end{aligned} \tag{3.16}$$

(by Lemma 5, which implies $x_{k+1} \in U(x^\star, r)$, and $\lim_{k \to \infty} x_k = x^\star$.

That completes the induction and the proof of Theorem 6. □

Remark 7. If estimate (2.3) holds as equality, then our Theorem 7 reduces to Theorem 3.1 in [8]. Otherwise, i.e., if (2.3) holds a strict inequality, then our result improves Theorem 3.1 in [8] under the same computational cost, since in practice the evaluation of function L requires that of L_0. Let h, q, r_1 be as h_0, q_0, r respectively by simply replacing L_0 by L.

Then, we have:

$$r_1 < r \tag{3.17}$$

and

$$q_0 < q. \tag{3.18}$$

Since, r_1 and q were used in [8], it follows from (3.17) and (3.18) that in this case a larger convergence ball is obtained and smaller ratio than in [8].

We state the local convergence result for the Gauss–Newton method (1.3) using only the weaker center–Lipschitz condition with L_0 average on $U(x^\star, r_0)$.

Theorem 8. *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be continuously Fréchet–differentiable on $U(x^\star, r_0)$, where $x^\star$ is a solution of (1.2) and $r_0 > 0$.*

Moreover, assume operator $F'(x^\star)$ is of full rank and F' satisfies the center–Lipschitz condition with L_0 average on $U(x^\star, r_0)$.

Furthermore, assume function H_0 has a minimal zero r on $[0, r_0]$, which also satisfies:

$$b \int_0^r L_0(t)\ dt < 1,$$

where, b is defined in Theorem 6, and H_0 has the following form:

$$H_0(p) = \left(\int_0^p (2\ p - t)\ L_0(t)\ dt + (\sqrt{2}\ b\ c + p) \int_0^p L_0(t)\ dt \right) b - p. \tag{3.19}$$

Then, sequence $\{x_k\}$ $(k \geq 0)$ generated by the Gauss–Newton method (1.3) is well defined, remains in $U(x^\star, r)$ for all $k \geq 0$, and converges to $x^\star$, provided that $x_0 \in U(x^\star, r)$

with $x_0 \neq x^\star$.

Moreover, the following estimates hold for all $k \geq 1$*:*

$$\begin{aligned} \| x_k - x^\star \| &\leq \alpha_0 \ \| x_{k-1} - x^\star \|^2 + \beta_0 \ \| x_{k-1} - x^\star \| \\ &\leq \overline{q}^k \ \| x_0 - x^\star \|, \end{aligned} \tag{3.20}$$

where,

$$\alpha_0 = \frac{b \displaystyle\int_0^{s(x_0)} (2\ s(x_0) - t)\ L_0(t)\ dt}{\left(1 - b \displaystyle\int_0^{s(x_0)} L_0(t)\ dt\right)\ s(x_0)^2},$$

$$\beta_0 = \frac{\sqrt{2}\ b^2\ c \displaystyle\int_0^{s(x_0)} L_0(t)\ dt}{\left(1 - b \displaystyle\int_0^{s(x_0)} L_0(t)\ dt\right)\ s(x_0)},$$

and

$$0 < \overline{q} = \alpha_0\ s(x_0) + \beta_0 < 1.$$

Proof. We shall show (3.20) for all $k \geq 1$.

As in the proof of Theorem 6, using (1.3) for $k = 1$, (2.1), and (2.2), we get:

$$\begin{aligned}
&\| x_1 - x^\star \| \\
&\leq \| (F'(x_0)^T\ F'(x_0))^{-1}\ F'(x_0)^T \| \times \\
&\| \int_0^1 (F'(x_0) - F'(x_0 + \tau\ (x^\star - x_0))\ (x_0 - x^\star)\ d\tau \| + \\
&\| (F'(x^\star)^T\ F'(x^\star))^{-1}\ F'(x^\star)^T - (F'(x_0)^T\ F'(x_0))^{-1}\ F'(x_0)^T \|\ \| F(x^\star) \|
\end{aligned}$$

$$\begin{aligned}
&\leq b \int_0^1 \| (F'(x_0) - F'(x^\star)) + (F'(x^\star) - F'(x_0 + \tau\,(x^\star - x_0)) \|\ \| x_0 - x^\star \|\ d\tau + \\
&\| (F'(x^\star)^T\ F'(x^\star))^{-1}\ F'(x^\star)^T - (F'(x_0)^T\ F'(x_0))^{-1}\ F'(x_0)^T \|\ \| F(x^\star) \|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{b}{1 - b \displaystyle\int_0^{s(x_0)} L_0(t)\ dt} \int_0^1 \left(\int_0^{s(x_0)} L_0(t)\ dt + \int_0^{\tau\ s(x_0)} L_0(t)\ dt \right) s(x_0)\ d\tau \\
&+ \frac{\sqrt{2}\ b^2\ c \displaystyle\int_0^{s(x_0)} L_0(t)\ dt}{1 - b \displaystyle\int_0^{s(x_0)} L_0(t)\ dt}
\end{aligned}$$

$$\begin{aligned}
&= \frac{b}{s(x_0)^2 \left(1 - b \displaystyle\int_0^{s(x_0)} L_0(t)\ dt\right)} \left(\int_0^{s(x_0)} (2\ s(x_0) - t)\, L_0(t)\, dt\ \| x_0 - x^\star \|^2 + \right. \\
&\left. \sqrt{2}\, b\ c\, s(x_0) \int_0^{s(x_0)} L_0(t)\, dt\ \| x_0 - x^\star \| \right)
\end{aligned}$$

$$= \overline{q}\ \| x_0 - x^\star \|.$$

Therefore, $x_1 \in U(x^\star, r)$. Clearly, (3.20) holds for $k = 0$. Assume that $x_k \in U(x^\star, r)$, (3.20) holds for $k > 0$, and $s(x_k)$ decreases monotonically. Then, using Lemmas 3, and 5, we have in turn:

$$\begin{aligned} &\| x_{k+1} - x^\star \| \\ &\leq \frac{b}{s(x_k)^2 \left(1 - b \int_0^{s(x_k)} L_0(t)\, dt\right)} \left(\int_0^{s(x_k)} (2\, s(x_k) - t)\, L_0(t)\, dt \, \| x_k - x^\star \|^2 + \right. \\ &\left. \sqrt{2}\, b\, c\, s(x_k) \int_0^{s(x_k)} L_0(t)\, dt \, \| x_k - x^\star \| \right) \\ &\leq \frac{b}{s(x_0)^2 \left(1 - b \int_0^{s(x_0)} L_0(t)\, dt\right)} \left(\int_0^{s(x_0)} (2\, s(x_0) - t)\, L_0(t)\, dt \, \| x_k - x^\star \|^2 + \right. \\ &\left. \sqrt{2}\, b\, c\, s(x_0) \int_0^{s(x_0)} L_0(t)\, dt \, \| x_k - x^\star \| \right) \\ &\leq \overline{q} \, \| x_k - x^\star \| \leq \overline{q}^{k+1} \, \| x_0 - x^\star \| . \end{aligned}$$

That completes the induction, and the proof of Theorem 8. □

Remark 9. If estimate (2.3) holds as equality, then our Theorem 8 reduces to Theorem 3.2 in [8]. Otherwise, i.e., if (2.3) holds a strict inequality, then our result improves Theorem 3.2 in [8] under less computational cost, since the evaluation of L is more expensive than the evaluation of L_0. Let $H, \overline{\overline{q}}, r_2$ be as $H_0, \overline{q}, r$ respectively by simply replacing L_0 by L.

Then, we have:

$$r_2 < r$$

and

$$\overline{q} < \overline{\overline{q}}.$$

4. Applications

In this section we apply the results of Section 3 in a concrete case. Assume: $L_0(t) = L_0$, and $L(t) = L$ on $[0, \infty)$. That is consider the Kantorovich–type case. Then in the case of Theorem 6, and also using the notation introduced in Remark 7, we can easily obtain:

$$r = \frac{2\,(1 - \sqrt{2}\, b^2\, c\, L_0)}{(2\, L_0 + L)\, b},$$

$$q_0 = \frac{(L\, s(x_0) + 2\, \sqrt{2}\, b\, c\, L_0)\, b}{2\,(1 - b\, L_0\, s(x_0))},$$

$$r_1 = \frac{2\,(1 - \sqrt{2}\, b^2\, c\, L)}{3\, b\, L} \leq r \tag{4.1}$$

and

$$q = \frac{(s(x_0) + 2\, \sqrt{2}\, b\, c)\, b\, L}{2\,(1 - b\, L\, s(x_0))} \geq q_0. \tag{4.2}$$

Note that strict inequality holds in (4.1), and (4.2), if $L_0 < L$.

We provide a numerical example using the above values.

Example2. Return back to Example 1. Then, we have $c = 0$, $b = \sqrt{2}$, and for $x_0 = .2$,

$$r_1 = .122626481, \qquad q = .595665552$$

$$r = .162473616 \quad \text{and} \quad q_0 = .414155282.$$

That is, we conclude that estimates (4.1), and (4.2) hold as strict inequalities.

Below, we provide a comparison between error bounds obtained in Theorem 3.1 [8], and Theorem 6 in this study.

Comparison table

k	$q^{k+1} \parallel x_0 - x^\star \parallel$	$q_0^{k+1} \parallel x_0 - x^\star \parallel$
0	.119130759	.082831056
1	.070960689	.03430492
2	.042268004	.014207564
3	.025177097	.005884138
⋮	⋮	⋮
20	.000006323	.000000004
22	.000002243	0

Finally, we provide an example where (2.2) holds, but (2.1) is violated.

Example3. Let $\mathcal{X} = \mathbb{R}$, and $\mathcal{Y} = \mathbb{R}^2$. Define

$$g_1(x) = \int_0^x \left(1 + \frac{x}{4}\ \sin\frac{\pi}{x}\right) dx, \quad g_2(x) = \frac{1}{8}\ x^2 \quad \text{for all} \quad x \in \mathcal{X}$$

and

$$g = (g_1, g_2)^T.$$

Then, we get

$$g_1'(x) = \begin{cases} 1 + \dfrac{x}{4}\ \sin\dfrac{\pi}{x}, & if \quad x \neq 0 \\ 1, & if \quad x = 0 \end{cases}$$

and

$$g_2'(x) = \frac{x}{4} \quad \text{for all} \quad x \in \mathcal{X}.$$

Clearly, $x^\star = 0$ is a solution of (1.2) (with F is replaced by g).

We also have

$$\parallel g'(x) - g'(x^\star) \parallel = \frac{1}{2} \parallel x - x^\star \parallel \quad \text{for all} \quad x \in \mathcal{X}.$$

That is, we can set

$$L_0(t) = \frac{1}{2}.$$

Then, since $b = \sqrt{2}$, and $c = 0$, Theorem 8 for any $x_0 \in U(x^\star, \dfrac{2\sqrt{5}}{5})$, guarantees that Gauss–Newton method converges to $x^\star$.

However, there is no positive integrable function L such that (2.1) holds. Indeed, we have

$$\| g_1'(x) - g_1'(x_\tau) \| = \frac{1}{4} \; |x \; \sin \frac{\pi}{x} - \tau \; x \; \sin \frac{\pi}{x}| = \frac{1}{4 \; (2 \; k + 1)}$$

for $x = \dfrac{1}{k}$, $\tau = \dfrac{2}{2 \; k + 1}$, $k = 1, 2, \cdots$

That is, if there was a positive integrable function L, such that (2.1) holds on $U(x^\star, r)$ for some $r > 0$, then, there exists $k_0 > 1$, such that

$$\begin{aligned} \int_0^r L(t) \; dt \;\; &\geq \;\; \sum_{k=k_0}^{\infty} \int_{2/(2k+1)}^{1/k} L(t) \; dt \\ &\geq \;\; \| g'(x) - g'(x_\tau) \| \\ &\geq \;\; \| g_1'(x) - g_1'(x_\tau) \| \geq \sum_{k=k_0}^{\infty} \frac{1}{4 \; (2 \; k + 1)} = \infty, \end{aligned}$$

which is a contradiction.

References

[1] I.K. Argyros, *A convergence analysis of Newton–like methods for singular equations using outer or generalized inverses*, Applicationes Mathematicae, **32** (2005), 37–49.

[2] I.K. Argyros, *A unifying local–semilocal convergence analysis and applications for two–point Newton–like methods in Banach space*, J. Math. Anal. and Appl., **298** (2004), 374–397.

[3] I.K. Argyros, *Convergence and applications of Newton–type iterations*, Springer Verlag Publ., New York, 2008.

[4] J.M. Gutierrez, *A new semilocal convergence theorem for Newton's method*, **79** (1997), 131–145.

[5] Z. Huang, *The convergence ball of Newton's method and the uniqueness ball of equations under Hölder continuous derivatives*, Comput. Appl. Math., **47** (2004), 247–251.

[6] L.V. Kantorovich, G.P. Akilov, *Functional analysis in normed spaces*, Pergamon Press, New York, 1982.

[7] C. Li, X. Wang, *Convergence of Newton's method and uniqueness of the solution of equations in Banach spaces II*, Acta Math. Sinica, New Series, **18** (2003), 405–412.

[8] C. Li, W. Zhang, X. Jin, *Convergence and uniqueness properties of Gauss–Newton's method*, Comput. Math. Appl., **47** (2004), 1057–1067.

[9] S. Smale, *Newton's method estimates from data at a point*, The merging of disciplines: New directions in Pure, Applied and Computational Mathematics, R. Ewing, K. Gross and C. Martin eds, New–York, Springer Verlag Publ., (1986), 185–196.

[10] X. Wang, *Convergence of Newton's method and uniqueness of the solution of equations in Banach space*, IMA J. Numer. Anal., **20** (2000), 123–134.

[11] Y. Yuan, W. Sun, *Optimization theory and methods. Nonlinear Programming*, Springer Optimization and Its Applications, Springer, New York, 2006.

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An Analytic Solution of MHD Viscous Flow With Thermal Radiation Over a Permeable Flat Plate

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Abstract. The problem of thermal radiation on MHD viscous flow over a stationary permeable wall is investigated. The governing partial differential equations are transformed to a system of ordinary differential equations with the help of similarity transformations. The resulting nonlinear differential equations are then solved analytically by a purely analytic technique, namely, homotopy analysis method. The effect of wall suction/injection on velocity field and temperature distribution is discussed in detail with the help of graphs.

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1. Introduction

The MHD flow with continuous heat transfer has many practical applications in industrial manufacturing processes. However, of late, the radiation effect on MHD flow and heat transfer problems has become more important industrially. At high operating temperatures, the radiation effect can be quite significant. Many processes in engineering areas occur at high temperature and the knowledge of radiation heat transfer becomes very much important for the design of the pertinent equipment. Nuclear power plants, gas turbines and the various propulsion devices for aircraft, missiles, satellites and space vehicles are examples of such engineering areas. When the difference between the surface temperature and the ambient temperature is large, the radiation effect becomes important. In the aspect of convection radiation, Viskanta and Grosh [1] considered the effect of thermal radiation on the temperature distribution and the heat transfer in an absorbing and emitting media flowing over a wedge by using the Rosseland diffusion approximation. This approximation leads to a considerable simplification in the expression for radiant flux. In [1] the temperature differences within the flow were assumed to be sufficiently small such that

T^4 may be expressed as a linear function of temperature, i.e., $T^4 \cong 4T_\infty^3 T - 3T_\infty^4$. The thermal radiation of gray fluid, which is emitting and absorbing radiation in non-scattering medium, has been examined by Ali et al. [2], Ibrahim [3] ,Mansour [4], Hossain et al. [5]-[6], Elbashbeshy [7] and Elbashbeshy and Dimian [8] .The thermal radiation effect on a micropolar fluid was studied by Raptis [9] Recently, Raptis et al. [10] investigated the effect of thermal radiation on a flow of an electrically conducting viscous fluid. In [10] the authors considered the rigid stationary plate and computed a numerical solution of the problem. In the present study we extend the work of Raptis et al. [10] for the case of permeable stationary wall. The objective of this investigation is two fold; firstly, to investigate the effect of suction/injection on velocity and temperature profiles, secondly, to present complete analytic solution to the governing nonlinear equations. A newly developed analytic technique, namely, homotopy analysis method [11] is used to get the explicit analytic solution.

Currently, Liao [11] introduced an analytic technique for highly nonlinear problems in science and engineering. Liao named it as "homotopy analysis method". The homotopy analysis method is very useful to fluid mechanics problems and Liao himself proved the validity of the method by solving number of nonlinear problems in fluid mechanics (see for instance [12]-[20]). In recent years, the popularity of the method has grown considerably and number of researchers have successfully applied it to many nonlinear problems (see [21]-[29]).

The paper is organized into five sections. Section 2 contains the mathematical description of the problem. Analytic solution of the governing equations and the issue of convergence of the solution series is discussed in section 3. Graphical representation of results and their discussion is given in section 4 and finally some concluding remarks are given in section 5.

2. Mathematical description of the problem

We consider an incompressible, viscous and electrically conducting fluid bounded by a permeable semi-infinite flat plate situated at $y = 0$. The fluid is assumed to be flowing with uniform free stream velocity U(x) at infinity. A uniform magnetic field of strength B_0 is applied perpendicular to the plate. After neglecting the induced magnetic field and the radiative heat flux in the $x-$direction we get the continuity, momentum and the energy equations [10]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + U\frac{dU}{dx} + \frac{\sigma B_0^2}{\rho}(U-u) \tag{2.2}$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{k}{\rho c_p}\frac{\partial^2 \theta}{\partial y^2} - \frac{1}{\rho c_p}\frac{\partial q_r}{\partial y}, \tag{2.3}$$

subject to the boundary conditions

$$\begin{aligned} &u = 0,\ v = -V_0^*,\ \theta = \theta_0,\ \text{at } y = 0,\\ &u \to U(x),\ \theta = \theta_\infty,\ \text{as } y \to \infty. \end{aligned} \tag{2.4}$$

where V_0^* is constant suction/injection velocity (positive values of V_0^* correspond to suction and the negative values correspond to the constant injection at the plate); ρ is the density, ν is the kinematic viscosity, σ is the electric conductivity of the fluid, θ is the temperature, k is the thermal conductivity, c_p is the specific heat of the fluid under constant

pressure and q_r is the radiation heat flux.
We assume the velocity of the free stream of the form

$$U(x) = ax + bx^2 \tag{2.5}$$

and where a and b are constants.
By using the Rosseland approximation for radiation for an optically thick layer([1]-[2]) and the following transformations

$$\eta = \sqrt{\frac{a}{\nu}}y,\ u = axf'(\eta) + bx^2 g'(\eta),\ v = -\sqrt{a\nu}f(\eta) - \frac{2bx}{\sqrt{\frac{a}{\nu}}}g(\eta),$$
$$\theta = \theta_0 + (\theta_\infty - \theta_0)\left[T(\eta) + \frac{2bx}{a}\tau(\eta)\right], \tag{2.6}$$

the system(2.1) - (2.4) reduces to

$$f''' + ff'' - f'^2 + M(1 - f') + 1 = 0, \tag{2.7}$$
$$g''' + fg'' - 3f'g' + 2f''g + M(1 - g') + 3 = 0, \tag{2.8}$$
$$(3K + 4)T'' + 3K\,\mathrm{Pr}\, fT' = 0, \tag{2.9}$$
$$(3K + 4)\tau'' + 3K\,\mathrm{Pr}(-f'\tau + gT' + f\tau') = 0, \tag{2.10}$$

with boundary conditions

$$f = w,\ f' = 0,\ g = 0,\ g' = 0,\ T = 0,\ \tau = 0, \text{at } \eta = 0,$$
$$f' \to 1,\ g' \to 1,\ T \to 1,\ \tau \to 0, \text{as } \eta \to \infty. \tag{2.11}$$

3. ANALYTIC SOLUTION

We use homotopy analysis method to solve the system (2.7) - (2.11) analytically. Due to the boundary conditions (2.11) , one can express the solution series of $f(\eta)$,$g(\eta)$, $T(\eta)$ and $\tau(\eta)$ in the following form:

$$f(\eta) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A_{i,j}\eta^j e^{-i\eta}, \tag{3.1}$$

$$g(\eta) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} B_{i,j}\eta^j e^{-i\eta} \tag{3.2}$$

or

$$T(\eta) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} C_{i,j}\eta^j e^{-i\eta}, \tag{3.3}$$

$$\tau(\eta) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} D_{i,j}\eta^j e^{-i\eta}, \tag{3.4}$$

respectively, where $A_{i,j}$, $B_{i,j}$, $C_{i,j}$, and $D_{i,j}$ are constant coefficients. They provided us with the so-called rule of solution expression, which plays an important role in the frame work of homotopy analysis method. According to the boundary conditions (2.11) and the foregoing rule of solution expression defined in (3.1)- (3.4) , we choose the initial approximations for $f(\eta)$, $g(\eta)$, $T(\eta)$ and $\tau(\eta)$ of the form

$$f_0(\eta) = w - 1 + \eta + e^{-\eta} \tag{3.5}$$
$$g_0(\eta) = -1 + \eta + e^{-\eta}, \tag{3.6}$$
$$T_0(\eta) = 1 - e^{-\eta}, \tag{3.7}$$

$$\tau_0(\eta) = e^{-\eta} - e^{-2\eta}, \tag{3.8}$$

respectively and the auxiliary linear operators

$$\mathcal{L}_1 \equiv \frac{\partial^3}{\partial\eta^3} - \frac{\partial}{\partial\eta}, \tag{3.9}$$

$$\mathcal{L}_2 \equiv \frac{\partial^2}{\partial\eta^2} + \frac{\partial}{\partial\eta}, \tag{3.10}$$

which satisfy the properties

$$\mathcal{L}_1\left[C_1 + C_2 e^{\eta} + C_3 e^{-\eta}\right] = 0, \tag{3.11}$$

and

$$\mathcal{L}_2\left[C_4 - C_5 e^{-\eta}\right] = 0, \tag{3.12}$$

where C_i $(i = 1, 2, \cdots, 5)$ are arbitrary constants, the subscripts 1 and 2 indicate that the operator corresponds to velocity and temperature functions respectively.

We construct the so-called zeroth-order deformation equations

$$(1-p)\,\mathcal{L}_1\left[F(\eta;p) - f_0(\eta)\right] = p\hbar_1 N_1\left[F(\eta;p)\right], \tag{3.13}$$

$$(1-p)\,\mathcal{L}_1\left[G(\eta;p) - g_0(\eta)\right] = p\hbar_1 N_2\left[F(\eta;p), G(\eta;p)\right], \tag{3.14}$$

$$(1-p)\,\mathcal{L}_2\left[\Gamma(\eta;p) - T_0(\eta)\right] = p\hbar_2 N_3\left[\Gamma(\eta;p), F(\eta;p)\right], \tag{3.15}$$

$$(1-p)\,\mathcal{L}_2\left[\Lambda(\eta;p) - \tau_0(\eta)\right] = p\hbar_2 N_4\left[F(\eta;p), G(\eta;p), \Gamma(\eta;p), \Lambda(\eta;p)\right], \tag{3.16}$$

subject to the boundary conditions

$$\begin{aligned} &F(0,p) = V_0,\ G(0,p) = 0,\ \frac{\partial F(\eta,p)}{\partial\eta}\,|_{\eta=0} = 0,\ \frac{\partial G(\eta,p)}{\partial\eta}\,|_{\eta=0} = 0, \\ &\Gamma(0,p) = 0, \Lambda(0,p) = 0,\ \frac{\partial F(\eta,p)}{\partial\eta}\,|_{\eta=\infty} = 1,\ \frac{\partial G(\eta,p)}{\partial\eta}\,|_{\eta=\infty} = 1, \\ &\Gamma(\infty,p) = 1,\ \Lambda(\infty,p) = 0, \end{aligned} \tag{3.17}$$

where $p \in [0,1]$ is the embedding parameter, $\hbar_1$ and $\hbar_2$ are the non-zero auxiliary parameters corresponding to the velocity and temperature profiles respectively and the nonlinear operators $N_1\left[F(\eta;p)\right]$, $N_2\left[F(\eta;p), G(\eta;p)\right]$, $N_3\left[\Gamma(\eta;p), F(\eta;p)\right]$, and $N_4\left[F(\eta;p), G(\eta;p), \Gamma(\eta;p), \Lambda(\eta;p)\right]$ are defined through

$$N_1\left[F(\eta;p)\right] = \frac{\partial^3 F}{\partial\eta^3} + F\frac{\partial^2 F}{\partial\eta^2} - \left(\frac{\partial F}{\partial\eta}\right)^2 + M\left(1 - \frac{\partial F}{\partial\eta}\right) + 1, \tag{3.18}$$

$$N_2\left[F(\eta;p), G(\eta;p)\right] = \frac{\partial^3 G}{\partial\eta^3} + F\frac{\partial^2 G}{\partial\eta^2} - 3\frac{\partial F}{\partial\eta}\frac{\partial G}{\partial\eta} + 2\frac{\partial^2 F}{\partial\eta^2}G + M\left(1 - \frac{\partial G}{\partial\eta}\right) + 3, \tag{3.19}$$

$$N_3\left[\Gamma(\eta;p), F(\eta;p)\right] = (3K+4)\frac{\partial^2\Gamma}{\partial\eta^2} + 3K\,\mathrm{Pr}\,F\frac{\partial\Gamma}{\partial\eta}, \tag{3.20}$$

$$\begin{aligned} N_4\left[F(\eta;p), G(\eta;p), \Gamma(\eta;p), \Lambda(\eta;p)\right] &= (3K+4)\frac{\partial^2\Lambda}{\partial\eta^2} \\ &+3K\,\mathrm{Pr}\left(-\frac{\partial F}{\partial\eta}\Lambda + G\frac{\partial\Gamma}{\partial\eta} + F\frac{\partial\Lambda}{\partial\eta}\right). \end{aligned} \tag{3.21}$$

When $p = 0$ we have the initial guess approximations

$$F(\eta;0) = f_0(\eta),\ G(\eta;0) = g_0(\eta),\ \Gamma(\eta;0) = T_0(\eta),\ \ \Lambda(\eta;0) = \tau_0(\eta). \tag{3.22}$$

When $p = 1$ equations (3.13 -3.17) are the same as (2.7 -2.11) respectively, therefore at $p = 1$ we get the final solutions

$$F(\eta;1) = f(\eta), \ G(\eta;1) = g(\eta), \ \Gamma(\eta;1) = T(\eta), \ \Lambda(\eta;1) = \tau(\eta). \tag{3.23}$$

The initial guess approximations $f_0(\eta), g_0(\eta), T_0(\eta)$, and $\tau_0(\eta)$, the linear operators $\mathcal{L}_1$, $\mathcal{L}_2$ and the auxiliary parameters $\hbar_1$ and $\hbar_2$ are assumed to be selected such that equations (3.13 − 3.17) have solution at each point $p \in [0,1]$, and also with the help of Maclaurin's series and due to eq. (3.22), $F(\eta;p), G(\eta;p), T(\eta;p)$, and $\tau(\eta;p)$ can be expressed as

$$F(\eta;p) = f_0(\eta) + \sum_{k=1}^{\infty} f_k(\eta)\, p^k, \tag{3.24}$$

$$G(\eta;p) = g_0(\eta) + \sum_{k=1}^{\infty} g_k(\eta)\, p^k, \tag{3.25}$$

$$\Gamma(\eta;p) = T_0(\eta) + \sum_{k=1}^{\infty} T_k(\eta)\, p^k, \tag{3.26}$$

$$\Lambda(\eta;p) = \tau_0(\eta) + \sum_{k=1}^{\infty} \tau_k(\eta)\, p^k, \tag{3.27}$$

where

$$f_k(\eta) = \frac{1}{k!}\frac{\partial^k F(\eta;p)}{\partial p^k}\Big|_{p=0}, \quad g_k(\eta) = \frac{1}{k!}\frac{\partial^k G(\eta;p)}{\partial p^k}\Big|_{p=0},$$
$$T_k(\eta) = \frac{1}{k!}\frac{\partial^k \Gamma(\eta;p)}{\partial p^k}\Big|_{p=0}, \quad \tau_k(\eta) = \frac{1}{k!}\frac{\partial^k \Lambda(\eta;p)}{\partial p^k}\Big|_{p=0}. \tag{3.28}$$

Assume that the auxiliary parameters $\hbar_1$ and $\hbar_2$ are so properly chosen that the series (3.24) -(3.27) are convergent at $p = 1$, then due to (3.27) we have

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \tag{3.29}$$

$$g(\eta) = g_0(\eta) + \sum_{m=1}^{\infty} g_m(\eta), \tag{3.30}$$

$$T(\eta) = T_0(\eta) + \sum_{m=1}^{\infty} T_m(\eta), \tag{3.31}$$

$$\tau(\eta) = \tau_0(\eta) + \sum_{m=1}^{\infty} \tau_m(\eta). \tag{3.32}$$

Equations (3.29) -(3.32) provide us with a relationship between the initial guess approximations $f_0(\eta), g_0(\eta), T_0(\eta)$, and $\tau_0(\eta)$ and the unknown solutions $f(\eta), g(\eta), T(\eta)$, and $\tau(\eta)$ respectively. In order to get the governing equations for $f_m(\eta), g_m(\eta), T_m(\eta)$, and $\tau_m(\eta)$ $(m \geq 1)$, we first differentiate m times the two sides of eqs. (3.13) -(3.17) with respect to the embedding parameter p at $p = 0$ and then divide them by $m!$. In this way we get

$$\mathcal{L}_1[f_m - \chi_m f_{m-1}] = \hbar_1 P_m(\eta), \tag{3.33}$$

$$\mathcal{L}_1[g_m - \chi_m g_{m-1}] = \hbar_1 Q_m(\eta), \tag{3.34}$$

$$\mathcal{L}_2[T_m - \chi_m T_{m-1}] = \hbar_2 R_m(\eta), \tag{3.35}$$

$$\mathcal{L}_2\left[\tau_m - \chi_m \tau_{m-1}\right] = \hbar_2 W_m(\eta), \tag{3.36}$$

with the boundary conditions

$$\begin{aligned} f_m(0) = 0, \quad f'_m(0) = 0, \quad g'_m(0) = 0, \quad T_m(0) = 0, \quad \tau_m(0) = 0, \\ f'_m(\infty) = 0, \quad g'_m(\infty) = 0, \quad T_m(\infty) = 0, \quad \tau_m(\infty) = 0, \end{aligned} \tag{3.37}$$

where

$$P_m(\eta) = f'''_{m-1} - M f'_{m-1} + (1-\chi_m)(M+1) + \sum_{k=0}^{m-1}\left[f_{m-1-k} f''_k - f'_{m-1-k} f'_k\right], \tag{3.38}$$

$$\begin{aligned} Q_m(\eta) &= g'''_{m-1} - M g'_{m-1} + (1-\chi_m)(M+3) \\ &+ \sum_{k=0}^{m-1}\left[f_{m-1-k} g''_k - 3 f'_{m-1-k} g'_k + 2 f''_{m-1-k} g_k\right], \end{aligned} \tag{3.39}$$

$$R_m(\eta) = (3K+4)\, T''_{m-1} + 3K\,\mathrm{Pr}\sum_{k=0}^{m-1} f_{m-1-k} T'_k, \tag{3.40}$$

$$W_m(\eta) = (3K+4)\,\tau''_{m-1} + 3K\,\mathrm{Pr}\sum_{k=0}^{m-1}\left[f_{m-1-k}\tau'_k + g_{m-1-k} T'_k - f'_{m-1-k}\tau_k\right], \tag{3.41}$$

and, for k being any integer

$$\begin{aligned} \chi_k &= 0 \quad \text{if} \quad k \leq 1, \\ &= 1 \quad \text{if} \quad k > 1. \end{aligned} \tag{3.42}$$

We emphasize here that eqs. (3.33) -(3.36) are linear for all $m \geq 1$. Also, the left-hand sides of all (3.33) -(3.36) are governed by the same linear operators $\mathcal{L}_1$ and $\mathcal{L}_2$, for all $m \geq 1$. These linear equations can be easily solved by means of symbolic computation software such as MATHEMATICA, MATLAB, MAPLE and so on. We solve the system ((3.33) -(3.37) for first few values of m and find that the solution expressions for $f_m(\eta)$, $g_m(\eta)$, $T_m(\eta)$, and $\tau_m(\eta)$ can be written as

$$f_m(\eta) = \sum_{n=0}^{2m+2}\sum_{q=0}^{2m+2-n} a^q_{m,n}\eta^q e^{-n\eta}, \tag{3.43}$$

$$g_m(\eta) = \sum_{n=0}^{2m+2}\sum_{q=0}^{2m+2-n} b^q_{m,n}\eta^q e^{-n\eta}, \tag{3.44}$$

$$T_m(\eta) = \sum_{n=0}^{2m+2}\sum_{q=0}^{2m+2-n} c^q_{m,n}\eta^q e^{-n\eta}, \tag{3.45}$$

$$\tau_m(\eta) = \sum_{n=1}^{2m+2}\sum_{q=0}^{2m+2-n} d^q_{m,n}\eta^q e^{-n\eta}. \tag{3.46}$$

In this way we get the explicit analytic solution

$$f(\eta) = \lim_{M\to\infty}\sum_{m=0}^{M}\sum_{n=0}^{2m+2}\sum_{q=0}^{2m+2-n} a^q_{m,n}\eta^q e^{-n\eta}, \tag{3.47}$$

$$g(\eta) = \lim_{M\to\infty} \sum_{m=0}^{M} \sum_{n=0}^{2m+2} \sum_{q=0}^{2m+2-n} b_{m,n}^{q} \eta^{q} e^{-n\eta}, \tag{3.48}$$

$$T(\eta) = \lim_{M\to\infty} \sum_{m=0}^{M} \sum_{n=0}^{2m+2} \sum_{q=0}^{2m+2-n} c_{m,n}^{q} \eta^{q} e^{-n\eta}, \tag{3.49}$$

$$\tau(\eta) = \lim_{M\to\infty} \sum_{m=0}^{M} \sum_{n=1}^{2m+2} \sum_{q=0}^{2m+2-n} d_{m,n}^{q} \eta^{q} e^{-n\eta}, \tag{3.50}$$

of the original eqs. (2.7) -(2.10).

Therefore, at the Mth-order approximation, the solution can be expressed as follows

$$f(\eta) \approx \sum_{m=0}^{M} \sum_{n=0}^{2m+2} \sum_{q=0}^{2m+2-n} a_{m,n}^{q} \eta^{q} e^{-n\eta}, \tag{3.51}$$

$$g(\eta) \approx \sum_{m=0}^{M} \sum_{n=0}^{2m+2} \sum_{q=0}^{2m+2-n} b_{m,n}^{q} \eta^{q} e^{-n\eta}, \tag{3.52}$$

$$T(\eta) \approx \sum_{m=0}^{M} \sum_{n=0}^{2m+2} \sum_{q=0}^{2m+2-n} c_{m,n}^{q} \eta^{q} e^{-n\eta}, \tag{3.53}$$

$$\tau(\eta) \approx \sum_{m=0}^{M} \sum_{n=1}^{2m+2} \sum_{q=0}^{2m+2-n} d_{m,n}^{q} \eta^{q} e^{-n\eta} \tag{3.54}$$

As mentioned by Liao [11] that whenever the solution series obtained by homotopy analysis method converges it will be one of the solution of the original equations. The convergence of the solution series depends upon the choice of initial approximations, the auxiliary linear operators and the nonzero auxiliary parameters. Once if the initial guess approximations and the auxiliary linear operators have been selected then the convergence of the solution series will strictly depend upon the auxiliary parameters only. Therefore, the convergence of the solution series is determined by the values of such kind of parameters. The admissible values of the parameters $\hbar_1$ and $\hbar_2$ are determined by the so-called $\hbar-$curves. In order to find the allowed values of $\hbar_1$ and $\hbar_2$ to make the series (3.47) -(3.50) convergent we have plotted the $\hbar-$curves corresponding to $f''(0)$, $g''(0)$, $T'(0)$, and $\tau'(0)$ in figs. 1 and 2.

In fig. 1, the $\hbar-$curves have been plotted for $f(\eta)$ and $g(\eta)$. Notice that for $\hbar_1 \in (-0.16, -0.12)$ both curves have their line segments parallel to the $\hbar_1-$axis. If $\hbar_1$ is chosen from this interval, then the series (3.47) and (3.48) will converge, also our analysis shows that these series are convergent for $\hbar_1 = -0.14$. Similarly, the series (3.49) and (3.50) are found to be convergent at a same value of $\hbar_2 = -0.10$. To show that the series (3.47) -(3.50) are uniformly convergent series, we have calculated the differences between their successive terms at particular orders of approximation (see table 1). We have define the successive absolute differences at $\eta = 0$ in the following way

$$\Delta_i f'' = \left|f_i'' - f_{i-1}''\right|,$$

$$\Delta_i g'' = \left|g_i'' - g_{i-1}''\right|,$$

$$\Delta_i T' = \left|T_i' - T_{i-1}'\right|,$$

$$\Delta_i \tau' = \left|\tau_i' - \tau_{i-1}'\right|.$$

From table 1 it is clear that by increasing the order of approximation the contribution of the higher order terms is decreasing and after a certain order of approximations the contribution of the next terms becomes negligible which confirms the convergence of the solutions series.

4. Graphical results and discussion

To see the effect of wall suction/injection on velocity and temperature we have plotted the graphs. In figs. 3 and 4 the velocity f' is plotted against η for different values of suction and injection parameter w. It is observed that the boundary layer thickness decreases by increasing the suction velocity and the effect is totally reversed in the case of injection, increasing injection at the wall causes to increase the layer thickness. In fig. 5, it is shown that there is a direct effect of suction on the velocity $f(\eta)$. By increasing suction at the wall the velocity $f(\eta)$ also increases at the plate. Similar effect of suction/injection is observed on the velocities $g(\eta)$ and $g'(\eta)$ as shown in figs. $(6)-(8)$. However, it is noted that the suction/injection effects are strong in $f(\eta)$ and $f'(\eta)$ as compared with $g(\eta)$ and $g'(\eta)$.

In figs. 9 and 10 the temperature $T(\eta)$ is plotted against η for different values of the suction and injection velocity respectively. Clearly, the thermal boundary layer thickness decreases by increasing w and increases by decreasing w as shown in figs. 9 and 10. Similar effects of suction are shown in fig. 11 for the temperature $\tau(\eta)$.

5. Concluding remarks

In this communication we investigated the effect of suction/injection on MHD viscous flow with thermal radiation. Explicit purely analytic solution for velocity and temperature distribution are obtained by homotopy analysis method. The solution is explicit and totally analytic valid for all values of the dimensionless parameters involved in the problem. Convergence of the solution has been shown through a table of absolute differences of the successive terms of the solution series at different orders. The effect of suction is to decrease the boundary layer thickness and the thermal boundary layer thickness whereas the effect of injection is reverse to it which is in accordance with the results present in literature. We also remark here that the homotopy analysis method is a very useful analytic technique to solve highly nonlinear problems.

References

[1] R. Viskanta, R.J. Grosh, Int. J. Heat Mass Transfer 5 (1962) 795 – 806.
[2] M.M. Ali, T.S. Chen, B.F., AIAA J. 22 (1984) 1797 – 1803.
[3] F.S. Ibrahim, Astrophys. Space Sci. 168 (1990) 263 – 276.
[4] M.A. Mansour, Astrophys. Space Sci. 166 (1990) 269 – 275.
[5] M.A. Hossain, M.A. Alim, D. Rees, Int. J. Heat Mass Transfer 42 (1999) 181 – 191.
[6] M.A. Hossain, K. Khanafer, K. Vafai, Int. J. Thermal Sci. 40 (2001) 115 – 124.
[7] E.M.A. Elbashbeshy, Can. J. Phys. 78 (2000) 1107 – 1112.
[8] E.M.A. Elbashbeshy, F.M. Dimian, Appl. Math. Comp. 132 (2002) 445 – 454.
[9] A. Raptis, Int. J. Heat Mass Transfer 41 (1998) 2865 – 2876.
[10] A. Raptis, C. Perdikis, H.S. Takhar, Appl. Math. Comp. 153 (2004) 645 – 649.
[11] SJ. Liao, *Beyond perturbation: Introduction to homotopy analysis method*, Boca Raton: Chapman & Hall/CRC Press 2003.
[12] SJ. Liao, Comm. NonLinear Sci. Numer. Simul. 11 (2006) 326 – 339.
[13] SJ. Liao, J. Fluid Mech. 385 (1999) 101 – 128.
[14] SJ. Liao, Int. J. NonLinear Mech. 34 (1999) 759 – 778.
[15] SJ. Liao, Int. J. NonLinear Mech. 37 (2002) 1 – 18.
[16] SJ. Liao, A. Campo, J. Fluid Mech. 453 (2002) 411 – 425.

[17] SJ. Liao, K.F. Cheung, J. Eng. Math. 45 (2) (2003) 105 – 116.
[18] SJ. Liao, Int. J. Heat Mass Transfer 147 (2004) 499 – 513.
[19] SJ. Liao, J. Fluid Mech. 488 (2003) 189 – 212.
[20] S.J. Liao, I. Pop, Int. J. Heat Mass Transfer 47 (1) (2004) 75 – 85.
[21] A. Ali, A. Mehmood, Comm. NonLinear Sci. Numer. Simul. (In press).
[22] A. Mehmood, A. Ali, Int. Comm. Heat Mass Transfer 33 (2006) 1243-1252
[23] F.M. Allan, M.I. Syam, J. Comp. Appl. Math. 182 (2005) 362 – 371.
[24] S. Li, SJ. Liao, Appl. Math. Comp. 169 (2) (2005) 854 – 865.
[25] Q. Sun, Appl. Math. Comp. 169 (1) (2005) 355 – 365.
[26] Y. Tan, H. Xu, SJ. Liao, Chaos, Solitons & Fractals (In press).
[27] C. Wang, Y. Wu, W. Wu, Wave Motion 41 (4) (2005) 329 – 337.
[28] H. Xu, SJ. Liao, J. Non-Newtonian Fluid Mech. 129 (2005) 46 – 55.
[29] Xu. H, Appl. Math. Comp. 158 (2004) 433 – 443.

Table 1: Successive differences at different orders.

Table 1				
	$\hbar_1 = -0.14, \hbar_2 = -0.1, M = 0.5, w = 0.5, K = 0.5, \Pr = 0.7, \eta = 0$			
i	$\Delta_i f''$	$\Delta_i g''$	$\Delta_i T'$	$\Delta_i \tau'$
5	0.05623560	0.13780800	0.014116700	0.023131500
10	0.02579640	0.05093700	0.005753070	0.002770360
15	0.01129380	0.02010250	0.000430896	0.000286050
20	0.00510152	0.00835626	0.000204585	0.000440658
25	0.00232260	0.00359375	0.000024761	0.000014915

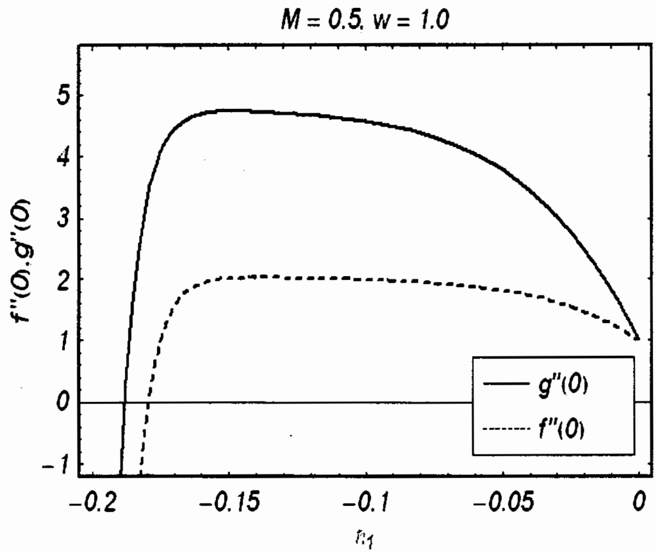

FIGURE 1. $\hbar$—curves corresponding to the velocity components.

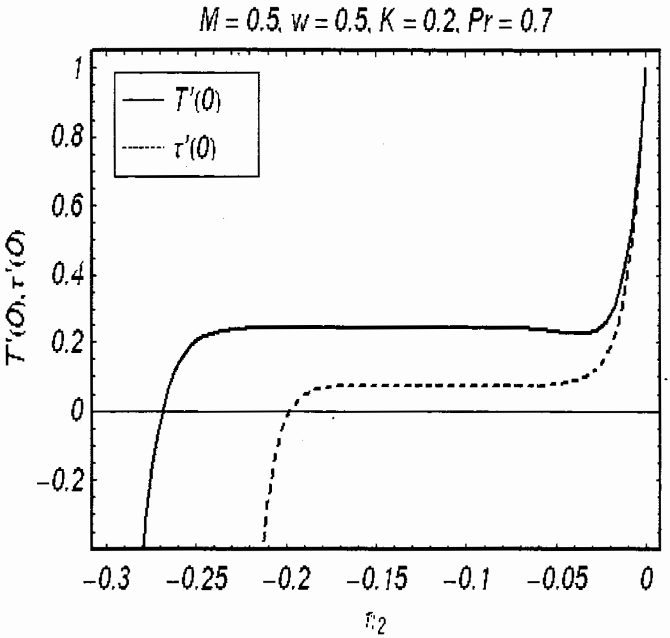

FIGURE 2. $\hbar$−curves for temperature.

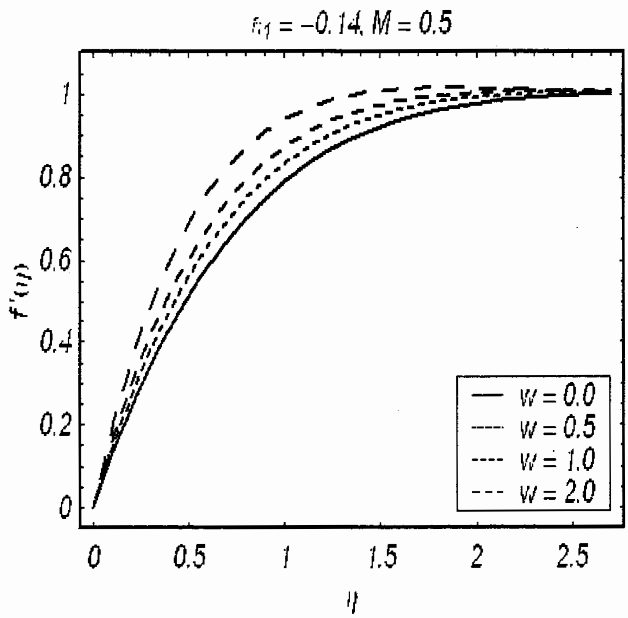

FIGURE 3. Effect of constant suction on the velocity component $f'(\eta)$

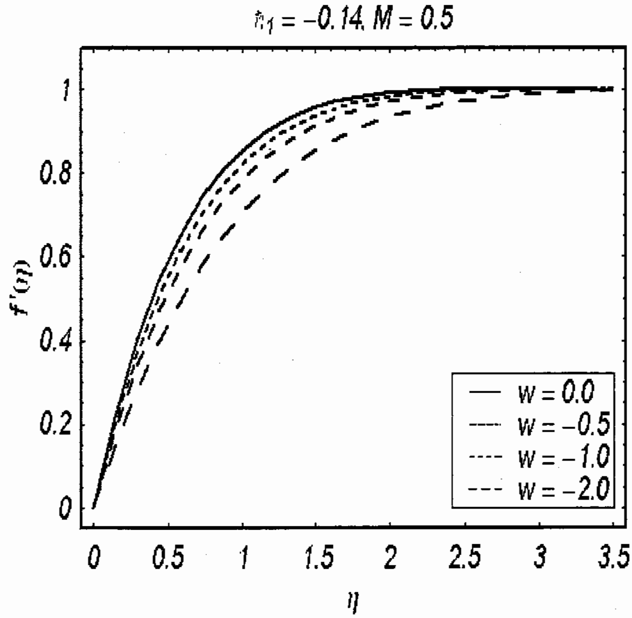

FIGURE 4. Velocity component $f'(\eta)$ in the case of constant injection.

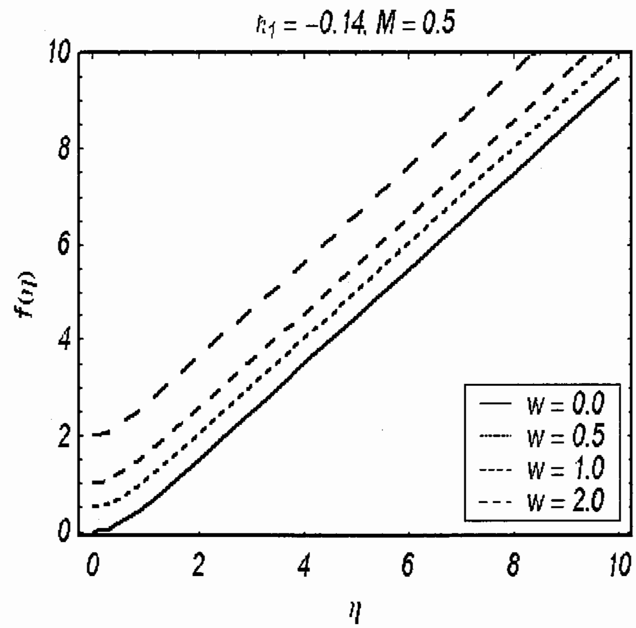

FIGURE 5. $f(\eta)$ for different values of the suction parameter w.

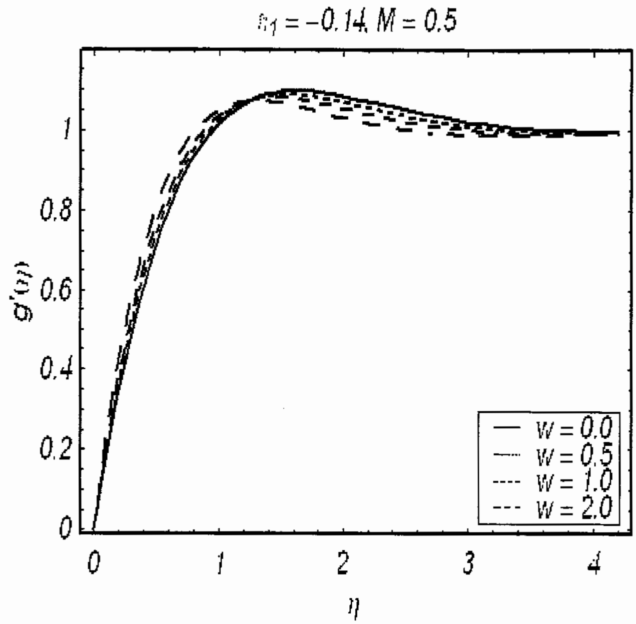

FIGURE 6. Variation of the velocity component $g'(\eta)$ for different values of the suction velocity w.

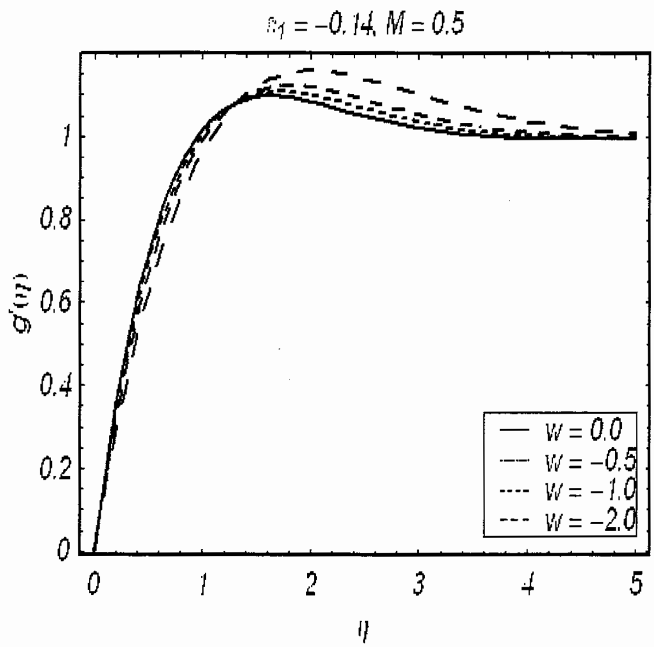

FIGURE 7. Injection effects on the velocity $g'(\eta)$.

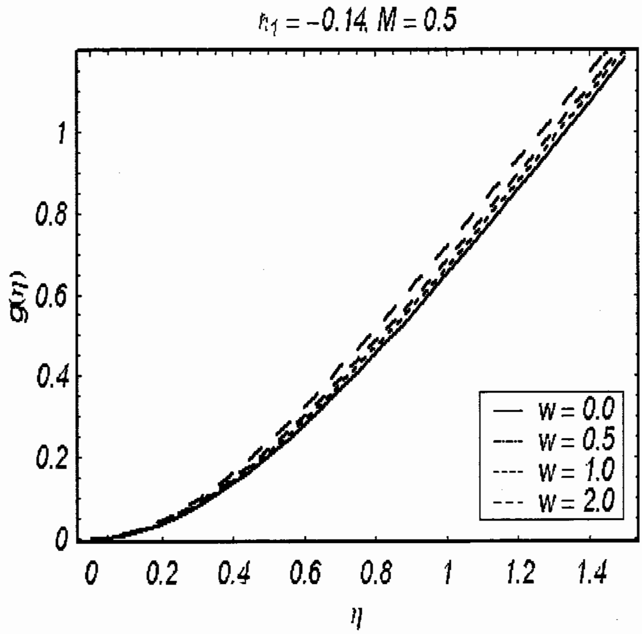

FIGURE 8. $g(\eta)$ for different values of the suction parameter w.

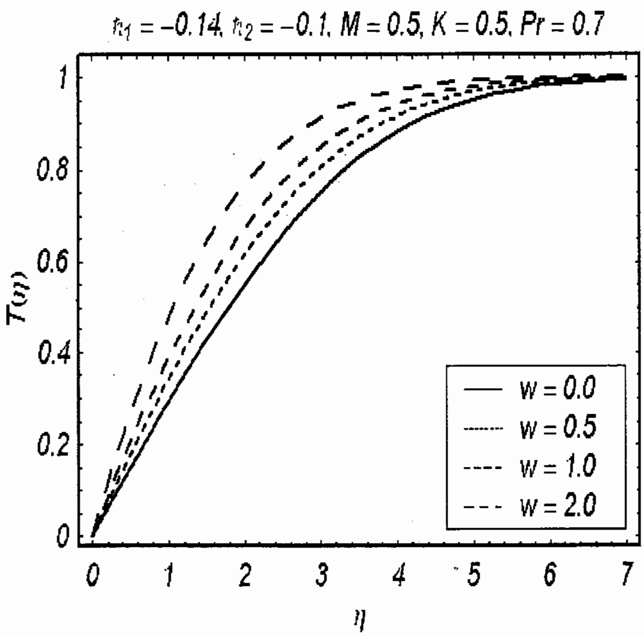

FIGURE 9. Temperature profiles at different w.

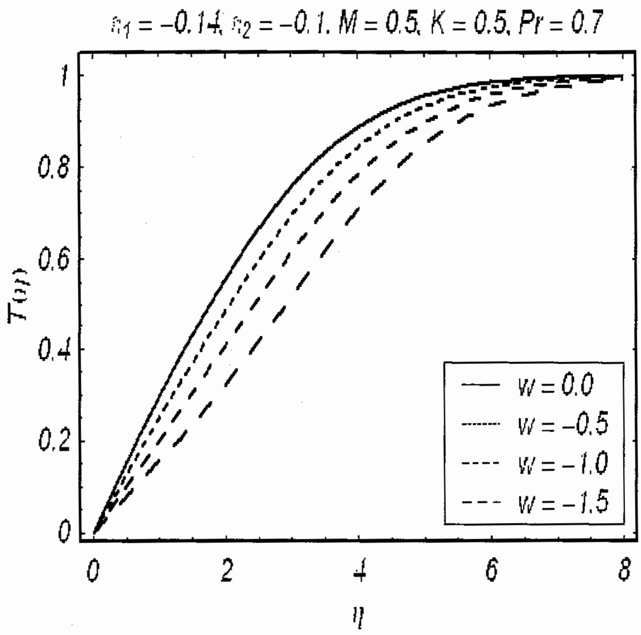

FIGURE 10. Injection effects on the temperature distribution.

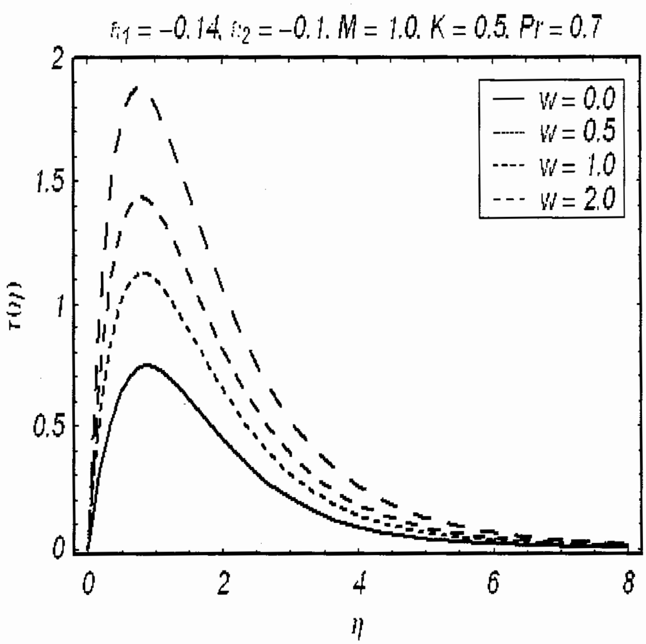

FIGURE 11. Effect of constant suction on $\tau(\eta)$.

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Hadamard-Type Inequalities for $s-$Convex Functions I

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Abstract. In this paper we give a refined upper bound for unit, or smaller intervals and refinement of Hermite Hadamard Inequality for $s-$convex functions in second sense. We also establish several Hadamard type Inequalities for differentiable and twice differentiable functions based on concavity and $s-$convexity with applications for some special means.

AMS (MOS) Subject Classification Codes:[2000]26D15, 26D10
Key Words: Hadamard's inequality; $s-$convex functions; concave functions; Beta function.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

$\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq \; tf(x) \; + \; (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$. Geometrically, this means that if P, Q and R are three distinct points on graph of f with Q between P and R, then Q is on or below chord PR.
In the paper [6], H. Hudzik and L. Maligranda considered, among others, the class of

functions which are $s-$convex in the second sense. This class is defined as follows:
A function $f : [0, \infty) \to \mathbb{R}$ is said to be $s-$convex in the second sense if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \tag{1.2}$$

holds for all $x, y \in [0, \infty)$, $t \in [0, 1]$ and for some fixed $s \in (0, 1]$. It may be noted that every 1-convex function is convex. In the same paper [6] H. Hudzik and L. Maligranda discussed a few results connecting with $s-$convex functions in second sense and some new results about Hadamard's inequality for $s-$convex functions are discussed in [1, 2, 8], while on the other hand there are many important inequalities connecting with 1-convex (convex) functions [4], but one of these is the classical Hermite-Hadamard inequality defined by [10]

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}$$

for $[a, b] \subseteq \mathbb{R}$.
In [5], S. S. Dragomir et al. proved a variant of Hermite-Hadamard's inequality for $s-$convex functions in second sense.

Theorem 1. *Suppose that $f : [0, \infty) \to [0, \infty)$ is $s-$convex function in the second sense, where $s \in (0, 1]$, and let $a, b \in [0, \infty)$, $a < b$. If $\in L^1[a, b]$, then the following inequality holds*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{s+1}. \tag{1.3}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.3). Their result was improved in [7], where Jagers gave both the upper and lower bound for the constant $c(s)$ in the inequality

$$c(s)\, f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx.$$

He proved that

$$\frac{2^{s+1}-1}{s+2} \leq c(s) \leq 2^{\frac{s-1}{s+1}}\left(\frac{2^s-1}{s}\right)^{\frac{s}{s+1}} \leq \frac{2^{s+1}-2^{s-1}-1}{s+1}.$$

In [3, 4] S. S. Dragomir et al. discussed inequalities for differentiable and twice differentiable functions connecting with the H-H Inequality on the basis of the following Lemmas.

Lemma 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° with $f'' \in L^1[a, b]$, then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{(b-a)^2}{2}\int_0^1 t(1-t)\, f''(ta+(1-t)b)\,dt.$$

Lemma 3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable function on I°, $a, b \in I^\circ$ with $a < b$ and $f' \in L^1[a, b]$, then*

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{(b-a)^2}{4}\int_0^1 (1-t)\left[f'\left(ta+(1-t)\frac{a+b}{2}\right) + f'\left(tb+(1-t)\frac{a+b}{2}\right)\right] dt.$$

We give here definition of Beta function of Euler type which will be helpful in our next discussion, which is for $x, y > 0$ defined as

$$\beta(x+1, y+1) = \int_0^1 t^x (1-t)^y \, dt.$$

This paper is organized as follows. After this Introduction, in section 2 we discuss some $s-$Hermite Hadamard type inequalities for differentiable functions, in section 3 we give applications of the results from section 2 for special means and in section 4 we will discuss refinement of $s-$Hermite Hadamard inequality and its refined upper bound for unit, or smaller, intervals.

2. Inequalities For differentiable functions

Theorem 4. *Let $f : I \to \mathbb{R}$, $I \subset [0, \infty)$, be a differentiable function on I° such that $f \in L^1[a,b]$, where $a, b \in I$, $a < b$. If $| f' |^q$ is s-convex on $[a,b]$ for some fixed $s \in (0,1]$ and $q \geq 1$, then*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\
&\leq 2^{-\frac{1}{p}} \frac{\{|f'(a)|^q + (s+1)|f'\left(\frac{a+b}{2}\right)|^q\}^{\frac{1}{q}}}{\{(s+1)(s+2)\}^{\frac{1}{q}}} + \\
&\quad 2^{-\frac{1}{p}} \frac{\{| f'(b) |^q + (s+1) | f'(\frac{a+b}{2}) |^q\}^{\frac{1}{q}}}{\{(s+1)(s+2)\}^{\frac{1}{q}}} \qquad (2.1) \\
&= 2^{-\frac{1}{p}} \left[\left(\beta(s+1,2)\,|f'(a)|^q + \beta(s+2,1) \left| f'\left(\frac{a+b}{2}\right) \right|^q \right)^{\frac{1}{q}} + \right. \\
&\quad \left. \left(\beta(s+1,2)|f'(b)|^q \, \beta(s+2,1) + \left| f'\left(\frac{a+b}{2}\right) \right|^q \right)^{\frac{1}{q}} \right]
\end{aligned}$$

Proof. By Lemma 3

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\
&\leq \frac{(b-a)^2}{4} \left[\int_0^1 (1-t) \left| f'\left(t\,a + (1-t)\frac{a+b}{2}\right) \right| dt + \right. \\
&\quad \left. \int_0^1 (1-t) \left| f'\left(t\,b + (1-t)\frac{a+b}{2}\right) \right| dt \right] \qquad (2.2)
\end{aligned}$$

$|f'|$ is $s-$convex on $[a,b]$ for $t \in [0,1]$

$$\left| f'\left(t\,a + (1-t)\frac{a+b}{2}\right) \right| \leq t^s\,|f'(a)| + (1-t)^s \left| f'\left(\frac{a+b}{2}\right) \right|$$

$$\begin{aligned}
&\int_0^1 (1-t)\left|f'\left(t\,a+(1-t)\frac{a+b}{2}\right)\right| dt \\
&\qquad \leq |f'(a)| \int_0^1 t^s(1-t)\,dt + \left|f'\left(\frac{a+b}{2}\right)\right| \int_0^1 (1-t)^{1+s}\,dt \\
&\qquad = \beta(s+1,2)\,|f'(a)| + \beta(1,s+2)\left|f'\left(\frac{a+b}{2}\right)\right| \\
&\qquad = \frac{|f'(a)| + (s+1)\left|f'\left(\frac{a+b}{2}\right)\right|}{(s+1)(s+2)}
\end{aligned}$$

Now

$$\begin{aligned}
&\int_0^1 (1-t)\left|f'\left(t\,a+(1-t)\frac{a+b}{2}\right)\right| dt \\
&\qquad = \int_0^1 (1-t)^{1-\frac{1}{q}}(1-t)^{\frac{1}{q}}\left|f'\left(t\,a+(1-t)\frac{a+b}{2}\right)\right| dt
\end{aligned}$$

By Hölder's Inequality for $q > 1$ with $p = \frac{q}{q-1}$

$$\begin{aligned}
&\int_0^1 (1-t)\left|f'\left(t\,a+(1-t)\frac{a+b}{2}\right)\right| dt \\
&\qquad \leq \left(\int_0^1 (1-t)\left|f'\left(t\,a+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}} \left(\int_0^1 (1-t)\,dt\right)^{\frac{1}{p}} \\
&\qquad = 2^{-\frac{1}{p}}\left(\int_0^1 (1-t)\left|f'\left(t\,a+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\qquad \leq 2^{-\frac{1}{p}}\left[\frac{|f'(a)|^q + (s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q}{(s+1)(s+2)}\right]^{\frac{1}{q}} \qquad (2.3) \\
&\qquad = 2^{-\frac{1}{p}}\left[|f'(a)|^q\,\beta(s+1,2) + \beta(1,s+2)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Analogously

$$\begin{aligned}
&\int_0^1 (1-t)\left|f'\left(t\,b+(1-t)\frac{a+b}{2}\right)\right| dt \\
&\qquad \leq 2^{-\frac{1}{p}}\left[\frac{|f'(b)|^q + (s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q}{(s+1)(s+2)}\right]^{\frac{1}{q}} \qquad (2.4) \\
&\qquad = 2^{-\frac{1}{p}}\left[|f'(b)|^q\,\beta(s+1,2) + \beta(1,s+2)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}
\end{aligned}$$

By using (2.3) and (2.4) in (2.2) we get (2.1). □

Theorem 5. *Let $f : I \to \mathbb{R}$, $I \subset [0,\infty)$, be a differentiable function on I° such that $f' \in L^1[a,b]$, where $a, b \in I$, $a < b$. If $|f'|^q$ is concave on $[a,b]$ for $q > 1$ with $p = \frac{q}{q-1}$, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}}\left[f'\left(\frac{3a+b}{4}\right) + f'\left(\frac{a+3b}{4}\right)\right] \qquad (2.5)$$

Proof. Similarly as in Theorem 4 by using Hölder's Inequality for $q > 1$ with $p = \frac{q}{q-1}$ we obtain

$$\int_0^1 (1-t)\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right| dt$$
$$\leq \left(\int_0^1 (1-t)^p\, dt\right)^{\frac{1}{p}} \left(\int_0^1 \left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$= (p+1)^{-\frac{1}{p}} \left(\int_0^1 \left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}} \qquad (2.6)$$

$\mid f' \mid^q$ is concave on $[a,b]$, by Integral Jensen's Inequality (cf. [9]) we obtain

$$\begin{aligned}\int_0^1 \left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q dt &= \int_0^1 t^0\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q dt \\ &\leq \left(\int_0^1 t^0\, dt\right)\left|f'\left(\frac{\int_0^1 \left(ta+(1-t)\frac{a+b}{2}\right) dt}{\left(\int_0^1 t^0\, dt\right)}\right)\right|^q \\ &= \left|f'\left(\int_0^1 \left(ta+(1-t)\frac{a+b}{2}\right)\right)\right|^q \\ &= \left|f'\left(\frac{3a+b}{4}\right)\right|^q. \end{aligned} \qquad (2.7)$$

Analogously

$$\int_0^1 \left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt \leq \left|f'\left(\frac{a+3b}{4}\right)\right|^q \qquad (2.8)$$

By using (2.6) – (2.8) in (2.2) we get (2.5). □

Theorem 6. *Let $f : I \to \mathbb{R}$, $I \subset [0,\infty)$, be a differentiable function on I° such that $f \in L^1[a,b]$, where $a, b \in I$, $a < b$. If $|f'|^q$ is $s-$convex on $[a,b]$ for some fixed $s \in (0,1]$ and $q > 1$ with $p = \frac{q}{q-1}$, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[\left(|f'(a)|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(\mid f'(b)\mid^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right] \qquad (2.9)$$
$$\leq \frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}}\left[\left(|f'(a)|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(|f'(b)|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]$$

Proof. We proceed similar to proof of Theorem 5.
By $s-$convexity of $|f'|^q$ we obtain

$$\int_0^1 \left|f'\left(t\,a+(1-t)\frac{a+b}{2}\right)\right|^q dt \leq \frac{|f'(a)|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}. \tag{2.10}$$

Analogously

$$\int_0^1 \left|f'\left(t\,b+(1-t)\frac{a+b}{2}\right)\right|^q dt \leq \frac{|f'(b)|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}. \tag{2.11}$$

By using (2.10), (2.11) and (2.6) in (2.2) we get (2.9).
And the second inequality follows from the facts
$s \in (0,1]$ and $q > 1$ we have $\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \leq 1$. □

Theorem 7. *Let $f : I \to \mathbb{R}$, $I \subset [0,\infty)$, be a differentiable function on I° such that $f \in L^1[a,b]$, where $a,b \in I$, $a < b$. If $|\ f'\ |^q$ is $s-$concave on $[a,b]$ for some fixed $s \in (0,1]$ and $q > 1$ with $p = \frac{q}{q-1}$, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}}\, 2^{\frac{s-1}{q}} \left[\left|f'\left(\frac{3a+b}{4}\right)\right| + \left|f'\left(\frac{a+3b}{4}\right)\right|\right] \tag{2.12}$$

Proof. we proceed similarly as in Theorem 6.
By $s-$concavity of $|f'|^q$ we obtain

$$\int_0^1 \left|f'\left(t\,a+(1-t)\frac{a+b}{2}\right)\right|^q dt \ \leq\ 2^{s-1} \left|f'\left(\frac{3a+b}{4}\right)\right|^q.$$

Analogously

$$\int_0^1 \left|f'\left(t\,b+(1-t)\frac{a+b}{2}\right)\right|^q dt \ \leq\ 2^{s-1} \left|f'\left(\frac{a+3b}{4}\right)\right|^q.$$

Now (2.12) is immediate from (2.2). □

Variants of these results for twice differentiable functions are given below. These can be proved in a similar way based on Lemma 2.

Theorem 8. *Let $f : I \to \mathbb{R}$, $I \subset [0,\infty)$, be twice differentiable function on I° such that $f'' \in L^1[a,b]$, where $a,b \in I, a < b$. If $|f''|^q$ is $s-$convex on $[a,b]$ for some fixed $s \in (0,1]$ and $q \geq 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{2\times 6^{\frac{q-1}{q}}}\left[\frac{|f''(a)|^q + |f''(b)|^q}{(s+2)(s+3)}\right]^{\frac{1}{q}}.$$

$$= \frac{(b-a)^2}{2\times 6^{\frac{q-1}{q}}}\left[\beta(s+2,2)\{|\ f''(a)\ |^q + |\ f''(b)\ |^q\}\right]^{\frac{1}{q}}.$$

Theorem 9. *Let $f : I \to \mathbb{R}$, $I \subset [0,\infty)$, be twice differentiable function on I° such that $f'' \in L^1[a,b]$, where $a,b \in I$, a,b. If $|\ f''\ |^q$ is concave on $[a,b]$ for $q > 1$ with $p = \frac{q}{q-1}$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{2}\left|f''\left(\frac{a+b}{2}\right)\right| [\beta(p+1,p+1)]^p.$$

Theorem 10. *Let $f : I \to \mathbb{R}$, $I \subset [0,\infty)$, be twice differentiable function on I° such that $f'' \in L^1[a,b]$, where $a,b \in I$, $a < b$. If $|f''|^q$ is s−convex on $[a,b]$ for some fixed $s \in (0,1]$ and $q > 1$ with $p = \frac{q}{q-1}$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{2}[|f''(a)|^q + |f''(b)|^q]^{\frac{1}{q}}(s+1)^{-\frac{1}{q}}[\beta(p+1,p+1)]^p.$$

Theorem 11. *Let $f : I \to \mathbb{R}$, $I \subset [0,\infty)$, be twice differentiable function on I° such that $f'' \in L^1[a,b]$, where $a,b \in I$, $a < b$. If $|\ f''\ |^q$ is s−concave on $[a,b]$ for some fixed $s \in (0,1]$ and $q > 1$ with $p = \frac{q}{q-1}$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq 2^{\frac{s-1-q}{q}}(b-a)^2\left|f''\left(\frac{a+b}{2}\right)\right|[\beta(p+1,p+1)]^p.$$

Remark 12. For $s = 1$, relations (2.1), (2.5), (2.9) and (2.12) provide the right estimate of left classical Hadmard difference, that is, the new improvements of left Hadamard inequality.

Remark 13. For $s = 1$, relations in Theorems 8-11 provide the right estimate of right classical Hadmard difference, that is, the new improvements of right Hadamard inequality.

3. Applications for special Means

Let us recall the following means for two positive numbers.

(1) *The Arithmetic mean*

$$A \equiv A(a,b) = \frac{a+b}{2}, \quad a,b > 0$$

(2) *The Harmonic mean*

$$H \equiv H(a,b) = \frac{2ab}{a+b}, \quad a,b > 0$$

(3) *The p−Logarithmic mean*

$$L_p \equiv L_p(a,b) = \begin{cases} a, & \text{if } a = b;\ a,b > 0 \\ \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & \text{if } a \neq b. \end{cases}$$

(4) The *Identric mean*

$$I \equiv I(a,b) = \begin{cases} a, & \text{if } a = b;\ a,b > 0 \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & \text{if } a \neq b. \end{cases}$$

(5) *The Logarithmic mean*

$$L \equiv L(a,b) = \begin{cases} a, & \text{if } a = b;\ a,b > 0 \\ \frac{b-a}{\ln b - \ln a}, & \text{if } a \neq b. \end{cases}$$

The following inequality is well known in the literature:

$$H \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 14. *Let $p > 1$, $0 < a < b$ and $q = \frac{p}{p-1}$. Then one has the inequality.*

$$\left|H^{-1}(a,b) \; - \; L^{-1}(a,b)\right| \leq \frac{(b-a)^2}{6\, a^3\, b^3}\, A^{1/q}\left(a^{3q}, b^{3q}\right) \tag{3.1}$$

Proof. By Theorem 8 applied for the mapping $f(x) = \frac{1}{x}$ for $s = 1$ we have

$$\left|\frac{\frac{1}{a}+\frac{1}{b}}{2} - \frac{\ln b \; - \; \ln a}{b-a}\right| \leq \frac{(b-a)^2}{2 \times 6^{1/p}}\left[\frac{\frac{2^q}{a^{3q}} + \frac{2^q}{b^{3q}}}{12}\right]^{1/q},$$

which is equivalent to (3.1). □

Another result which is connected with $p-$Logarithmic mean $L_p(a,b)$ is the following one:

Proposition 15. *Let $p > 1$, $0 < a < b$ and $q = \frac{p}{p-1}$, then*

$$\left|A(a^p, b^p) \; - \; L_p^p(a,b)\right| \leq \frac{p\,(p-1)(b-a)^2}{12}\, A^{1/q}\left(a^{q(p-2)}, b^{q(p-2)}\right)$$

Proof. Follows by Theorem 8, setting $f(x) = x^p$ for $s = 1$. □

Another result which is connected with $p-$Logarithmic mean $L_p(a,b)$ is the following one:

Proposition 16. *Let $p > 1$, $0 < a < b$ and $q = \frac{p}{p-1}$, then*

$$\frac{A(a,b)}{I(a,b)} \leq \exp\left[\frac{3^{-1/q}}{2}\left\{\left(a^{-q} + 2A^{-q}(a,b)\right)^{1/q} + \left(b^{-q} + 2A^{-q}(a,b)\right)^{1/q}\right\}\right]$$

Proof. Follows by Theorem 4, setting $f(x) = -\ln x$ for $s = 1$. □

Remark 17. By selecting some other convex functions, in the same way as above, we can find out some new relations connecting to some special means.

4. Refinement And New Refined Upper Bound For s-Hermite Hadamard Inequality

To find new refined upper bound we integrate (1.2) w.r.t t over $[a,b] \subseteq [0,1]$

$$\frac{1}{x-y}\int_{x\,a+(1-a)\,y}^{x\,b+(1-b)\,y} f(u)\,du \leq \; (b-a)\left[L_s^s(a,b)\; f(x) \; + \; L_s^s(1-a, 1-b)\; f(y)\right],$$

where,

$$L_p^p(\alpha,\beta) = \frac{\beta^{p+1} - \alpha^{p+1}}{(\beta-\alpha)(p+1)}, \qquad \alpha \neq \beta,\; p > 0.$$

For better right bound of Hermite Hadamard Inequality for s-convex function in second, we compare the above bound with usual one, $\frac{f(a)+f(b)}{s+1}$.

Suppose the above is less than the usual upper bound, that is,

$\frac{b^{s+1}-a^{s+1}}{s+1} f(x) - \frac{(1-b)^{s+1}-(1-a)^{s+1}}{s+1} f(y) \leq \frac{f(x)+f(y)}{s+1}$

or, $[b^{s+1} - a^{s+1}]\, f(x) + [(1-a)^{s+1} - (1-b)^{s+1}]\, f(y) \leq f(x) + f(y)$.

Consider $b = a + \lambda$ for $\lambda > 0$ such that its cube and higher powers approaching zero.

$[(a+\lambda)^{s+1} - a^{s+1}]\, f(x) + [(1-a)^{s+1} - (1-a-\lambda)^{s+1}]\, f(y) \leq f(x) + f(y)$

So, all we need, for the above being true, is that

$(a+\lambda)^{s+1} - a^{s+1} \leq 1; \quad (1-a)^{s+1} - (1-a-\lambda)^{s+1} \leq 1$

$i.e,\quad a^{s+1}\left[\left(1+\frac{\lambda}{a}\right)^{s+1} - 1\right] \leq 1; \quad (1-a)^{s+1}\left[1 - \left(1 - \frac{\lambda}{1-a}\right)^{s+1}\right] \leq 1$

By binomial expansion,

$$a^{s+1}\left[\tfrac{s+1}{a}\,\lambda+\tfrac{s(s+1)}{2a^2}\,\lambda^2\right]\le 1;\qquad (1-a)^{s+1}\left[\tfrac{s+1}{1-a}\,\lambda-\tfrac{s(s+1)}{2(1-a)^2}\,\lambda^2\right]\le 1$$

$$\frac{s}{2}a^{s-1}\,\lambda^2+a^s\,\lambda-\frac{1}{s+1}\le 0;\qquad ,-\frac{s}{2}(1-a)^{s-1}\,\lambda^2+(1-a)^s\,\lambda-\frac{1}{s+1}\le 0 \quad (4.1)$$

From (4.1) we get $\lambda\le 1$.
This means we have improved the upper bound of Hermite Hadamard inequality for $s-$convex function in second sense, when the distance between a and b is almost one. The most interesting thing is that all linking work is with the interval $[0,1]$ better than other. This discussion gives the following result.

Theorem 18. *Let $f:[a,a+\lambda]\to\mathbb{R}$ be $s-$convex function in second sense for $0<\lambda\le 1$, $0\le a<1$ and $s\in(0,1]$, then*

$$2^{s-1}f\left(\frac{2a+\lambda}{2}\right)\ \le\ \frac{1}{\lambda}\int_a^{a+\lambda}f(t)\,dt\le\frac{1}{s+1}\left[\left\{\frac{s}{2}a^{s-1}\,\lambda^2+a^s\,\lambda\right\}f(x)+\left\{-\frac{s}{2}(1-a)^{s-1}\,\lambda^2+(1-a)^s\,\lambda\right\}f(y)\right].$$

The following result is related with the improvement of inequality (1.3).

Theorem 19. *Suppose that $f:[0,\infty)\to[0,\infty)$ is an s-convex function in the second sense, where $s\in(0,1]$, and let $a,b\in[0,\infty)$, $a<b$. If $f\in L^1[a,b]$, then*

$$\frac{f(a)+f(b)}{s+1}-\frac{1}{b-a}\int_a^b f(x)\,dx\ \ge\ \left|\int_0^1|t^s f(a)+(1-t)^s f(b)|\,dt-\int_0^1 |\ f(t\,a+(1-t)\,b)\ |\ dt\right| \quad (4.2)$$

and

$$\frac{1}{b-a}\int_a^b f(x)\,dx-2^{s-1}f\left(\frac{a+b}{2}\right)\ \ge\ 2^{s-1}\left|\frac{1}{2^s}\int_0^1|f(t\,a+(1-t)\,b)+f(t\,b+(1-t)\,a)|\,dt+\left|f\left(\frac{a+b}{2}\right)\right|\right| \quad (4.3)$$

Proof. From inequality (1.2)

$$\begin{aligned} t^s f(a)+(1-t)^s f(b)-f(t\,a+(1-t)\,b) &= |\ t^s f(a)+(1-t)^s f(b)-f(t\,a+(1-t)\,b)\ | \\ &\ge |\ |\ t^s f(a)+(1-t)^s f(b)\ |-|\ f(t\,a+(1-t)\,b)\ |\ |. \end{aligned}$$

Integrating w.r.t t over [0,1]

$$\begin{aligned} \frac{f(a)+f(b)}{s+1}-\int_0^1 f(t\,a+(1-t)\,b)\,dt &\ge \int_0^1 |\ |\ t^s f(a)+(1-t)^s f(b)\ |-|\ f(t\,a+(1-t)\,b)\ |\ |\ dt. \\ &\ge \left|\int_0^1\{|t^s f(a)+(1-t)^s f(b)|-|f(t\,a+(1-t)\,b)|\}\,dt\right|, \end{aligned}$$

which is equivalent to (4.2).
Again by definition

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2^s}.$$

$$\begin{aligned}\frac{f(x)+f(y)}{2^s} - f\left(\frac{x+y}{2}\right) &= \left|\frac{f(x)+f(y)}{2^s} - f\left(\frac{x+y}{2}\right)\right| \\ &\geq \left|\left|\frac{f(x)+f(y)}{2^s}\right| - \left|f\left(\frac{x+y}{2}\right)\right|\right|\end{aligned}$$

By setting, $x \mapsto t\,a + (1-t)\,b \quad and \quad y \mapsto t\,b + (1-t)\,a \quad for \quad t \in [0,1]$, we have

$$\frac{f(t\,a+(1-t)\,b)+f(t\,b+(1-t)\,a)}{2^s} - f\left(\frac{a+b}{2}\right) \geq \left|\left|\frac{f(t\,a+(1-t)\,b)+f(t\,b+(1-t)\,a)}{2^s}\right| - \left|f\left(\frac{a+b}{2}\right)\right|\right|.$$

Integrating w.r.t t over [0,1]

$$\frac{1}{2^s}\left[\int_0^1 f(t\,a+(1-t)\,b)\,dt + \int_0^1 f(t\,b+(1-t)\,a)\,dt\right] - f\left(\frac{a+b}{2}\right) \geq \left|\int_0^1 \left\{\left|\frac{f(t\,a+(1-t)\,b)+f(t\,b+(1-t)\,a)}{2^s}\right| - \left|f\left(\frac{a+b}{2}\right)\right|\right\} dt\right|.$$

From here we get (4.3). □

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REFERENCES

[1] M. Alomari and M. Darus, *On Co-ordinated $s-$convex functions,* Inter. Math. Forum, 3(40) (2008) 1977-1989.

[2] M. Alomari and M. Darus, *Hadamard-type inequalities for $s-$convex functions,* Inter. Math. Forum, 3(40) (2008) 1965-1970.

[3] S. S. Dragomir, On some inequalities for differentiable convex functions and applications, (submitted)

[4] S. S. Dragomir, C. E. M. Pierce, Selected Topics on Hermite-Hadamard Inequalities and Applications. RGMIA, Monographs.

[5] S. S. Dragomir, S. Fitzpatrick, The Hadamard's inequality for $s-$convex functions in the second sense, Demonstratio Math. 32 (4) (1999) 687-696.

[6] H. Hudzik, L. Maligranda, Some remarks on $s-$convex functions, Aequationes math. 48 (1994) 100-111.

[7] B. Jagers, On a Hadamard-type inequality for $s-$convex functions. http://wwwhome.cs.utwente.nl/ jagersaa/alphaframes/Alpha.pdf.

[8] U. S. Kirmaci et al., *Hadamard-type inequalities for $s-$convex functions,* Appl. Math. Comp., 193(2007), 26-35.

[9] D. S. Mitrinović, J. E. Pečarić, A. M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publishers, 1993, p. 106, 10, 15.

[10] J. Pečarić, F. Proschan, Y. L. Tong, *Convex functions, partial orderings and statistical applications*, Academic Press, Inc., 1992, p. 137.

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Sharp Function Estimates for Multilinear Commutators of Integral Operators

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Abstract. In this paper, we prove the sharp function inequality for some multilinear commutators related to certain integral operators. By using the sharp inequality, we obtain the weighted L^p-norm inequality for the multilinear commutators. The integral operators include the Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

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1. Introduction

As the development of singular integral operators, their commutators have been well studied (see [1-4]). Let T be the Calderón-Zygmund singular integral operator, a classical result of Coifman, Rocherberg and Weiss (see [2]) states that commutator $[b,T](f) = T(bf) - bT(f)$(where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$. In [4][11-13], the sharp estimates for some multilinear commutators of the Calderón-Zygmund singular integral operators are obtained. The main purpose of this paper is to prove the sharp function inequality for some multilinear commutators related to certain integral operators. By using the sharp inequality, we obtain the weighted L^p-norm inequality for the multilinear commutators. The integral operators include the Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

2. Notations and Results

First let us introduce some notations (see [3][14][15]). In this paper, Q will denote a cube of R^n with sides parallel to the axes, and for a cube Q let $f_Q = |Q|^{-1} \int_Q f(x)dx$ and the sharp function of f is defined by

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy.$$

It is well-known that (see [3])

$$f^{\#}(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - C| dy.$$

We say that b belongs to $BMO(R^n)$ if $b^{\#}$ belongs to $L^\infty(R^n)$ and define $||b||_{BMO} = ||b^{\#}||_{L^\infty}$. It has been known that (see [14])

$$||b - b_{2^kQ}||_{BMO} \leq Ck||b||_{BMO}.$$

Let M be the Hardy-Littlewood maximal operator, that is

$$M(f)(x) = \sup_{x\in Q} |Q|^{-1} \int_Q |f(y)|dy.$$

We write that $M_p(f) = (M(|f|^p))^{1/p}$ for $0 < p < \infty$. For $b_j \in BMO(R^n)(j = 1, \cdots, m)$, set

$$||\vec{b}||_{BMO} = \prod_{j=1}^m ||b_j||_{BMO}.$$

Given some functions b_j $(j = 1, \cdots, m)$ and a positive integer m and $1 \leq j \leq m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{BMO} = ||b_{\sigma(1)}||_{BMO} \cdots ||b_{\sigma(j)}||_{BMO}$.

In this paper, we will study some multilinear commutators as follows.

Definition 1. Suppose $b_j (j = 1, \cdots, m)$ are the fixed locally integrable functions on R^n. Let $F_t(x, y)$ define on $R^n \times R^n \times [0, +\infty)$. Set

$$F_t(f)(x) = \int_{R^n} F_t(x, y)f(y)dy$$

and

$$F_t^{\vec{b}}(f)(x) = \int_{R^n} \prod_{j=1}^m (b_j(x) - b_j(y))F_t(x, y)f(y)dy,$$

for every bounded and compactly supported function f. Let H be the Banach space $(H, ||\cdot||)$ such that, for each fixed $x \in R^n$, $F_t(f)(x)$ and $F_t^{\vec{b}}(f)(x)$ may be viewed as the mappings from $[0, +\infty)$ to H. The multilinear commutator related to F_t is defined by

$$T_{\vec{b}}(f)(x) = ||F_{(\cdot)}^{\vec{b}}(f)(x)||,$$

where F_t satisfies: for fixed $\varepsilon > 0$

$$||F_t(x, y)|| \leq C|x - y|^{-n}$$

and

$$||F_t(y, x) - F_t(z, x)|| + ||F_t(x, y) - F_t(x, z)|| \leq C|y - z|^\varepsilon |x - z|^{-n-\varepsilon},$$

if $2|y - z| \leq |x - z|$. We also define that $T(f)(x) = ||F_{(\cdot)}(f)(x)||$.

Note that when $b_1 = \cdots = b_m$, $T_{\vec{b}}$ is just the m order commutator (see [1][13]). It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1-2][4][11-13]). Our main purpose is to establish the sharp inequality for the multilinear commutator.

Now we state our theorems as following.

Theorem 2. *Let $b_j \in BMO(R^n)$ for $j = 1, \cdots, m$. Suppose that T is bounded on $L^q(R^n)$ for all $1 < q < \infty$. Then for any $1 < r < \infty$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$ and any $x \in R^n$,*

$$(T_{\vec{b}}(f))^{\#}(x) \le C||\vec{b}||_{BMO}\left(M_r(f)(x) + \sum_{j=1}^{m}\sum_{\sigma\in C_j^m} M_r(T_{\vec{b}_{\sigma^c}}(f))(x)\right).$$

Theorem 3. *Let $b_j \in BMO(R^n)$ for $j = 1, \cdots, m$. Suppose that T is bounded on $L^q(R^n)$ for all $1 < q < \infty$. Then $T_{\vec{b}}$ is bounded on $L^p(R^n)$ for $1 < p < \infty$.*

3. Proofs of Theorems

To prove the theorems, we need the following lemma, which is well known.

Lemma 4. *Let $1 < q < \infty$, $b_j \in BMO(R^n)$ for $j = 1, \cdots, k$. Then*

$$\frac{1}{|Q|}\int_Q \prod_{j=1}^{k} |b_j(y) - (b_j)_Q|dy \le C\prod_{j=1}^{k} ||b_j||_{BMO}$$

and

$$\left(\frac{1}{|Q|}\int_Q \prod_{j=1}^{k} |b_j(y) - (b_j)_Q|^q dy\right)^{1/q} \le C\prod_{j=1}^{k} ||b_j||_{BMO}.$$

Proof. **Proof of Theorem 1.** It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q |T_{\vec{b}}(f)(x) - C_0|dx \le C||\vec{b}||_{BMO}\left(M_r(f)(\tilde{x}) + \sum_{j=1}^{m}\sum_{\sigma\in C_j^m} M_r(T_{\vec{b}_{\sigma^c}}(f)(\tilde{x}))\right).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$.

We first consider the **Case** $m = 1$. Write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$,

$$F_t^{b_1}(f)(x) = (b_1(x) - (b_1)_{2Q})F_t(f)(x) - F_t((b_1 - (b_1)_{2Q})f_1)(x) - F_t((b_1 - (b_1)_{2Q})f_2)(x).$$

Then,

$$\begin{aligned}
& |T_{b_1}(f)(x) - T(((b_1)_{2Q} - b_1)f_2)(x_0)| \\
= & \left| ||F_t^{b_1}(f)(x)|| - ||F_t(((b_1)_{2Q} - b_1)f_2)(x_0)|| \right| \\
\le & ||F_t^{b_1}(f)(x) - F_t(((b_1)_{2Q} - b_1)f_2)(x_0)|| \\
\le & ||(b_1(x) - (b_1)_{2Q})F_t(f)(x)|| + ||F_t((b_1 - (b_1)_{2Q})f_1)(x)|| \\
& + ||F_t((b_1 - (b_1)_{2Q})f_2)(x) - F_t((b_1 - (b_1)_{2Q})f_2)(x_0)|| \\
= & A(x) + B(x) + C(x).
\end{aligned}$$

For $A(x)$, by Hölder's inequality with exponent $1/r + 1/r' = 1$, we get

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q A(x)dx \\
= &\frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_{2Q}||T(f)(x)|dx \\
\le &C\left(\frac{1}{|2Q|}\int_{2Q}|b_1(x)-(b_1)_{2Q}|^{r'}dx\right)^{1/r'}\left(\frac{1}{|Q|}\int_Q |T(f)(x)|^r dx\right)^{1/r} \\
\le &C||b_1||_{BMO}M_r(T(f))(\tilde{x}).
\end{aligned}
$$

For $B(x)$, choose s, q such that $1 < s, q < \infty$ and $r = qs$, by the boundedness of T on $L^q(R^n)$ and Hölder's inequality, we obtain

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q B(x)dx \\
= &\frac{1}{|Q|}\int_Q [T((b_1-(b_1)_{2Q})f_1)(x)]dx \\
\le &\left(\frac{1}{|Q|}\int_{R^n}[T((b_1-(b_1)_{2Q})f\chi_{2Q})(x)]^q dx\right)^{1/q} \\
\le &C\frac{1}{|Q|^{1/q}}\left(\int_{R^n}|b_1(x)-(b_1)_{2Q}|^q|f(x)\chi_{2Q}(x)|^q dx\right)^{1/q} \\
\le &C|Q|^{-1/q+1/qs'+1/qs}\left(\frac{1}{|2Q|}\int_{2Q}|b_1(x)-(b_1)_{2Q}|^{qs'}dx\right)^{1/qs'}\times \\
\cdot &\left(\frac{1}{|2Q|}\int_{2Q}|f(x)|^{qs}dx\right)^{1/qs} \\
\le &C||b_1||_{BMO}M_r(f)(\tilde{x}).
\end{aligned}
$$

For $C(x)$, by Minkowski's inequality, we obtain, for $x \in Q$,

$$
\begin{aligned}
C(x) &= \left|\left|\int_{R^n}(b_1(y)-(b_1)_{2Q})f_2(y)(F_t(x,y)-F_t(x_0,y))dy\right|\right| \\
&\le \int_{(2Q)^c}|b_1(y)-(b_1)_{2Q}||f(y)|||F_t(x,y)-F_t(x_0,y)|||dy \\
&\le C\int_{(2Q)^c}|b_1(y)-(b_1)_{2Q}||f(y)|\frac{|x_0-x|^\varepsilon}{|x_0-y|^{n+\varepsilon}}dy \\
&\le C\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}|b_1(y)-(b_1)_{2Q}||f(y)|\frac{|x_0-x|^\varepsilon}{|x_0-y|^{n+\varepsilon}}dy \\
&\le C\sum_{k=1}^\infty 2^{-k\varepsilon}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^r dy\right)^{1/r}\times \\
&\quad\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b_1(y)-(b_1)_{2Q}|^{r'}dy\right)^{r'} \\
&\le C\sum_{k=1}^\infty k2^{-k\varepsilon}||b_1||_{BMO}M_r(f)(\tilde{x}) \\
&\le C||b_1||_{BMO}M_r(f)(\tilde{x}),
\end{aligned}
$$

thus

$$\frac{1}{|Q|}\int_Q C(x)dx \le C||b_1||_{BMO}M_r(f)(\tilde{x}).$$

Now, we consider the **Case** $m \ge 2$, we have known that, for $b = (b_1, \cdots, b_m)$,

$$
\begin{aligned}
& F_t^{\vec{b}}(f)(x) = \int_{R^n}\left[\prod_{j=1}^m (b_j(x) - b_j(y))\right] F_t(x,y)f(y)dy \\
= & \int_{R^n} [(b_1(x) - (b_1)_{2Q}) - (b_1(y) - (b_1)_{2Q})]\cdots[(b_m(x) - (b_m)_{2Q}) - (b_m(y) - \\
& (b_m)_{2Q})]F_t(x,y)f(y)dy \\
= & \sum_{j=0}^m \sum_{\sigma\in C_j^m} (-1)^{m-j}(b(x) - (b)_{2Q})_\sigma \int_{R^n}(b(y) - (b)_{2Q})_\sigma F_t(x,y)f(y)dy \\
= & (b_1(x) - (b_1)_{2Q})\cdots(b_m(x) - (b_m)_{2Q})F_t(f)(x) \\
& + (-1)^m F_t((b_1 - (b_1)_{2Q})\cdots(b_m - (b_m)_{2Q})f)(x) \\
& + \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x) - (b)_{2Q})_\sigma \int_{R^n}(b(y) - b(x))_{\sigma^c}F_t(x,y)f(y)dy \\
= & (b_1(x) - (b_1)_{2Q})\cdots(b_m(x) - (b_m)_{2Q})F_t(f)(x) \\
& + (-1)^m F_t((b_1 - (b_1)_{2Q})\cdots(b_m - (b_m)_{2Q})f)(x) \\
& + \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x) - (b)_{2Q})_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x),
\end{aligned}
$$

thus

$$
\begin{aligned}
& |T_{\vec{b}}(f)(x) - T(((b_1)_{2Q} - b_1)\cdots((b_m)_{2Q} - b_m))f_2)(x_0)| \\
= & \Big|\,||F_t^{\vec{b}}(f)(x)|| - ||F_t(((b_1)_{2Q} - b_1)\cdots((b_m)2Q - b_m)f_2)(x_0)||\,\Big| \\
\le & ||F_t^{\vec{b}}(f)(x) - F_t(((b_1)_{2Q} - b_1)\cdots((b_m)_{2Q} - b_m)f_2)(x_0)|| \\
\le & ||(b_1(x) - (b_1)_{2Q})\cdots \\
& (b_m(x) - (b_m)_{2Q})F_t(f)(x)|| \\
+ & \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} ||(\vec{b}(x) - (b_m)_{2Q})_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x)|| \\
+ & ||F_t((b_1 - (b_1)_{2Q})\cdots(b_m - (b_m)_{2Q})f_1)(x)|| \\
+ & ||F_t((b_1 - (b_1)_{2Q})\cdots(b_m - (b_m)_{2Q})f_2)(x) - F_t((b_1 - (b_1)_{2Q})\cdots \\
& (b_m - (b_m)_{2Q})f_2)(x_0)|| \\
= & I_1(x) + I_2(x) + I_3(x) + I_4(x).
\end{aligned}
$$

For $I_1(x)$, by Hölder's inequality with exponent $1/p_1 + \cdots + 1/p_m + 1/r = 1$, where $1 < p_j < \infty$, $j = 1, \cdots, m$, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q I_1(x)dx \\
\leq\ & \frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_{2Q}|\cdots|b_m(x)-(b_m)_{2Q}||T(f)(x)|dx \\
\leq\ & \prod_{j=1}^{m}\left(\frac{1}{|Q|}\int_Q |b_j(x)-(b_j)_{2Q}|^{p_j}\right)^{1/p_j}\left(\frac{1}{|Q|}\int_Q |T(f)(x)|^r dx\right)^{1/r} \\
\leq\ & C||\vec{b}||_{BMO}M_r(T(f))(\tilde{x}).
\end{aligned}$$

For $I_2(x)$, by the Minkowski's and Hölder's inequality, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q I_2(x)dx \\
=\ & \frac{1}{|Q|}\int_Q \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} ||(b(x)-(b)_{2Q})_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x)||dx \\
\leq\ & \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|}\int_Q |(b(x)-(b)_{2Q})_\sigma||T_{\vec{b}_{\sigma^c}}(f)(x)|dx \\
\leq\ & C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|2Q|}\int_{2Q}|(b(x)-(b)_{2Q})_\sigma|^{r'}dx\right)^{1/r'}\left(\frac{1}{|Q|}\int_Q |T_{\vec{b}_{\sigma^c}}(f)(x)|^r dx\right)^{1/r} \\
\leq\ & C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}M_r(T_{\vec{b}_{\sigma^c}}(f))(\tilde{x}).
\end{aligned}$$

For $I_3(x)$, choose $1 < s, q < \infty$ with $r = qs$, by the boundedness of T on $L^q(R^n)$ and Hölder's inequality, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q I_3(x)dx \\
\leq\ & \left(\frac{1}{|Q|}\int_{R^n}|T(\prod_{j=1}^{m}(b_j-(b_j)_{2Q})f\chi_{2Q})(x)|^q dx\right)^{1/q} \\
\leq\ & C\frac{1}{|Q|^{1/q}}\left(\int_{R^n}|\prod_{j=1}^{m}(b_j(x)-(b_j)_{2Q})|^q|f(x)\chi_{2Q}(x)|^q dx\right)^{1/q} \\
\leq\ & C|Q|^{-1/q+1/qs'+1/qs}\left(\frac{1}{|2Q|}\int_{2Q}|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})|^{qs'}dx\right)^{1/qs'} \\
&\times\left(\frac{1}{|2Q|}\int_{2Q}|f(x)|^{qs}dx\right)^{1/qs} \\
\leq\ & C||\vec{b}||_{BMO}M_r(f)(\tilde{x}).
\end{aligned}$$

For $I_4(x)$, choose $1 < p_j < \infty\ j = 1, \cdots, m$ such that $1/p_1 + \cdots + 1/p_m + 1/r = 1$, we obtain, by Hölder's inequality,

$$\begin{aligned} I_4(x) &= ||F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q}) f_2)(x) - F_t((b_1 - (b_1)_{2Q}) \cdots \\ &\quad (b_m - (b_m)_{2Q}) f_2)(x_0)|| \\ &\leq \int_{R^n} |\prod_{j=1}^m (b_j(y) - b(b_j)_{2Q})||f_2(y)\chi_{(2Q)^c}(y)|||F_t(x,y) - F_t(x_0,y)||dy \\ &\leq C\int_{(2Q)^c} |\prod_{j=1}^m (b_j(y) - (b_j)_{2Q})||f(y)|\frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon}}dy \\ &\leq C\sum_{k=1}^\infty \int_{2^{k+1}Q\setminus 2^kQ} |x-x_0|^\varepsilon |x_0-y|^{-(n+\varepsilon)}|\prod_{j=1}^m (b_j(y) - (b_j)_{2Q})||f(y)|dy \\ &\leq C\sum_{k=1}^\infty 2^{-k\varepsilon}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^r dy\right)^{1/r} \\ &\quad \prod_{j=1}^m \left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b_j(y) - (b_j)_{2Q}|^{p_j}dy\right)^{1/p_j} \\ &\leq C\sum_{k=1}^\infty k^m 2^{-k\varepsilon}\prod_{j=1}^m ||b_j||_{BMO} M_r(f)(\tilde{x}) \\ &\leq C||\vec{b}||_{BMO} M_r(f)(\tilde{x}), \end{aligned}$$

thus

$$\frac{1}{|Q|}\int_Q I_4(x)dx \leq C||\vec{b}||_{BMO} M_r(f)(\tilde{x}).$$

This completes the proof of the theorem.

□

Proof. **Proof of Theorem 2.** Choose $1 < r < p$ in Theorem 1. We first consider the case m=1, we have

$$\begin{aligned} ||T_{b_1}(f)||_{L^p} &\leq ||M(T_{b_1}(f))||_{L^p} \leq C||(T_{b_1}(f))^\#||_{L^p} \\ &\leq C||M_r(T(f))||_{L^p} + C||M_r(f)||_{L^p} \\ &\leq C||T(f)||_{L^p} + C||M_r(f)||_{L^p} \\ &\leq C||f||_{L^p} + C||f||_{L^p} \\ &\leq C||f||_{L^p}. \end{aligned}$$

When $m \geq 2$, we may get the conclusion of Theorem 2 by induction. This finishes the proof. □

4. Applications

Now we give some applications of the theorems in this paper.

Application 1. Littlewood-Paley operator.

Fixed $\varepsilon > 0$. Let ψ be a fixed function which satisfies the following properties:

(1) $\int_{R^n}\psi(x)dx = 0$,

(2) $|\psi(x)| \leq C(1+|x|)^{-(n+1)}$,

(3) $|\psi(x+y) - \psi(x)| \leq C|y|^\varepsilon(1+|x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$.

The Littlewood-Paley multilinear commutator is defined by

$$g_\psi^{\vec{b}}(f)(x) = \left(\int_0^\infty |F_t^{\vec{b}}(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x) = \int_{R^n} \prod_{j=1}^m (b_j(x) - b_j(y))\psi_t(x-y)f(y)dy$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Set $F_t(f)(y) = f * \psi_t(y)$. We also define that

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

which is the Littlewood-Paley operator(see [15]). Let H be the space

$$H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t\right)^{1/2} < \infty\right\},$$

then, for each fixed $x \in R^n$, $F_t^{\vec{b}}(f)(x)$ may be viewed as the mappings from $[0, +\infty)$ to H, and it is clear that

$$g_\psi^{\vec{b}}(f)(x) = ||F_t^{\vec{b}}(f)(x)||, \quad g_\psi(f)(x) = ||F_t(f)(x)||.$$

It is easily to see that g_ψ satisfies the conditions of Theorems 1 and 2 (see [5-7]), thus Theorems 1 and 2 hold for $g_\psi^{\vec{b}}$.

Application 2. Marcinkiewicz operator.

Fixed $0 < \gamma \le 1$. Let Ω be homogeneous of degree zero on R^n with $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$. The Marcinkiewicz multilinear commutator is defined by

$$\mu_\Omega^{\vec{b}}(f)(x) = \left(\int_0^\infty |F_t^{\vec{b}}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x) = \int_{|x-y|\le t} \prod_{j=1}^m (b_j(x) - b_j(y)) \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

Set

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy;$$

We also define that

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

which is the Marcinkiewicz operator(see [16]). Let H be the space

$$H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t^3\right)^{1/2} < \infty\right\}.$$

Then, it is clear that

$$\mu_\Omega^{\vec{b}}(f)(x) = ||F_t^{\vec{b}}(f)(x)||, \quad \mu_\Omega(f)(x) = ||F_t(f)(x)||.$$

It is easily to see that μ_Ω satisfies the conditions of Theorems 1 and 2 (see [8][16]), thus Theorems 1 and 2 hold for $\mu_\Omega^{\vec{b}}$.

Application 3. Bochner-Riesz operator.

Let $\delta > (n-1)/2$, $B_t^\delta(f)\hat{}(\xi) = (1-t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(z) = t^{-n}B^\delta(z/t)$ for $t > 0$. Set

$$F_{\delta,t}^{\vec{b}}(f)(x) = \int_{R^n} \prod_{j=1}^{m}(b_j(x) - b_j(y))B_t^\delta(x-y)f(y)dy.$$

The maximal Bochner-Riesz multilinear commutator is defined by

$$B_{\delta,*}^{\vec{b}}(f)(x) = \sup_{t>0} |B_{\delta,t}^{\vec{b}}(f)(x)|.$$

We also define that

$$B_{\delta,*}(f)(x) = \sup_{t>0} |B_t^\delta(f)(x)|$$

which is the maximal Bochner-Riesz operator(see [10]). Let H be the space $H = \{h : ||h|| = \sup_{t>0} |h(t)| < \infty\}$, then

$$B_{\delta,*}^{\vec{b}}(f)(x) = ||B_{\delta,t}^{A}(f)(x)||, \quad B_*^\delta(f)(x) = ||B_t^\delta(f)(x)||.$$

It is easily to see that $B_{\delta,*}^{\vec{b}}$ satisfies the conditions of Theorems 1 and 2 (see [9]), thus Theorems 1 and 2 hold for $B_{\delta,*}^{\vec{b}}$.

REFERENCES

[1] J. Alvarez, R. J. Babgy, D. S. Kurtz and C. Pérez, *Weighted estimates for commutators of linear operators*, Studia Math., **104**(1993), 195-209.

[2] R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math., **103**(1976), 611-635.

[3] J. Garcia-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland Math., Amsterdam, **16**, 1985.

[4] G. E. Hu and D. C. Yang, *A variant sharp estimate for multilinear singular integral operators*, Studia Math., **141**(2000), 25-42.

[5] L. Z. Liu, *Weighted weak type estimates for commutators of Littlewood-Paley operator*, Japanese J. of Math., **29**(**1**)(2003), 1-13.

[6] L. Z. Liu, *A sharp endpoint estimate for multilinear Littlewood-Paley operator*, Georgian Math. J., **11**(2004), 361-370.

[7] L. Z. Liu, *Sharp endpoint inequality for multilinear Littlewood-Paley operator*, Kodai Math. J., **27**(2004), 134-143.

[8] L. Z. Liu, *A sharp estimate for multilinear Marcinkiewicz integral operator*, The Asian J. of Math., **9**(**2**)(2005), 177-184.

[9] L. Z. Liu and S.Z. Lu, *Weighted weak type inequalities for maximal commutators of Bochner-Riesz operator*, Hokkaido Math. J., **32**(**1**)(2003), 85-99.

[10] S. Z. Lu, *Four Lectures on Real H^p Spaces, World Scientific*, River Edge, NI, 1995.

[11] C. Pérez, *Endpoint estimate for commutators of singular integral operators*, J. Func. Anal., **128**(1995), 163-185.

[12] C. Pérez and G. Pradolini, *Sharp weighted endpoint estimates for commutators of singular integral operators*, Michigan Math. J., **49**(2001), 23-37.

[13] C. Pérez and R.Trujillo-Gonzalez, *Sharp Weighted estimates for multilinear commutators*, J. London Math. Soc., **65**(2002), 672-692.

[14] E. M. Stein, *Harmonic analysis: real variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton NJ, 1993.

[15] A. Torchinsky, *Real variable methods in harmonic analysis*, Pure and Applied Math., 123, Academic Press, New York, 1986.

[16] A. Torchinsky and S. Wang, *A note on the Marcinkiewicz integral*, Colloq. Math.,**60/61**(1990), 235-24.

[17] B. S. Wu and L. Z. Liu, *A sharp estimate for multilinear Bochner-Riesz operator*, Studia Sci. Math. Hungarica, **40**(**1**)(2004), 47-59.

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Numerical Solution of Gas Dynamics Equation using Second Order Dynamic Mesh Technique

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Abstract. Second order dynamic mesh technique is employed to solve gas dynamic equations which represent the different aspects of hyperbolic (nonlinear) equations. The value of the density, velocity, pressure and internal energy of the gas at t= 0.15s are computed using dynamic mesh technique and uniform mesh method. Their graphical comparison is given in £gures 1-4. We observe that the dynamic mesh graphs are more smooth than uniform mesh graphs. Therefore it is clear that the dynamic mesh technique gives better results than standard uniform mesh method.

1. Introduction

In this paper we have considered one dimensional gas dynamics equations representing laws of conservation of mass, momentum and energy along with the equation of state of the gas. This problem is considered as a case study for solving hyperbolic (non linear) conservation laws because it depicts the next level of complexity after the Berger's equations. In dynamic mesh technique a £xed number of mesh points move automatically to minimize the error in the solution. Second order dynamic mesh technique based on equidistribution principle is used here because it is more ef£cient than uniform mesh methods for solving time-dependent partial differential equations.

2. Gas Dynamics Equation

The one dimensional gas dynamics equations in conservation form are
Continuity equation

$$\rho_t + m_x = 0 \tag{2.1}$$

Momentum equation

$$m_t + \left[\frac{m^2}{\rho} + P\right]_x = 0 \tag{2.2}$$

Energy Equation

$$E_t + \left[\frac{m}{\rho}(E+P)\right]_x = 0 \tag{2.3}$$

In vector form, they can be written as

$$u_t + [f(u)]_x = 0 \tag{2.4}$$

Where

$$u = \begin{bmatrix} \rho \\ m \\ E \end{bmatrix}, F(u) = \begin{bmatrix} m \\ \frac{m^2}{\rho} + p \\ \frac{m}{\rho}(E+P) \end{bmatrix}$$

And ρ stands for density, m for momentum, P for pressure and E is the total energy per unit volume.

The equation of state for an ideal gas to express the internal energy per unit mass e as a function of P and ρ is given by

$$e = \frac{P}{\rho(\gamma-1)}, \gamma > 1 \tag{2.5}$$

Where γ is the ratio of speci£c heats.

We take γ = 1.4, a value corresponding to a diatomic gas.

Bernoullie's equation is given by

$$E = e\rho + \frac{m^2}{2\rho}$$

Using equation 2.5 it becomes

$$E = \frac{P}{\gamma-1} + \frac{m^2}{2\rho}$$

$$or P = (\gamma-1)\left[E - \frac{m^2}{2\rho}\right] \tag{2.6}$$

We solve equations(2.1), (2.2), (2.3) and ((2.6)) which involves three independent variables ρ, m and E, t is time and x are the position coordinates, using the following initial and boundary conditions.

$$\rho(x,0) = \begin{bmatrix} 1.0000 & \text{if x} < 0 \\ 0.5635 & \text{if x} = 0 \\ 0.1250 & \text{if x} > 0 \end{bmatrix}$$

m(x,0) = 0 for all

$$E(x,0) = \begin{bmatrix} 2.5000 & \text{if x} < 0 \\ 1.375 & \text{if x} = 0 \\ 0.2500 & \text{if x} > 0 \end{bmatrix}$$

ρ (-0.5,t) = 1.0000, ρ (0.5,t) = 0.1250

m (-0.5,t) = 0.0000 = m (0.5,t)

E (-0.5,t) = 2.5000, E(0.5,t) = 0.2500

3. Addition of Artificial Viscosity

If we solve equation(2.4) by using £nite differences for ux it reduces to a system of £rst order differential equations. The RKF45 program can be used to solve this system under the given boundary conditions but it has discontinuous solutions. To overcome this diff£-culty arti£cial viscosity, a term proportional to uxx is added to the system which makes the problem simpler. Then the system of equations (2.1), (2.2) and (2.3) becomes

$$\rho_t + mx = \lambda \rho_{xx} \tag{3.1}$$

$$m_t + \left[\frac{m^2}{\rho} + P\right]_x = \lambda m_{xx} \tag{3.2}$$

and

$$E_t + \left[\frac{m}{\rho}(E + P)\right]_x = \lambda E_{xx} \tag{3.3}$$

and in vector form, it can be written as

$$u_t + [F(u)]_x = \lambda u_{xx} \tag{3.4}$$

With boundary conditions as
ρ (-0.5,t) = 1.0000, ρ(0.5,t) = 0.1250
m (-0.5,t) = 0.000 = m (0.5,t) for all *t*
E (-0.5,t) = 2.5000, E(0.5,t) = 0.2500

$$u = \begin{bmatrix} \rho \\ m \\ E \end{bmatrix}, F(u) = \begin{bmatrix} m \\ \frac{m^2}{\rho} + p \\ \frac{m}{\rho}(E + P) \end{bmatrix}$$

It is well known that for hyperbolic conservation laws, even smooth initial conditions can produce solutions which eventually become discontinuous. Hence when we speak here of a solution of (2.4) we will mean a weak solution. In [6] Lax proves that if the solution u (x,t;λ) of (3.4) converges to a limit $\bar{u}$(x,t) as λ approaches 0^+, then $\bar{u}$(x,t) is a weak solution of (2.4). Further Foy [5] proves that the solution of (3.4) do indeed converge if the original shock waves are weak enough. Therefore the addition of arti£cial viscosity will not destroy the essential character of hyperbolic equations ((2.1)-(2.3)). We take λ= 5 x 10^{-4}.

4. Dynamic Mesh:

Case Study.
To apply dynamic mesh technique we write the gas dynamics equation 3.4 as

$$u_t = G \tag{4.1}$$

where $G = \lambda u_{xx} - [F(u)]_x$
and we take λ = 5 x 10^{-4}
Finite difference approximation for the £rst and second order derivatives on a moving grid are given by

$$u_x = \frac{u_{i+1} - u_{i-1}}{x_{i+1} - x_{i-1}} \tag{4.2}$$

$$u_{xx} = \left[\frac{u_{i+1}(x_i - x_{i-1}) - u_i(x_{i+1} - x_{i-1}) + u_{i-1}(x_{i+1} - x_i)}{(x_{i+1} - x_i)(x_i - x_{i-1})(x_{i+1} - x_{i-1})}\right] \tag{4.3}$$

By using transformation

x = x (ξ,t), xϵ[0,1]
x(0,t) = 0, x (1,t) = 1
equation (4.1) reduces to the quasi Lagrangian form as

$$\dot{u} - \frac{\partial u}{\partial x}\dot{x} = G \tag{4.4}$$

Using eq (4.2), it becomes

$$\dot{u} - \frac{u_{i+1} - u_{i-1}}{x_{i+1} - x_{i-1}}\dot{x} = G_i \tag{4.5}$$

i = 0,1,2,. . . ,n
and Gi is the discrete approximation of G. Monitor functions involving higher derivatives of u(x) are extremely complicated for use, so White [7] recommends to use the arc length function

$$M(x,t) = \sqrt{1 + u_x^2}$$

5. DISCRETIZATION OF MOVING MESH PARTIAL DIFFERENTIAL EQUATION (MMPDE)

MMPDE [1]

$$\dot{x} = \frac{\partial}{\partial \xi}\left(M\frac{\partial x}{\partial \xi}\right) \tag{5.1}$$

gives the node speed and it will be used to solve the system because it gives better results. For this purpose equation (5.1) is discretized in space with centered £nite differences on the uniform mesh.

$$\xi_i = \frac{i}{n}, i = 0, 1..., n$$

where n is positive integers and by using method of lines.
The discrete approximation of (5.1) is then

$$\dot{x} = \frac{E_i}{\tau} \tag{5.2}$$

Where τ = 1/λ and Ei is the discrete approximation of
$E = \frac{\partial}{\partial \xi}\left(M\frac{\partial x}{\partial \xi}\right)$at ξ = ξi given by

$$E_i = \frac{\bar{M}_{i+1} + \bar{M}_i}{2\left(\frac{1}{n}\right)^2}(x_{i+1} - x_i) - \frac{\bar{M}_i + \bar{M}_{i-1}}{2\left(\frac{1}{n}\right)^2}(x_i - x_i - 1) \tag{5.3}$$

Thus equations (5.2) becomes

$$\dot{x}_i = \frac{n^2}{2\tau}\left[(\bar{M}_i + \bar{M}_{i+1})x_{i+1} - (\bar{M}_{i-1} + 2\bar{M}_{i+1})x_i + (\bar{M}_{i-1} + \bar{M}_i)x_{i-1}\right] \tag{5.4}$$

$$i = 1, 2, \ldots, n-1$$

$$\dot{x}_1 = \frac{n^2}{2\tau}\left[(\bar{M}_i + \bar{M}_2)x_2 - (\bar{M}_0 + 2\bar{M}_1 + \bar{M}_2)x_1 - 0.5(\bar{M}_0 + \bar{M}_1)\right] \tag{5.5}$$

$$\dot{x}_i = \frac{n^2}{2\tau}\left[(\bar{M}_i + \bar{M}_{i+1})_{i+1} - (\bar{M}_{i-1} + 2\bar{M}_i + \bar{M}_{i+1})x_i + (\bar{M}_{i-1} + \bar{M}_i)x_{i-1}\right] \tag{5.6}$$

i = 2,...,n-2

$$\dot{x}_{n-1} = \frac{n^2}{2\tau}[0.5(\bar{M}_{n-1} + \bar{M}_n) - (\bar{M}_{n-2} + 2\bar{M}_{n-1} + \bar{M}_n)x_{n-1} + (\bar{M}_{n-2} + \bar{M}_{n-1})x_{n-2}] \quad (5.7)$$

where $\bar{M}_i$is the smoothed form of

$$M_i = \left[1 + \left(\frac{u_{i+1} - u_{i-1}}{x_{i+1} - x_{i-1}}\right)^2\right]^{1/2} \quad (5.8)$$

Mi must be smoothed in order to obtain reasonable accuracy
Let $\bar{M}_i = \left(\frac{M_i^*}{S_i^*}\right)^{1/2}$
where $M_i^* = \sum_{k=i-j}^{i+j}(M_k)^2 \left(\frac{\eta}{1+\eta}\right)^{|k-i|}$
and $S_i^* = \sum_{k=i-j}^{i+j} \left(\frac{\eta}{1+\eta}\right)^{|k-i|}$
Where $\eta > 0$ is the smoothing parameter, and j a non-negative integer, is the smoothing index. The summation is understood to contain only elements with indices between zero and n. Thus the problem is reduced to solving two sets of equations (4.5) and (5.2). The initial conditions for xi is a uniform mesh, i.e.

$$x_i(0) = \frac{i}{n}, \; i = 0, 1, 2, 3..., n$$

The boundary conditions being used are $\dot{x}(0) = 0$ and $\dot{x}(n) = 0$
The systems of ordinary differential equations are solved using ordinary differential equation solver RKF45. For calculation a relative and absolute tolerance of 10^{-8} is assumed. After testing various values and combinations for the parameters; η, J and τ the following values have been chosen since they given the most accurate result:
$\eta = 2$, J = 2 and $\tau = 10^{-3}$

6. Results

The one dimensional gas dynamics equation is solved using moving mesh method with n = 100 at t = 0.15s. Other dynamic mesh formulations have also been considered. However, better results were obtained using (5.1).
The values of density, velocity, pressure and internal energy of the gas at t = 0.15s using dynamic mesh technique and uniform mesh method are given in tables 1-4 and are plotted together in £gures 1-4. It is observed that the dynamic mesh technique yields equally accurate results as the uniform mesh method for signi£cantly smaller number of points for the gas dynamics equations considered in this paper.

7. Conclusion

In this paper we have used the dynamic (moving) mesh technique for solution of one dimensional gas dynamics equation. This technique employs the equidistribution principle. On investigation it is found that dynamic (moving) mesh technique gives the better results for gas dynamics equations.

REFERENCES

[1] Maqbool Ahmad Chaudhry and M. Ozair Ahmad, *Moving mesh method, Research Journal, University of Engineering and Technology, Lahore*, **16** (2005), 1-2.

[2] R. Li, W.B. Liu and H.P. Ma, *Moving mesh method with error-estimater based monitor and its application to static obstacle problem*, Journal of Scienti£c Components, **21(1)** (2004).

[3] placeCity Ames, W.F., *Numerical Methods for partial differential equations*, 3r Ed. Academic, placeState-New York (1988).

[4] Boor. C. DE., *Good approximation by Splines with variables knots II*, in Springer Lecturer Notes Series 363, Springer Verlag. placeStateBerlin. (1973).

[5] Foy, Linus Richard, *Steady state solution of hyperbolic systems of conservation laws with viscosity terms.* Communication on pure and applied Mathematics, **17** (1964), 177-188.

[6] Lax, P.D., *Weak solution of nonlinear hyperbolic equations an their numerical computation.* Communication on pure and applied Mathematics, **7** (1954), 159-193.

[7] A.B, *White on selection of equdistributing methods for two point boundary problems.* SIAMJ. Number. Anal., **16** (1979), 472-502.

[8] L.R. Petzold, *observation on an adaptive moving grid method for one dimensional systems of partial differential equations.* Appl. Number. Math., **3** (1987), 347-360.

[9] H. D. Ceniceros and T. Y. Hou, *An ef£cient dynamically daptive mesh for potentially singular solutions*, Journal of Computational Physics, **172** (2001), 609-639.

[10] B.N. Azarenok and S. A. Ivanenko, *Application of adaptive rids in numerical analysis of time-dependent problems in gas dynamics*, Comput. Maths. Math. Phys, **40** (2000), 1330-1349.

[11] B.N. Azarenok, *Variational barrier method of adaptive grid generation in hyperbolic problems of gas dynamics*, SIAM Journal on Numerical Analysis, **40** (2002), 651-682.

[12] H.D. Ceniceros and T. Y. Hou, *An ef£cient dynamically daptive mesh for potentially singular solutions*, Journal of Computational Physics **172** (2001), 609-639.

[13] T. Gener and D. Krner, *Dynamic mesh adaptive for supersonic reactive ¤ow Preprint*, (2001).

Table 1- Density of the gas

X	Finite Difference	Moving Mesh	Absolute Error
-0.500000	1.00000000	1.00000000	0.00000000
-0.400000	1.00000000	1.00000000	0.00000000
-0.300000	1.00000000	1.00000000	0.00000000
-0.200000	0.90522825	0.91599216	0.01076391
-0.100000	0.59331406	0.60146611	0.00815205
0.000000	0.43400938	0.43642349	0.00241411
0.100000	0.42314568	0.42465446	0.00150878
0.200000	0.24555786	0.26980234	0.02424448
0.300000	0.24500000	0.26450688	0.01950688
0.400000	0.12500000	0.12500000	0.00000000
0.500000	0.12500000	0.12500000	0.00000000

Table 2- Velocity of the gas

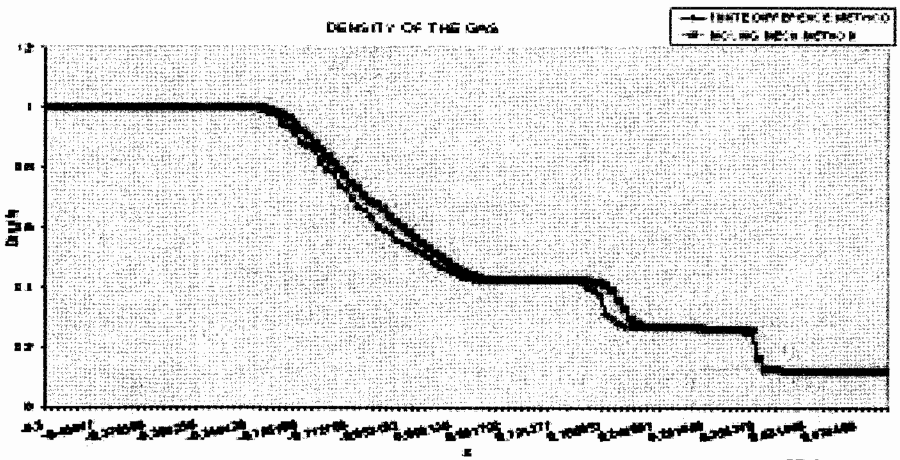
X	Finite Difference	Moving Mesh	Absolute Error
-0.500000	0.00000000	0.00000000	0.00000000
-0.400000	0.00000000	0.00000000	0.00000000
-0.300000	0.00000000	0.00000000	0.00000000
-0.200000	0.00565540	0.00000983	0.00564557
-0.100000	0.42905106	0.32891547	0.10013559
0.000000	0.90723119	0.74714813	0.16008306
0.100000	0.92746010	0.92765451	0.00019441
0.200000	0.92742284	0.92746329	0.00004045
0.300000	0.00000000	0.93505693	0.93505930
0.400000	0.00000000	0.00000003	0.00000003
0.500000	0.00000000	0.00000000	0.00000000

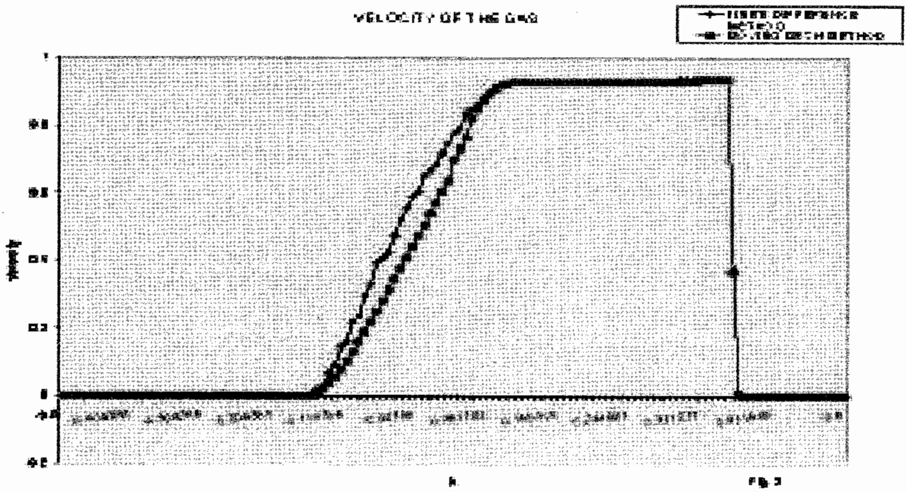
Table 3- Pressure of the gas

X	Finite Difference	Moving Mesh	Absolute Error
-0.500000	1.00000000	1.00000000	0.00000000
-0.400000	1.00000000	1.00000000	0.00000000
-0.300000	1.00000000	1.00000000	0.00000000
-0.200000	0.99332728	0.99694026	0.00361298
-0.100000	0.52936241	0.55750428	0.02814187
0.000000	0.31182478	0.3220423	0.01021752
0.100000	0.30312357	0.30311585	0.00000772
0.200000	0.30311914	0.30312362	0.00000448
0.300000	0.10000000	0.30200000	0.20200000
0.400000	0.10000000	0.10000000	0.00000000
0.500000	0.10000000	0.10000000	0.00000000

Table 4-Energy of the gas

X	Finite Difference	Moving Mesh	Absolute Error
-0.500000	2.50000000	2.50000000	0.00000000
-0.400000	2.50000000	2.50000000	0.00000000
-0.300000	2.50000000	2.50000000	0.00000000
-0.200000	2.49522477	2.49854773	0.00332296
-0.100000	2.16122261	2.21779172	0.05656911
0.000000	1.79618690	1.88824636	0.09205946
0.100000	1.79089372	1.78172459	0.00916913
0.200000	1.85360731	1.82745774	0.02614957
0.300000	2.00000000	2.85256890	0.85256890
0.400000	1.99999999	2.00000000	0.00000001
0.500000	2.00000000	2.00000000	0.00000000

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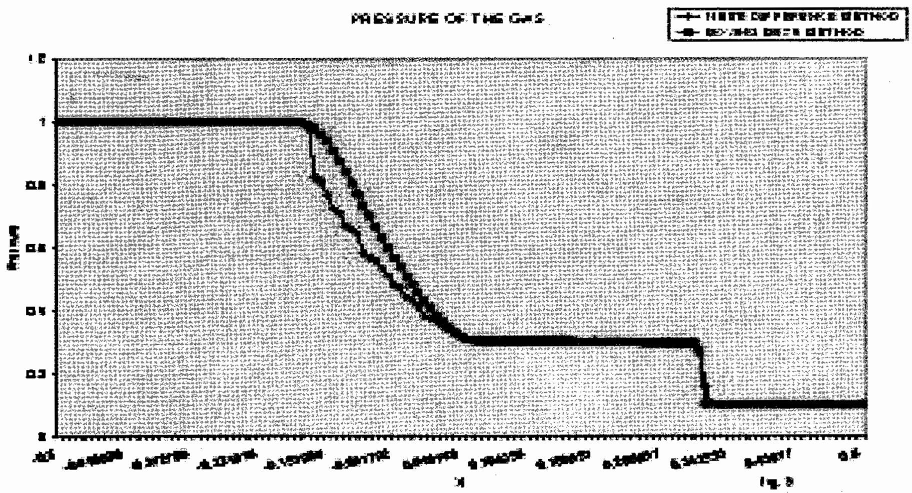
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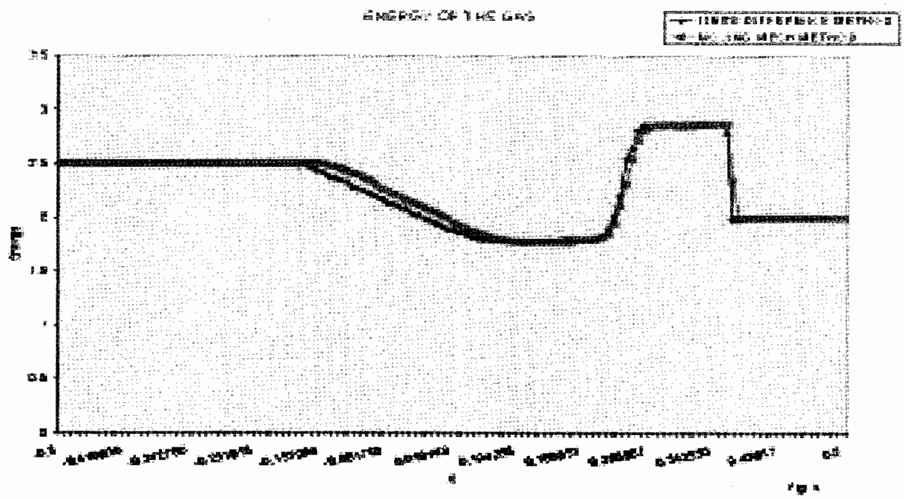
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