VOLUME 42 (2010) ISSN 1016-2526

PUNJAB UNIVERSITY JOURNAL OF MATHEMATICS

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF THE PUNJAB
LAHORE - 54590 PAKISTAN
www.pu.edu.pk/math/pujm

D. R. Choudary
School of Mathematical Sciences (SMS), Lahore-Pakistan

P. Corsini
Dept of Mathematics and CS Via delle Science 206
33100 Udine
Italy

Palle Jorgensen
Department of Mathematics
14 MLH
The University of Iowa
Iowa City, IA 52242-1419
USA

Young Bae Jun
Dept of Mathematics Educ.
Gyeongsang Natl University
Chinju 660-701
Korea

Abdul Qayyum M. Khaliq
Department of Mathematical Sciences
Middle Tennessee State University
Murfreesboro, TN 37132
USA

Manabu Sakai
Department of Mathematics and Computer Science
Kagoshima University
1-22-36, Kagoshima city
Kagoshima, 890-0065
Japan

Muhammad Sarfraz
Department of Information and Computer Science
King Fahd University of Petroleum and Minerals
KFUPM # 1510, Dhahran 31261
Saudi Arabia

Walter Schempp
Lehrstuhl fuer Mathematik I
University of Siegen
Walter-Flex-Strasse 3
57068 Siegen
Germany

Malik Zawwar Hussain
Department of Mathematics
University of the Punjab
Lahore-Pakistan

Muhammad Aslam Malik
Department of Mathematics
University of the Punjab
Lahore-Pakistan

Ratnasingham Shivaji
Department of Mathematics and Statistics
Mississippi State University
Mississippi State, MS 39762
USA

Hari M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4
Canada

Jieqing Tan
Director, Institute of Applied Mathematics
Hefei University of Technology
Hefei 230009
China

E.H. Twizell
School of Information Systems
Computing and Mathematics
Brunel University
Uxbridge, Middlesex
UB8 3PH
England

E.M.E. Zayed
Department of Mathematics
Zagazig University
Zagazig
Egypt

CONTENTS

Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 42 (2010) pp. 1-16

Magnetohydrodynamic Flow and Heat Transfer for A Peristaltic Motion of Carreau Fluid Through A Porous Medium

Nabil T.M.El-Dabe, Ashraf Fouad and Marwa M.Hussein
Department of Mathematics, Faculty of Education
Ain Shams University,
Heliopolis, Cairo, Egypt
Email: khaledelagamy2004@yahoo.com

Abstract. The two-dimensional peristaltic motion of magnetohydrodynamic flow and heat transfer for incompressible non-Newtonian fluid through a porous medium in uniform channel with a sinusoidal wave are studied. The system is influenced by uniform magnetic field. The problem is formulated and analyzed using a perturbation expansion in terms of a variant of the Weicsenberg number. Carreau flow is considered in this study to investigate the effect of porous medium. An analytic forms for axial velocity,pressure gradient and heat transfer have been obtained. The results were studied for various values of the physical parameters of the problem and illustrated graphically.

1. INTRODUCTION

Our purpose is to investigate the mechanism by which a fluid is transported through a duct when contraction waves propagate progressively along its wall. This valveless-pumping principle, which is called peristalsis [1], plays a role in many physiological processes with fluid transport and is also exploited in technology, e.g. in so-called "roller pumps".There are many investigations on peristaltic flow of Newtonian fluids have been carried out . Rath [2] has given a survey of this subject, with a probably complete summary of the bibliography unit. Studying peristaltic flows, especially with a view to applications in biomechanics and physiology, one should consider real material properties of the fluid being transported and determine the essential departures from the results of the theories for Newtonian fluids. These investigations are, also, interesting for technological applications, e.g. in the field of polymer processing. In this regard there are only few contributions in the literature.

The analysis of the mechanisms responsible for peristaltic transport have been studied by many authors. Latham's investigation[3], may be the first study in this field and since that time several theoretical and experimental investigations have been made to understand peristaltic action in both mechanical and physiological situations.

Some of these studies were made by Burns and Parkes [4], Barton and Raynor [5], Shapiro et al. [6], Lykoudis and Roos [7], Roos and Lykoudis [8], Shuka et al. [9], Elshehawey and Mekheimer [10].

Since most of physiological fluids in the human body behave like non-Newtonian fluids, some researches on non-Newtonian fluids were recently published.

Bohme and Friedrich [1] have investigated peristaltic flow of second-order viscoelastic liquid assuming that the reliant Reynolds number is small enough to neglect inertia forces, and that the ratio of the wave length and the channel height is large so that the pressure is constant over the cross-section. ELmisery, Elshehawey and Hakeem [11] have studied the peristaltic motion of an incompressible generalized Newtonian fluid in a planar channel of uniform geometry in case of long-wave approximation. The same problem for non-uniform channel has been studied by Elshehawey, EL Misery [12].

The effect of porous medium on the motion of the fluid have been studied by many authors, Elshehawey et al. [13] studied the effect of porous medium on peristaltic motion of a Newtonian fluid. Eldabe [14] studied magnetohydrodynamic flow through a porous medium fluid at a rear stagnation point. Eldabe et al. [15] studied MHD flow and heat transfer in a viscoelastic incompressible fluid confined between a horizontal stretching sheet and a parallel porous wall. Elshehawey et al. [16] studied the peristaltic motion of a Generalized Newtonian fluid through a porous medium.

Elshehawey et al. [17] studied the peristaltic motion of a Generalized Newtonian fluid under the affect of transverse magnetic field. This problem studied the effect of porous boundaries on peristaltic transport through a porous medium. Elshehawey and Sobh [18] studied the peristaltic viscoelastic fluid motion in a tube.

The main aim of this work is to study the effect a magnetic field and heat transfer on a peristaltic motion of Carreau fluid through a porous medium in uniform channel with a sinusoidal wave. The system is expressed by uniform magnetic field and heat transfer. By using Weicsenberg perturbation technique, in fact we have to choose the parameters for the Carreau fluid such that the Weicsenberg number $W \prec 1$, the wave number δ is neglected and the Reynold's number R_e is very small [17]. The velocity, pressure gradient and heat transfer have been obtained in explicit forms. The effects of the parameters of the problem on these solutions (the magnetic number, the Prandtl number, the Eckert number and the Weicsenberg number) are discussed and shown graphically.

2. Basic Equations

The basic equations of MHD motion neglecting displacement and free charges are

2.1. The continuity equation.

$$\nabla \cdot \underline{V} = 0 \tag{2. 1}$$

where $\underline{V}$ the velocity vector

2.2. The momentum equation.

$$\rho[\frac{\partial V}{\partial t} + (\underline{V} \cdot \nabla)\underline{V}\,] = -\nabla P + \nabla \cdot \underline{\tau} - \frac{\mu}{\epsilon_0}\underline{V} + \underline{J} \times \underline{B} \tag{2. 2}$$

where ρ is the density of the fluid, t is the time, P is the pressure of fluid, $\underline{\tau}$ is the extra stress tensor, μ is the viscosity coefficient, ϵ_0 is the permeability fluid, $\underline{J}$ is the current density and $\underline{B}$ the magnetic flux density.

2.3. Energy equation.

$$\rho c_p[\frac{\partial T}{\partial t} + (\underline{V} \cdot \nabla)T\,] = k\nabla^2 T + \tau \cdot (\nabla \underline{V}) \tag{2. 3}$$

where c_p, k and T are capacity, thermal coductivity and temperature of the fluid.

2.4. **Maxwell's equations.**

$$\nabla \times \underline{B} = \mu_e \underline{J} \tag{2. 4}$$

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \tag{2. 5}$$

$$\nabla \cdot \underline{B} = 0 \tag{2. 6}$$

where $\underline{E}$ denotes the electric field and μ_e is the magnetic permeability.

2.5. **Ohm's equation.**

$$\underline{J} = \sigma(\underline{E} + \underline{V} \times \underline{B}) \tag{2. 7}$$

where σ is the electric conductivity.

2.6. **Constitutive equation for Carreau fluid** [16] **is given by.**

$$\frac{\eta - \eta_\infty}{\eta_0 - \eta_\infty} = [1 + (\Gamma\gamma\cdot)^2]^{(\frac{n-1}{2})} \tag{2. 8}$$

where η_0 is the zero-shear-rate viscosity, η_∞ is the infinite-shear-rate viscosity, Γ is a time constant, n is the dimensionless power law index and $\gamma\cdot$ is defined by:

$$\gamma\cdot = \sqrt{\frac{1}{2}\sum_i \sum_j \gamma\cdot_{ij}\gamma\cdot_{ij}} = \sqrt{\frac{1}{2}\Pi_{\gamma\cdot_{ij}}} \tag{2. 9}$$

where $\Pi_{\gamma\cdot_{ij}}$ is second invariant of strain -rate tensor $\gamma\cdot_{ij}$.

In equation (2. 8) we shall consider the case for which $\eta_\infty = 0$ and $\Gamma\gamma\cdot < 1$, so τ_{ij} can be written as [16]:

$$\tau_{ij} = -\eta_0[1 + \frac{n-1}{2}(\Gamma\gamma\cdot)^2]\gamma\cdot_{ij} \tag{2. 10}$$

τ_{ij} are the components of the extra stress tensor.

3. Formulation of the Problem

We shall consider a two-dimensional channel of uniform thickness $2a$, filled with an incompressible Carreau fluid through a porous medium. A uniform magnetic flux density B_0 fixed relative to the fluid is imposed along Y-axis.

The walls of the channel are flexible and non-conducting, on which are imposed travelling sinusoidal waves of moderate amplitude.The geometry of the wall surface is defined as in fig. (1)

$$H(X,t) = a + b\,\sin\frac{2\pi}{\lambda}(X - ct), \tag{3. 1}$$

where b is the wave amplitude, λ is the wave length, c is the speed of the wave and X is the same direction of the wave propagation .

We choose moving coordinates (x, y), wave frame, which travel in the X-direction with the same speed as the wave, the unsteady flow in the laboratory frame (X, Y) can be treated as steady [6].The coordinates frame is related by :

$$x = X - ct, \; y = Y \tag{3. 2}$$

$$u = \overline{U} - c, \; v = \overline{V} \tag{3. 3}$$

where $\overline{U}$, $\overline{V}$and u, v are the velocity components in the corresponding coordinate systems.

Equation of continuity, Navier-Stoke's equation and heat equation, respectively, take the following forms [12], [14]:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{3. 4}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}-\frac{1}{\rho}(\frac{\partial \tau_{11}}{\partial x}+\frac{\partial \tau_{21}}{\partial y})-(\frac{\nu}{\epsilon_0}+\frac{\sigma}{\rho}B_0^2)u \tag{3. 5}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial y}-\frac{1}{\rho}(\frac{\partial \tau_{21}}{\partial x}+\frac{\partial \tau_{22}}{\partial y})-\frac{\nu}{\epsilon_0}v \tag{3. 6}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{K}{\rho c_p}(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2})+\frac{1}{\rho c_p}[\tau_{11}\frac{\partial u}{\partial x}+\tau_{22}\frac{\partial v}{\partial y}+\tau_{21}(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})] \tag{3. 7}$$

where ν is the kinematic viscosity. From equation (2. 10) for $(i,j=1,2)$ we get:

$$\tau_{11}=-\eta_0[1+\frac{n-1}{2}(\Gamma\gamma\cdot)^2]\gamma\cdot_{11}$$

$$\tau_{12}=-\eta_0[1+\frac{n-1}{2}(\Gamma\gamma\cdot)^2]\gamma\cdot_{12}$$

$$\tau_{22}=-\eta_0[1+\frac{n-1}{2}(\Gamma\gamma\cdot)^2]\gamma\cdot_{22} \tag{3. 8}$$

where:

$$\gamma\cdot_{11}=2\frac{\partial u}{\partial x},\ \gamma\cdot_{12}=\gamma\cdot_{21}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},\ \gamma\cdot_{22}=2\frac{\partial v}{\partial y} \tag{3. 9}$$

The appropriate boundary conditions are:

$$u=-c,\ v=-c\frac{dH}{dx},\ T=T_s\ at\ y=H(x)$$

$$\frac{\partial u}{\partial y}=0\ ,\ v=0,\ T=T_0\quad at\ y=0 \tag{3. 10}$$

Let us introduce the following non-dimensional variables

$$x^*=\frac{x}{\lambda},\ X^*=\frac{X}{\lambda},\ y^*=\frac{y}{a},\ Y^*=\frac{Y}{a},\ t^*=\frac{c}{\lambda}t,\ P^*=\frac{a^2}{c\lambda\eta_0}P$$

$$u^*=\frac{u}{c},\ \overline{U}^*=\frac{\overline{U}}{c},\ v^*=\frac{\lambda}{ca}v,\overline{V}^*=\frac{\lambda}{ca}\overline{V},\ T^*=\frac{T-T_s}{T_w-T_s}$$

$$\tau_{ij}^*=\frac{\lambda}{c\eta_0}\tau_{ij}\ i=j,\ \tau_{ij}^*=\frac{a}{c\eta_0}\tau_{ij}\ i\neq j,\ \gamma\cdot_{ij}^*=\frac{\lambda}{c}\gamma\cdot_{ij}\ i=j$$

$$\gamma\cdot_{ij}^*=\frac{a}{c}\gamma\cdot_{ij}\ i\neq j,\ \gamma\cdot^*=\frac{a}{c}\gamma\cdot,\ \epsilon_0^*=\frac{\epsilon_0}{a^2} \tag{3. 11}$$

Using (3. 11) in equations (3. 1) and (3. 4 - 3. 9) after dropping star, we obtain the following equations:

$$H(x)=1+\phi\ \sin 2\pi x \tag{3. 12}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{3. 13}$$

$$R_e\delta(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y})=-\frac{\partial P}{\partial x}-(\delta^2\frac{\partial \tau_{11}}{\partial x}+\frac{\partial \tau_{21}}{\partial y})-(\frac{1}{\epsilon_0}+M)u \tag{3. 14}$$

$$R_e\delta^3(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}) = -\frac{\partial P}{\partial y} - \delta^2(\frac{\partial \tau_{21}}{\partial x}+\frac{\partial \tau_{22}}{\partial y}) - \frac{\delta^2}{\epsilon_0}v \tag{3. 15}$$

$$\begin{aligned} R_e\delta(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}) &= \frac{1}{P_r}(\delta^2\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}) + E_c[\delta^2\tau_{11}\frac{\partial u}{\partial x}+\delta^2\tau_{22}\frac{\partial v}{\partial y} \\ &\quad +\tau_{21}(\frac{\partial u}{\partial y}+\delta^2\frac{\partial v}{\partial x})], \end{aligned} \tag{3. 16}$$

$$\begin{aligned} \tau_{11} &= -[1+\frac{n-1}{2}W^2\gamma^{\cdot 2}]\gamma^{\cdot}{}_{11}, \\ \tau_{12} &= -[1+\frac{n-1}{2}W^2\gamma^{\cdot 2}]\gamma^{\cdot}{}_{12}, \\ \tau_{22} &= -[1+\frac{n-1}{2}W^2\gamma^{\cdot 2}]\gamma^{\cdot}{}_{22} \end{aligned} \tag{3. 17}$$

$$\gamma^{\cdot}{}_{11} = 2\frac{\partial u}{\partial x},\ \gamma^{\cdot}{}_{12} = \gamma^{\cdot}{}_{21} = \frac{\partial u}{\partial y}+\delta^2\frac{\partial v}{\partial x},\ \gamma^{\cdot}{}_{22} = 2\frac{\partial v}{\partial y} \tag{3. 18}$$

where:

$$\begin{aligned} \phi &= \frac{a}{b}, \\ \delta &= \frac{a}{\lambda} \text{ is the wave number,} \\ R_e &= \frac{\rho a c}{\eta_0} \text{ is the Reynold number,} \\ M &= \frac{\sigma a^2 B_0^2}{\eta_0} \text{ is the magnetic number,} \\ P_r &= \frac{\eta_0}{K}c_p \text{ is the Prandtl number,} \\ E_c &= \frac{c}{c_p(T_w - T_s)} \text{ is the Eckert number,} \\ W &= \frac{c\Gamma}{a} \text{ is the Weicsenberg number.} \end{aligned}$$

The dimensionless boundary conditions are:

$$\begin{aligned} &u=-1,\ v=-\frac{dH}{dx},\ T=0 \qquad at\ y=H(x) \\ &\frac{\partial u}{\partial y}=0,\ v=0,\ T=1 \quad at\ y=0 \end{aligned} \tag{3. 19}$$

Using long wavelength approximation ($\delta = \frac{a}{\lambda} = 0$), equations (3. 14 - 3. 18) become:

$$\frac{\partial P}{\partial x} = -\frac{\partial \tau_{21}}{\partial y} - (\frac{1}{\epsilon_0}+M)u \tag{3. 20}$$

$$\frac{\partial P}{\partial y} = 0 \tag{3. 21}$$

$$\frac{1}{P_r}\frac{\partial^2 T}{\partial y^2} = -E_c\tau_{21}(\frac{\partial u}{\partial y}) \tag{3. 22}$$

$$\tau_{12} = \tau_{21} = -[1 + \frac{n-1}{2} W^2 \gamma^{\cdot}] \gamma^{\cdot}_{12}, \tag{3. 23}$$

$$\gamma^{\cdot}_{12} = \gamma^{\cdot}_{21} = \frac{\partial u}{\partial y} \tag{3. 24}$$

Eliminating the pressure from (3. 20)and (3. 21), we get:

$$-\frac{\partial^2 \tau_{21}}{\partial y^2} - (\frac{1}{\epsilon_0} + M)\frac{\partial u}{\partial y} = 0 \tag{3. 25}$$

From equations (3. 23) and (3. 24), we have:

$$\tau_{12} = \tau_{21} = -[1 + \frac{n-1}{2} W^2 \left(\frac{\partial u}{\partial y}\right)^2] \frac{\partial u}{\partial y} \tag{3. 26}$$

4. Rate of Volume Flow

The rate of volume flow in the fixed frame is given by:

$$Q(X,t) = \int_0^{H(X,t)} \overline{U}(X,Y,t) dY \tag{4. 1}$$

The rate of volume flow in the moving frame (wave frame) is given by:

$$q(x) = \int_0^{H(x)} u(x,y) dy \tag{4. 2}$$

With the help of equations (3. 2) and (3. 3), one can show that these two rates of volume flow are related by:

$$Q = q + cH(x) \tag{4. 3}$$

The time-mean flow over a period $\bar{t} = \frac{\lambda}{c}$ at a fixed position X is defined as:

$$\overline{Q} = \frac{1}{\bar{t}} \int_0^{\bar{t}} Q dt \tag{4. 4}$$

By using (2. 10),(4. 2) in (4. 3) we get :

$$\overline{Q} = q + ca \tag{4. 5}$$

Defining the dimensionless time-mean flows θ and F in the fixed and wave frame, respectively as:

$$\theta = \frac{\overline{Q}}{ac} and F = \frac{q}{ac} \tag{4. 6}$$

Equation (4. 4) can be rewritten as:

$$\theta = 1 + F \tag{4. 7}$$

where:

$$F = \int_0^{H(x)} u(x,y) dy \tag{4. 8}$$

5. METHOD OF SOLUTION

We expand the following quantities as power series in the small parameter W as follows:

$$u = W^0 u_o + W^2 u_1 + O(W^4)$$

$$\frac{\partial P}{\partial x} = W^0 \frac{\partial P_0}{\partial x} + W^2 \frac{\partial P_1}{\partial x} + O(W^4).$$
$$\tau_{12} = W^0 \tau_{12(0)} + W^2 \tau_{12(1)} + O(W^4)$$
$$T = W^0 T_o + W^2 T_1 + O(W^4)$$

$$F = W^0 F_o + W^2 F_1 + O(W^4) \quad (5.\ 1)$$

The use of expansion (5. 1) with equations(3. 19),(3. 20),(3. 22),(3. 26)and (4. 4) gives the systems of equations and after comparing the coefficient of $W^0$and W^2, we get:

$$\frac{\partial^2 u_0}{\partial y^2} - (\frac{1}{\epsilon_0} + M)u_0 = \frac{\partial P_0}{\partial x},\ F_0 = \int_0^{H(x)} u_0(x,y)dy \quad (5.\ 2)$$

$$\frac{\partial^2 u_1}{\partial y^2} - (\frac{1}{\epsilon_0} + M)u_1 = \frac{\partial P1}{\partial x} - \frac{3}{2}(n-1)(\frac{\partial^2 u_0}{\partial y^2})(\frac{\partial u_0}{\partial y})^2,\ F_1 = \int_0^{H(x)} u_1(x,y)dy \quad (5.\ 3)$$

$$\frac{\partial^2 T_0}{\partial y^2} = p_r E_c (\frac{\partial u_0}{\partial y})^2 \quad (5.\ 4)$$

$$\frac{\partial^2 T_1}{\partial y^2} = p_r E_c [2(\frac{\partial u_0}{\partial y})(\frac{\partial u_1}{\partial y}) + \frac{n-1}{2}(\frac{\partial u_0}{\partial y})^4] \quad (5.\ 5)$$

With corresponding boundary conditions:

$$u_0 = u_1 = -1,\ v_0 = v_1 = -\frac{dH}{dx},\ T_0 = T_1 = 0 \qquad at\ y = H(x)$$

$$\frac{\partial u_0}{\partial y} = \frac{\partial u_1}{\partial y} = 0,\ v_0 = v_1 = 0,\ T_0 = 1,\ T_1 = 0\ \ at\ y = 0 \quad (5.\ 6)$$

The solutions of equations (5. 2 - 5. 5) subject to the boundary conditions (5. 6) give the axial velocity component u, the pressure gradient $\frac{\partial P}{\partial x}$ and the temperature distribution T as:

$$u = b_0 + b_1 \cosh\frac{y}{\sqrt{N}} + W^2[b_2 + b_3 \cosh\frac{y}{\sqrt{N}} + b_4 \cosh\frac{3y}{\sqrt{N}} + b_5 y \sinh\frac{y}{\sqrt{N}}], \quad (5.\ 7)$$

$$\frac{\partial P}{\partial x} = (F \cosh\frac{H}{\sqrt{N}} + \sqrt{N}\sinh\frac{H}{\sqrt{N}})/(N^{\frac{3}{2}}\sinh\frac{H}{\sqrt{N}} - NH\cosh\frac{H}{\sqrt{N}}) - W^2\frac{3(n-1)}{16N^3}\beta f(H,N), \quad (5.\ 8)$$

$$T = \frac{b_6}{8}\cosh\frac{2y}{\sqrt{N}} - \frac{b_6}{2N}y^2 + b_7 y + b_8 + W^2[b_9 \cosh\frac{4y}{\sqrt{N}} + b_{10}\cosh\frac{2y}{\sqrt{N}} + b_{11} y \sinh\frac{y}{\sqrt{N}} + b_{12}y^2 + b_{13}y + b_{14}], \quad (5.\ 9)$$

where N, β and $(b_0, \ldots.., b_{14})$ are defined in the appendix.

6. Results and discussion

The momentum and energy equations of magnetohydrodynamic flow and heat transfer in a peristaltic motion of Generalized Newtonian fluid through a porous medium are solved analytically by using Weicsenberg perturbation technique, in fact we have to choose the parameters for the carreau fluid such that the Weicsenberg number $W < 1$, δ is neglected and the Reynold number R_e is very small [16]. Our system of linear partial differential equations are solved and the effects of the parameters of the problem on these solutions are shown graphically.

Fig. (2) illustrates that the velocity distribution u increases with increasing the permeability fluid ϵ_0 , but at $y < 0.57$ the vise verse occurs. It is found that the velocity distribution u decreases when the magnetic number M increases, but after that at $y = 0.57$, u starts to increase with increasing M in fig. (3). The effect of the parameter $\phi = \frac{a}{b}$ on the velocity distribution u is shown in fig. (4), where u increases as ϕ increases. From fig. (5) we have seen that the pressure gradient $\frac{\partial P}{\partial x}$ decreases as the permeability fluid ϵ_0 increases. In fig. (6) we note that the pressure gradient $\frac{\partial P}{\partial x}$ increases with increasing the magnetic number M. It is clear from fig (7) that the pressure gradient $\frac{\partial P}{\partial x}$ decreases as the parameter $\phi = \frac{a}{b}$ increases and the inverse effect occurs at $\theta = 0.5$. It seems from figs. (9) and (10) that the temperature T increases with increasing the magnetic number M and the parameter $\phi = \frac{a}{b}$. Figs.(8), (11) and (12) clear that the temperature T decreases when the permeability fluid ϵ_0 , the Prandtl number P_r and the Eckert number E_c increase.

7. Conclusion and Applications

In this work, we study of magnetohydrodynamics flow and heat transfer of the two-dimensional peristaltic motion for incompressible non-Newtonian fluid through a porous medium analytically. The governing partial differential equation of this problem, subject to the boundary conditions are solved by using Weicsenberg perturbation technique. The analytical forms for the velocity distribution u, the pressure gradient $\frac{\partial P}{\partial x}$ and the temperature T are obtained. The effects of the various physical parameters of the problem are discussed and have been shown graphically. It is seen that the velocity distribution u decreases or increases as ϵ_0 and M, but the pressure gradient $\frac{\partial P}{\partial x}$ and the temperature T increase with increasing the magnetic number M, the temperature T decreases when the permeability fluid ϵ_0 , the Prandtl number P_r and the Eckert number E_c increase and the velocity distribution u and the temperature T decrease with increasing ϕ.

The study of this phenomena is very important, because the study of flow through porous medium have many applications. It has an important role in agricultural, engineering, science and petroleum industry. For example, ground water hydrology, extracting pure petrol from crude oil and chemical engineering. There are examples of natural porous media such as wood, filter paper, cotton, leather and plastics. As a good biological examples on the porous medium the human lung gall bladder and the walls of vessels. The peristaltic motion has been found to involved in many biological organs such as esophagus, small and large intestine, stomach, the human ureter, lymphatic vessels and small blood vessels. Also, peristaltic transport occurs in many practical applications involving biomechanical systems such as finger pumps [16].

8. Appendix

$$
\begin{aligned}
N &= \frac{1}{(\frac{1}{\epsilon_0}+M)},\\
b_0 &= -1-\frac{(F+H)\cosh\frac{H}{\sqrt{N}}}{(\sqrt{N}\sinh\frac{H}{\sqrt{N}}-H\cosh\frac{H}{\sqrt{N}})},\\
b_1 &= \frac{(F+H)}{(\sqrt{N}\sinh\frac{H}{\sqrt{N}}-H\cosh\frac{H}{\sqrt{N}})},\\
\beta &= \frac{(F^3+H^3+3H^2F+3HF^2)}{(\sqrt{N}\sinh\frac{H}{\sqrt{N}}-H\cosh\frac{H}{\sqrt{N}})^3},\\
f(H,N) &= (H+\frac{\sqrt{N}}{4}\cosh\frac{3H}{\sqrt{N}}\sinh\frac{H}{\sqrt{N}}-\frac{\sqrt{N}}{12}\cosh\frac{H}{\sqrt{N}}\sinh\frac{3H}{\sqrt{N}}\\
&\quad -\sqrt{N}\cosh\frac{H}{\sqrt{N}}\sinh\frac{H}{\sqrt{N}})/(\sqrt{N}\sinh\frac{H}{\sqrt{N}}-H\cosh\frac{H}{\sqrt{N}}),\\
b_2 &= \frac{3(n-1)}{16N}\beta f(H,N),\\
b_3 &= \frac{3(n-1)}{16N^{\frac{3}{2}}}\beta(\frac{\sqrt{N}}{4}\cosh\frac{3H}{\sqrt{N}}-H\sinh\frac{H}{\sqrt{N}})/\\
&\quad (\cosh\frac{H}{\sqrt{N}})-\frac{3(n-1)}{16N^{\frac{3}{2}}}\beta f(H,N),\\
b_4 &= \frac{-3(n-1)}{64N}\beta,\\
b_5 &= \frac{3(n-1)}{16N^{\frac{3}{2}}}\beta,\\
b_6 &= p_r E_c b_1^2,\\
b_7 &= -b_6(\frac{1}{8H}\cosh\frac{2H}{\sqrt{N}}-\frac{H}{4N}-\frac{1}{8H}+\frac{1}{Hb_6}),\\
b_8 &= b_6(\frac{1}{b_6}-\frac{1}{8}),\\
b_9 &= \frac{b_6}{16b_1}(\frac{(n-1)}{16N^3}b^3+3b_4),
\end{aligned}
$$

$$
\begin{aligned}
b_{10} &= \frac{b_6}{8b_1}(2b_3-6b_4-3\sqrt{N}b_5-\frac{(n-1)}{64N^{\frac{5}{2}}}b^3),\\
b_{11} &= \frac{b_6 b_5}{2b_1},\\
b_{12} &= \frac{b_6}{8Nb_1}(\frac{3(n-1)}{4N^3}b^3-4b_3-12\sqrt{N}b_5),\\
b_{13} &= \frac{1}{H}(b_9(1-\cosh\frac{4H}{\sqrt{N}})+b_{10}(1-\cosh\frac{2H}{\sqrt{N}})-b_{11}H\sinh\frac{H}{\sqrt{N}}\\
&\quad -b_{12}H^2),\\
b_{14} &= -b_9-b_{10}.
\end{aligned}
$$

REFERENCES

[1] G. Bohme, R. Friedrich, " Peristaltic Flow of Viscoelastic Liquids" J. Fluid Mech., 128, (1983),109 – 122.
[2] Rath, H. J. Peristaltische Stromungen. Springer. (1980).
[3] T. W. Latham: M. S. Thesis, MIT, Cambridege, MA, (1966).
[4] J. C. Burns and T. Parkes: J. Fluid Mech., 29, (1967), 731.
[5] C. Barton and S. Raynor: Bull. Math. Biophys., 30, (1968).
[6] A. H. Shapiro, M. Y. Jaffrin and S. L. Weinberg, " Peristaltic Pumping with Long Wavelengths at Low Reynolds Number" J. fluid Mech., 37, (1969),799.
[7] P. S. Lykoudis and R. Roos: J. fluid Mech., 43, (1970), 661.
[8] R. Roos: J. and P. S. Lykoudis: J. fluid Mech., 46, (1971),625.
[9] J. B. Shukla, R. S. Parihar, R. P. Rao and S. P. Gupta: J. fluid Mech., 97, (1980), 255.
[10] E. F. El-Shehawey and k. S. Mekheimer, "Coupple-Stresses in Peristaltic Transport of Fluids" J. Phys. D: Apply. Phys., 27, (1994), 1163.
[11] A. M. Elmisery, E. F. El-Shehawey and A. A. Hakeem, " Peristaltic Motion of an Incompressible Generalized Newtonian Fluid in Planar Channel "J. Phys. Soc. Jpn., 65, (1996), 3524.
[12] E. F. Elshehawey, A. M. Elmisery and A. A. Hakeem, "Peristaltic Motion of a Generalized Newtonian Fluid in a Non-Uniform Channel "J. Phys. Jap., 67, (1998),431.
[13] E. F. Elshehawey and k. S. Mekheimer, S. Kaldas and N. Afifi: J. Biomathematics,13, (1999), 417.
[14] N. T. Eldabe: J. Phys. Soc. Jpn.,56, (1987), 1713.
[15] N. T. Eldabe and Sallam N. Sallam," Magenetohydrodynamic Flow and Heat Transfer in a Viscoelastic Incompressible Fluid Confined Between a Horizontal Stretching Sheet and a Parallel Porous Wall " Can. J. Phys., 76, (1998), 98.
[16] E. F. Elshehawey, Ayman M. F. Sobh and Elsayed M. E. Elbarbary, " Peristaltic Motion of a Generalized Newtonian Fluid Through a Porous Medium " J. Phys. Soc. Jpn., 69, (2000), 401 – 407.
[17] E. F. Elshehawey, Ayman M.F. Sobh and N. A. S. Afify, "Peristaltic Motion of a Generalized Newtonian Fluid under The Effect of Transferse Magnetic Field " Bulg. Acad. Sci., 53, (2000), 33.
[18] E. F. Elshehawey and Ayman M.F. Sobh:Int, "Peristaltic Viscoelastic Fluid Motion in a Tube"J. Math. and Math. Scim., 26, (2001),21.

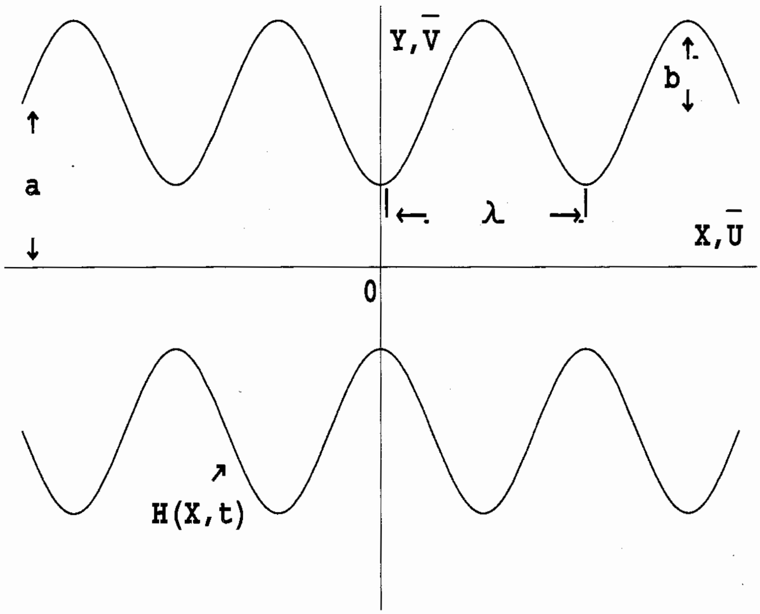


FIGURE 1

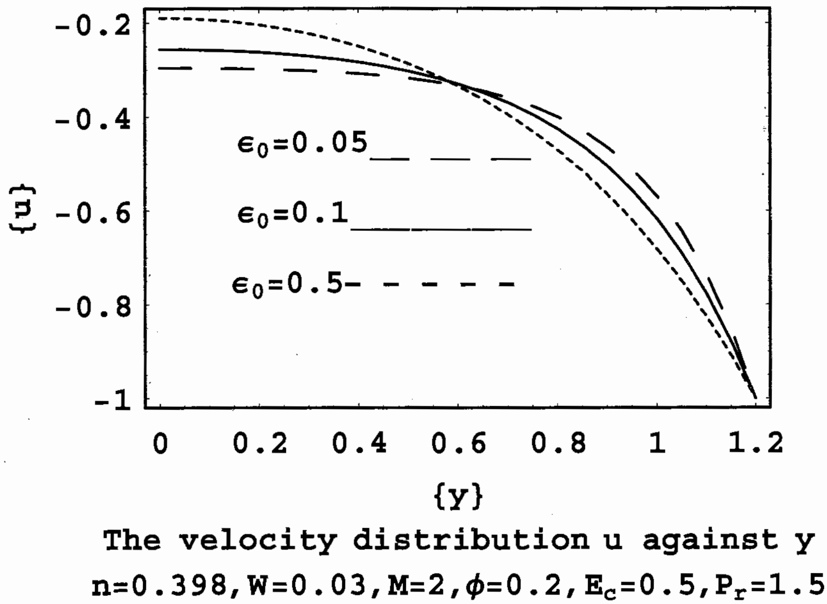
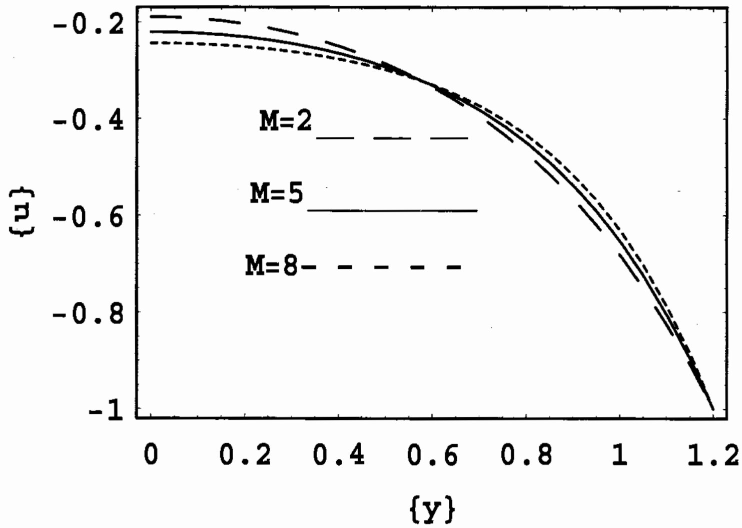


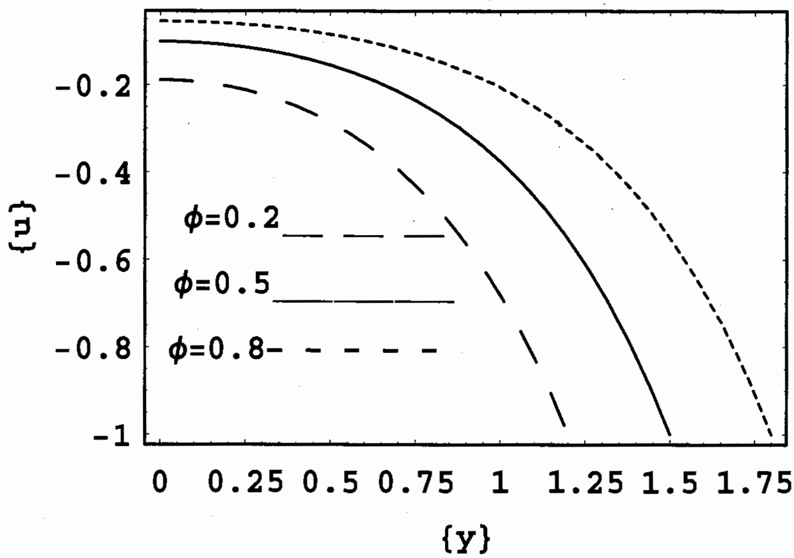
FIGURE 2



The velocity distribution u against y

$n=0.398, W=0.03, \epsilon_0=0.5, \phi=0.2, E_c=0.5, P_r=1.5$

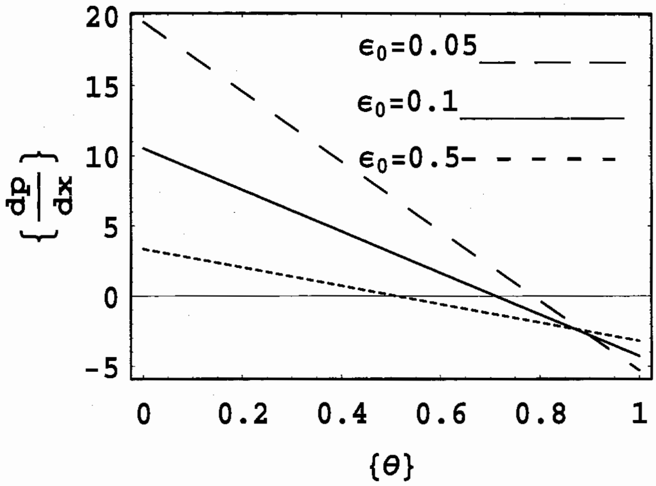
FIGURE 3



The velocity distribution u against y

$n=0.398, W=0.03, M=2, \epsilon_0=0.5, E_c=0.5, P_r=1.5$

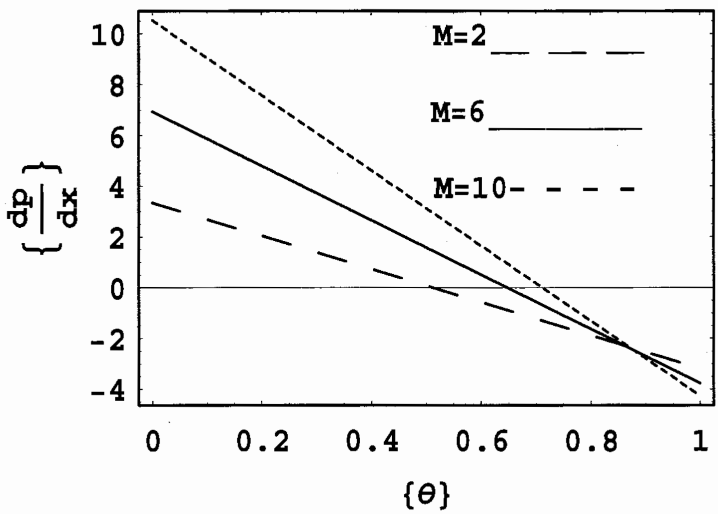
FIGURE 4



The pressure gradient $\frac{dp}{dx}$ against θ

$n=0.398, W=0.03, M=2, \phi=0.1, E_c=0.5, P_r=1.5$

FIGURE 5



The pressure gradient $\frac{dp}{dx}$ against θ

$n=0.398, W=0.03, \epsilon_0=0.5, \phi=0.1, E_c=0.5, P_r=1.5$

FIGURE 6

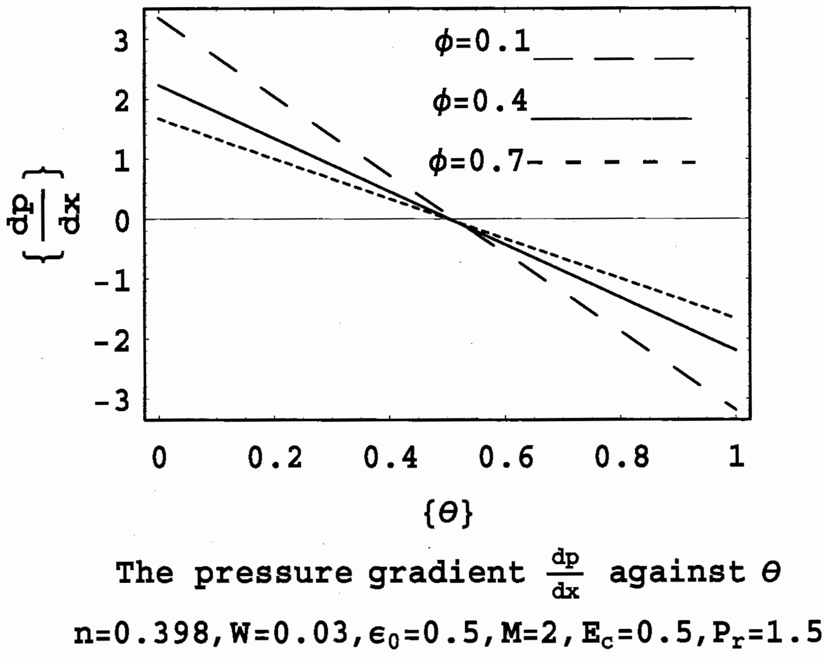


FIGURE 7

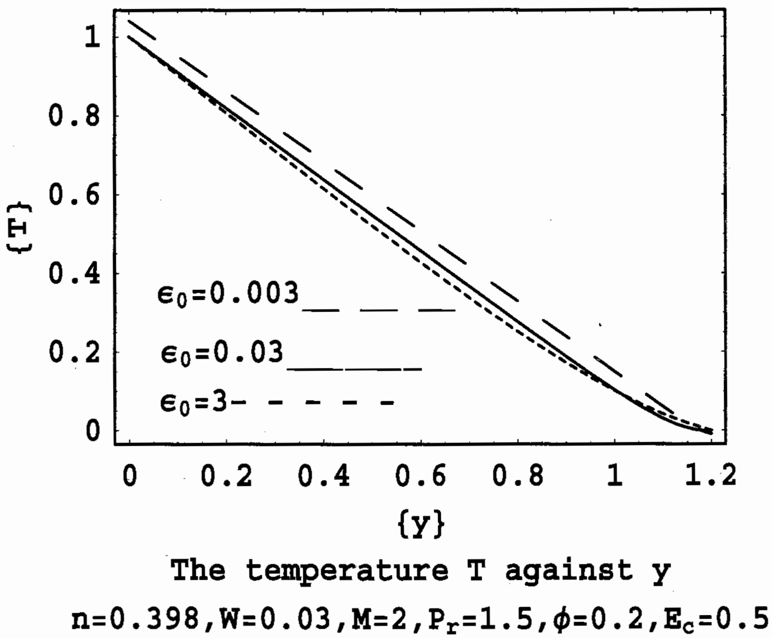
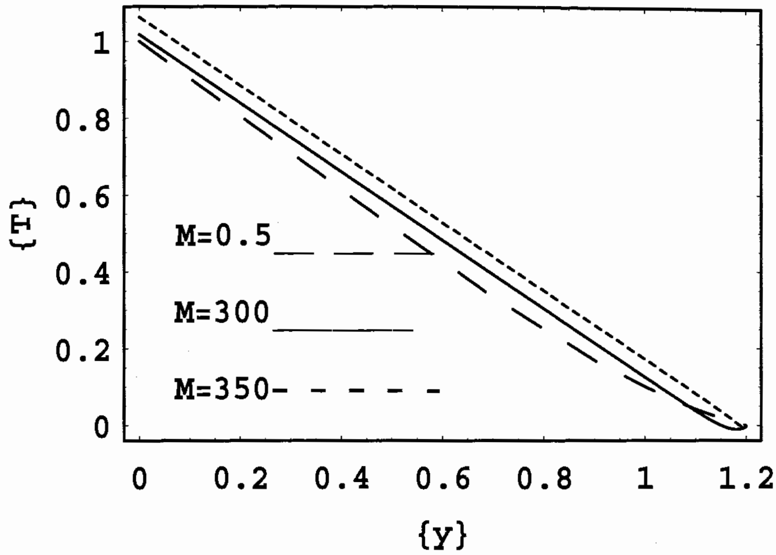


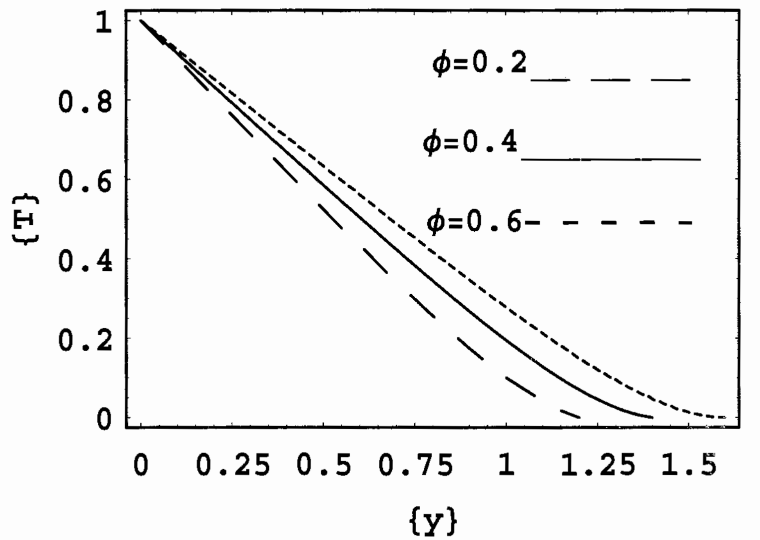
FIGURE 8



The temperature T against y

$n=0.398, W=0.03, E_c=.5, P_r=1.5, \phi=0.2, \epsilon_0=0.5$

FIGURE 9



The temperature T against y

$n=0.398, W=0.03, M=2, E_c=0.5, P_r=1.5, \epsilon_0=0.5$

FIGURE 10

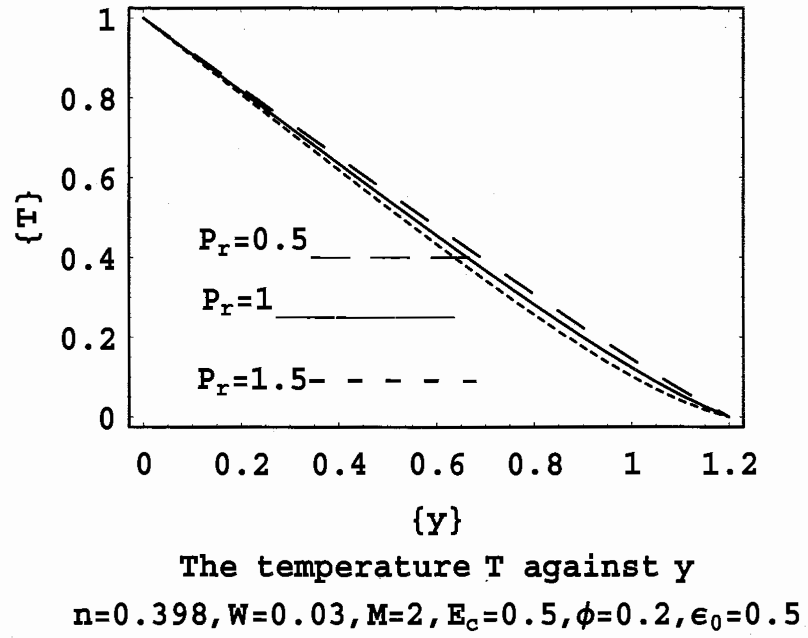


FIGURE 11

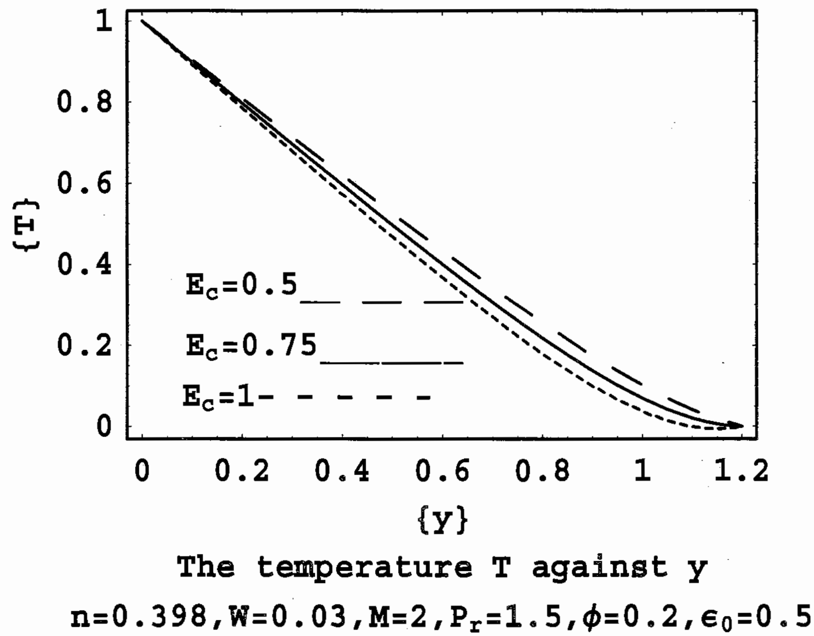


FIGURE 12

Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 42 (2010) pp. 17-24

Study of Fractal Properties of Climatic Time Series from Pakistan

M. Jawed Iqbal
Department of Mathematics
University of Karachi
Karachi, Pakistan
Email: jiqbalku@yahoo.com

Abstract. We study fractal properties of climatic time series from Pakistan in this paper The correlation dimension is calculated for investigating the fractal structure of summer rainfall series of Lahore, Peshawar and Karachi. Our computations show that time series of summer rainfall for each station exhibit a fractional brownian motion. In particular, it seems that such an evolution of the climate possesses a random fractal structure.

AMS (MOS) Subject Classification Codes: 28A80, 37F45, 26A18
Key Words: Fractal structures,correlation dimension, Average Mutual Information.

1. Introduction

Several studies employ fractal dimensional analysis for examining the predictability of geophysical time series (Mandelbrot, 1977, 1969; Rangarajan and Sant, 2004; Rangarajan, 1997). The so-called fractal structures are often studied under the name of fractal geometry (Mandelbrot, 1968, 1982). This entirely new non-euclidean geometry provides a way of describing a surprisingly wide range of complex structures and phenomena in quantitative terms and provides an avenue towards an understanding of their properties. Generally, fractals are structures which have the same degree of complexity on all length scales. Recent mathematical studies of dynamical systems, represented by the available data models, tend to show that the solution of simple nonlinear equations can exhibit complex temporal and spatial behaviour. One can now find a lot of examples of equations showing a variety of behaviour ranging from simple periodic to chaotic. In particular, during the last few decades there have emerged several attempts to use the paradigm of 'chaos' for a description and forecasting of climatic processes (Carl et al., 1995; Krupchatnikov, 1995; Sonechkin and Ivashchenko, 1996; Vasechkina et al., 1996; Wang and Fang, 1996; Shrirer et al., 1997; Sonechkin et al., 1997). Fractals are to chaos what geometry is to algebra. They are the usual geometric manifestation of the chaotic dynamics. They have been called the fingerprints of chaos (Sprott, 2003, 10. Iqbal and Quamar, 2007), but they are interesting and important in their own right, independent of their relation to chaos. Therefore, this paper employs dimension analysis for investigating the fractal structure of the time series of summer rainfall (for the months June to September) for Lahore, Peshawar, Hyderabad and Karachi from 1882 to 2003. Section 2 determines time delay for the time series of summer precipitation of different stations. Section 3 provides the detail of calculation for the correlation dimension of fractal structure of summer rainfall of different cities of Pakistan. Section 4 concludes the paper.

2. Determination of Delay Time

There are two possible ways to estimate the delay time from an observed time series. The first method is based on the calculation of the autocorrelation function of the data and selecting as τ the time of its first zero crossing. The reasoning behind this choice is that the time when the autocorrelation function reaches a zero value marks the point beyond where the $X\ (t+\tau)$ sample is not completely correlated from $X\ (t)$, i.e. τ is the earliest time at which the autocorrelation drops to a fraction of its initial value. The second approach involves the calculation from the data of a nonlinear autocorrelation function called Average Mutual Information (AMI) defined as (Fraser and Swinney, 1986). They argue that a better value for τ is the value that corresponds to the first local minimum of the mutual information where the mutual information is a measure of how much information can be predicted about one time series point given full information about the other. We continue by describing the algorithm for computing the mutual information, according to Frasier and Swinney.

Mutual information is a measure found in the field of information theory. Let P (X (1)), P (X (2))...P (X (n)) are the associated probabilities of X (1), X (2)...X (n)), the entropy H of the system is defined as that it is the average amount of information gained from measuring a series $X\ (i)$. It is given mathematically as:

$$H(X(i)) = \sum_i P(X(i)) \log_2 P(X(i)) \tag{2. 1}$$

For a logarithmic base of two, H is measured in bits. Mutual information measures the dependency of two series $X\ (i+\tau)$ on $X\ (i)$. Let

$$[s, q] = [X(i), X(i+\tau] \tag{2. 2}$$

and consider a coupled system (S, Q). Then for a signal s and corresponding measurement si , Eq. (2.1) becomes

$$H(Q\,|s_i) = -\sum_j P(q_j\,|s_i)\ \log_2[P(q_j\,|s_i)] \tag{2. 3}$$

$$H(Q|s_i) = \sum_j \frac{P(s_i, q_j}{P(s_i} log_2 \frac{P(s_i, q_j)}{P(s_i}(4) \tag{2. 4}$$

Where $P\ (qj \mid$ si) is the probability that a measurement of q will result in qj, subject to the condition that the measured value of s is si and $P\ (si,\ qj)$ is the joint probability density for measurements $P\ (si)$ and $P\ (qj)$. Next we take the average uncertainty of $H\ (Q \mid si)$ over si,

$$E[H(Q|S] = \sum_i P(s_i) H(Q|s_i)(5) \tag{2. 5}$$

$$E[H(Q|S] = \sum P(s_i, q_j) log_2 \frac{P(s_i, q_j)}{P(s_i})(6) \tag{2. 6}$$

$$E[H(Q|S] = H(S, Q) - H(S)(7) \tag{2. 7}$$

with

$$H(S,Q) = \sum P(s_i, q_i) log_2 P(s_i, q_i)(8) \tag{2. 8}$$

The reduction of the uncertainty of q by measuring s is called the mutual information I (S, Q), which can be expressed as

$$I(S,Q) = H(Q) - H(S|Q) \tag{2. 9}$$

$$I(S,Q) = H(Q) + H(S) - H(S,Q) \tag{2. 10}$$

where H (Q) is the uncertainty of q in isolation. If both S and Q are discrete time series, then

$$I(S,Q) = \sum P(s,q) log_2 \frac{P(s,q)}{P(s)P(q)} \tag{2. 11}$$

Replacing [s,q] by from Eq.(2.4), we get

$$I(\tau) = \sum P(X(i), X(i+\tau)) log_2 \frac{P(X(i), X(i+\tau)}{P(X(i))P(X(i+\tau)} \tag{2. 12}$$

The appropriate time delay τ is defined as the first minimum of the average mutual information I (τ). Then the values of X (i) and X ($i+\tau$) are independent enough of each other to be useful as coordinates in a time delay vector but not so independent as to have no connection with each other at all (for detail see, Khan, 2007).

Now we apply AMI technique to our climate data for the calculation of time delay. We plot AMI of the summer rainfall of Lahore after removing trend in Fig. 1 As the first minima occurs at $\tau = 2$, we infer that delay time for our climate data is 2 years. Similarly, Figs. 2-4 show that summer precipitation data of Karachi, Hyderabad and Peshawar have time delay 4, 1 and 4 respectively.

3. Fractal Structure of Summer Rainfall

We use correlation dimension for discriminating between chaotic and random behavior. We construct a function C(ε) that is the probability that two arbitrary points on the orbit are closer together than (ε). The correlation dimension is given by log(C) / log((ε)) in the limit ε tends to 0 and N tends to ∞. The correlation dimension is defined as the slope of the curve C(ε) versus ε. C(ε) is the correlation of the data set, or the probability that any two points in the set are separated by a distance ε. A noninteger result for the correlation dimension indicates that the data is probably fractal. C(ε) is calculated for every embedding dimension specified in the range and plotted against that range. For the truly random signals, the correlation dimension graph will look like a 45-degree straight line, indicating that no matter how you embed the noise, it will evenly fill that space. Chaotic (and periodic) signals, on the other hand, have a distinct spatial structure, and their correlation dimension will saturate as some point, as embedding dimension is increased.

We now calculate correlation dimension with different embedding dimension for summer rainfall of Lahore (see Fig. 5). Our calculation shows that the time series of Lahore's summer rainfall has non-integer values of correlation dimension whish shows that Lahore's summer rainfall is probably fractal. As correlation dimension is greater than 1.5, data of summer rainfall over Lahore exhibits Brownian motion.

Similarly, we summarize our calculations in Figures 6-8 for summer rainfall data of Karachi, Hyderabad and Peshawar. In fact, our calculations suggest that time series of these stations also exhibits Brownian motion.

4. Conclusions

To better understand the climate evolution, this paper examines the question of character of physical processes represented by the climate data. Well, how about a possible fractal structure of evolution of processes governing the major climate stations we are investigating?. Our computations show that the time series of summer rainfall for each station exhibit a fractional brownian motion. In particular, it seems that such an evolution of the climate possesses a random fractal structure.

References

[1] P. Carl, K. D. Worbs and I. Tschentscher, *On a dynamic systems approach to atmospheric model intercomparison, WCRP / WMO, No. 732 (1995),445-450.*

[2] A. M. Fraser and Swinney, *Independent coordinates for strange attractors from mutual information', Phys. Re, A33 (1986),1134.*

[3] M. J. Iqbal and J. Quama, *Nature of Processes Responsible for Rainfall Variability, M. Phil. Thesis, University of Karachi, Vol. 14, No. 2 2007.*

[4] M. S. Khan, *Investigation of Chaos Existence in the Time Series of Lahore Precipitation, Asia Pacific Journal on Environment and Development, 2007.*

[5] V. Krupchatnikov, *Global attractors for the Lorenz model on the sphere,Bull. Novosibirsk Comput. Cent. Ser. Numer. Model. Atmos., Ocean Environ. Stud., 2 (1995),31-40.*

[6] B. B. Mandelbrot, *Fractal: FormChance and Dimension*, Freeman, San Francisco, (1977).

[7] B. B. Mandelbrot, *The Fractal Geometry of Nature*, Freeman, San Francisco, (1982).

[8] B. B. Mandelbrot, *Fractional Brownian motions, fractional noises and applications*, SIAM Rev., **A33** (1968), 1134.

[9] B. B. Mandelbrot and J. R. Wallis, *Some long-run properties of geophysical records'*, Water Resources Res., , **5** (1969), 321-40.

[10] G. Rangarajan , *A climate predictability index and its applications*, Geophysical Research Letters, **24** (1997), 1239.

[11] G. Rangarajan and DA Sant, *Fractal dimensional analysis of Indian climatic dynamics, Chaos*, Solitons and Fractals, **19** (2004), 285.

[12] H. N. Shrirer, S. J. Fosmire, R. Wells and L. Suciu L, *Chaos and Time Series Analysis'*, Oxford University Press, (2003), pp. 226-227, 291. .

[13] J. C. Sprott, *Chaos and Time Series Analysis*, , Oxford University Press, **A33** (1993), 1134.

[14] D. M. Sonechkin and N. N. Ivashchenko, *Contemporary approach to the problem of existence of rithms in the atmosphere dynamics*, Trudy Gidrometeorol Nauchno-Issled Tsentr Rossiiskoi Federatsii, **329** (1996), 72-79 .

[15] D. M. Sonechkin, N. M. Datsenko and N. N. Ivashchenko, *Estimation of the global heating trend using the wavelet analysis*, FAO (Physics of the atmosphere and the oceans, **97** (1997), 184-194.

[16] E. F. Vasechkina,T. E. Timchenko and V¿ D. Yarin, *Forecasting of hydrological processes using the methods of self-organizing mathematical models*, Morskoi Gidrofizicheskii Zhurnal, **3** (1996), 48-57 .

[17] B. Wang and Z. Fang, *Chaotic oscillations of tropical climate: a dynamic system theory for ENSO* , J. Atmos. Sci, **53** (1996), 2786-2802.

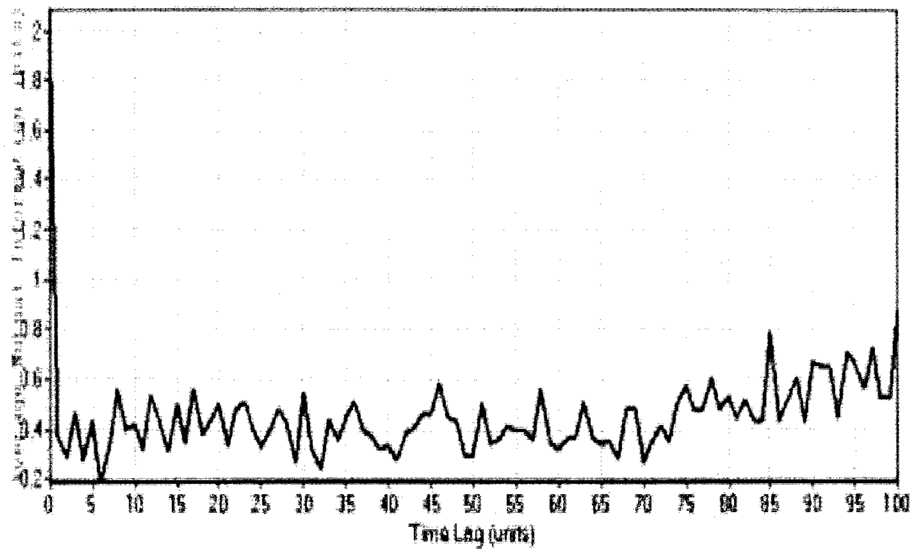


FIGURE 1. Average mutual information graph of summer rainfall of Lahore. Thee graph shows that the £rst AMI minimum was found at time lag 2 i.e. time delay, τ = 2 years.

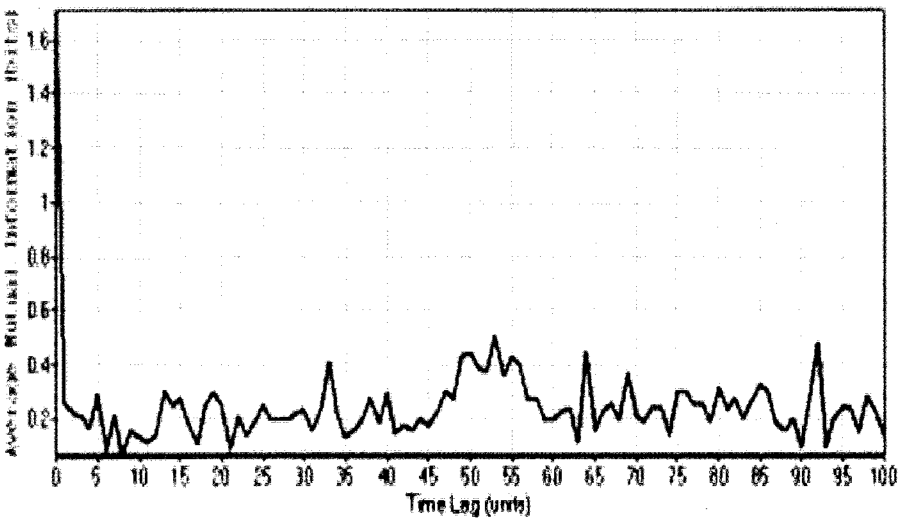


FIGURE 2. Average mutual information graph of summer rainfall of Karachi. Thee graph shows that the £rst AMI minimum was found at time lag 4 i.e. time delay, τ = 4 years.

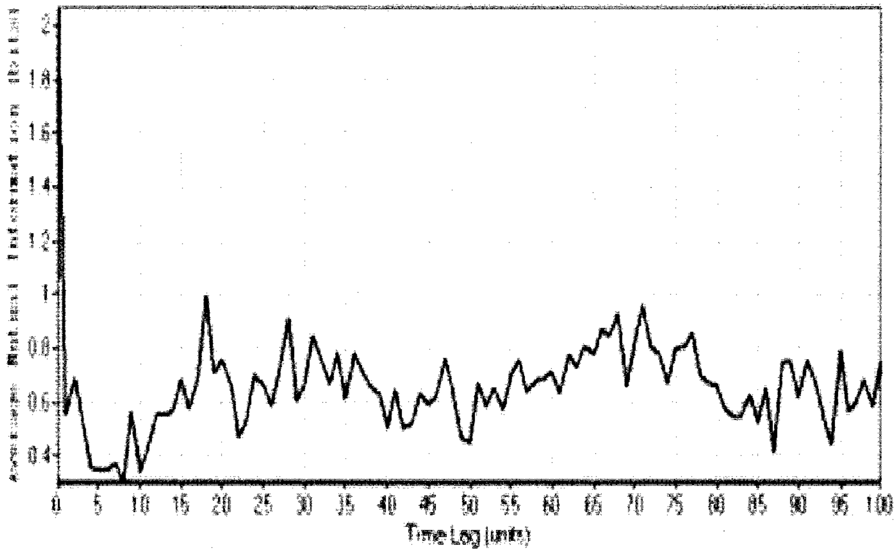


FIGURE 3. Average mutual information graph of summer rainfall of Hyderabad. Thee graph shows that the £rst AMI minimum was found at time lag 1 i.e. time delay, $\tau = 1$ year.

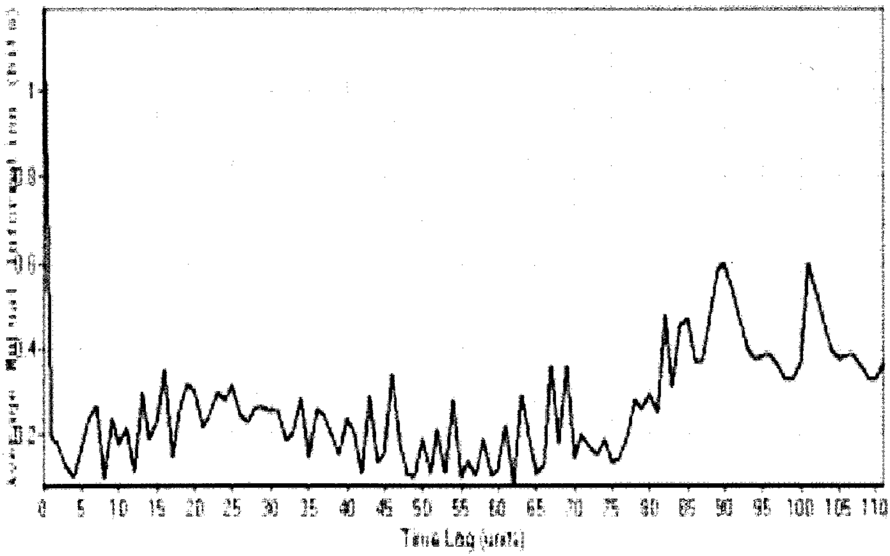


FIGURE 4. Average mutual information graph of summer rainfall of Peshawar. Thee graph shows that the £rst AMI minimum was found at time lag 4 i.e. time delay, $\tau = 4$ years.

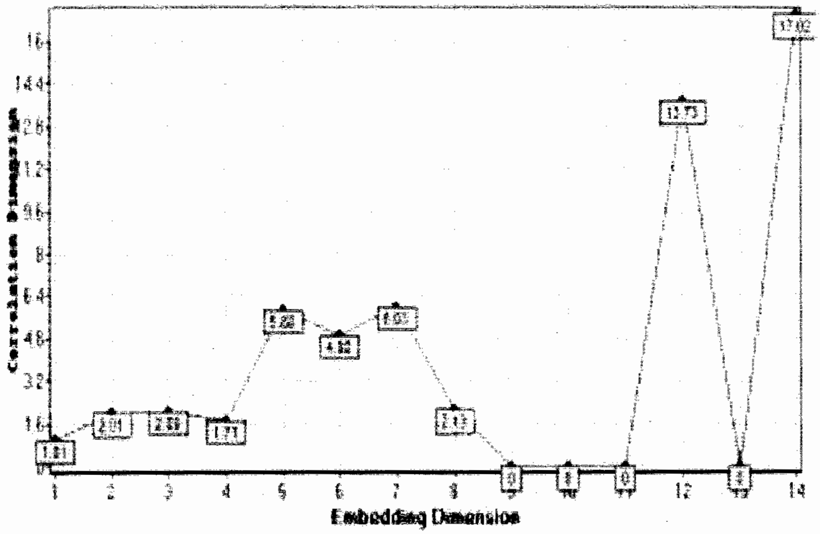


FIGURE 5. Plot of correlation dimension against embedding dimension for summer rainfall of Lahore. As correlation dimension is greater than 1.5, the time series of summer rainfall over Lahore exhibits Brownian motion.

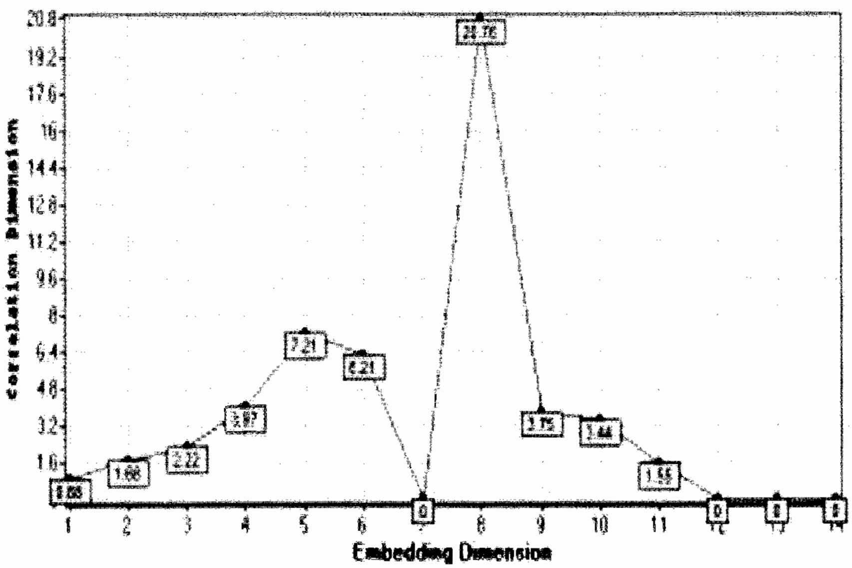


FIGURE 6. Plot of correlation dimension against embedding dimension for summer rainfall of Karachi. As correlation dimension is greater than 1.5, the summer rainfall over Karachi exhibits Brownian motion.

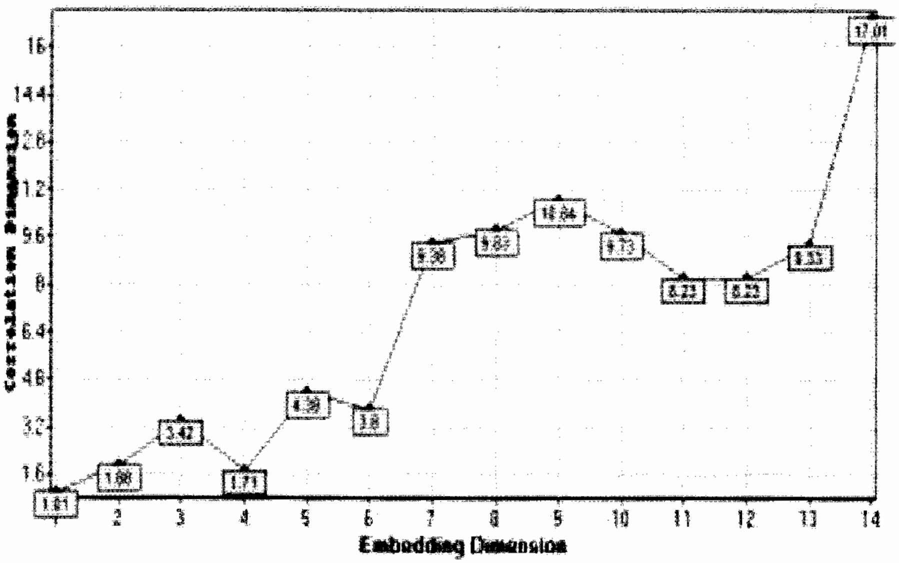


FIGURE 7. Plot of correlation dimension against embedding dimension for summer rainfall of Hyderabad As correlation dimension is greater than 1.5, summer rainfall over Hyderabad exhibits Brownian motion.

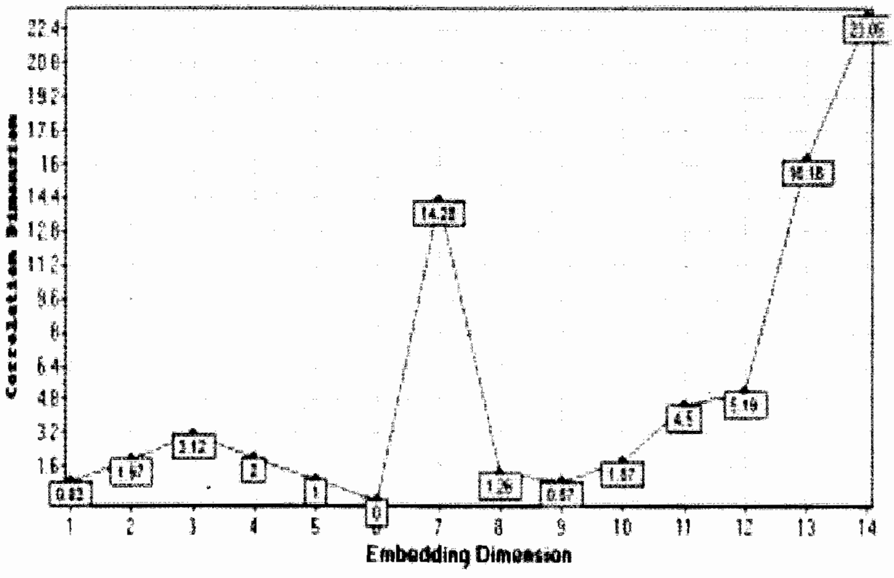


FIGURE 8. Plot of correlation dimension against embedding dimension for summer rainfall of Peshawar As correlation dimension is greater than 1.5, summer rainfall over Peshawar also exhibits Brownian motion.

Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 42 (2010) pp. 25-40

On Certain Subclasses of Analytic Functions of Complex Order Defined by Generalized Hypergeometric Functions

Halit Orhan
Department of Mathematics
Faculty of Science
Ataturk University
25240, Erzurum Turkey
Email: horhan@atauni.edu.tr

Dorina Răducanu
Faculty of Mathematics and Computer Science
Transilvania University of Braşov
50091, Iuliu Maniu, 50,Braşov, Romania
Email: dorinaraducanu@yahoo.com

Abstract. By making use of the generalized hypergeometric functions, in this paper we introduce and investigate certain new subclasses of analytic functions of complex order defined in the open unit disk. Coefficient inequalities, radii of close-to-convexity, starlikeness and convexity, closure theorems, integral means inequalities and several relations associated with (n, δ)-neighborhood for these classes are obtained.

AMS (MOS) Subject Classification Codes: 30C45
Key Words: Analytic functions, Generalized hypergeometric functions, Fekete-Szegö functional, Integral means, (n, δ)-neighborhood.

1. Introduction

Let $\mathcal{A}(n)$ denote the class of functions f of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathbb{N} := \{1, 2, \ldots\}) \tag{1. 1}$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$.

Denote by $\mathcal{T}(n)$ the subclass of $\mathcal{A}(n)$ consisting of functions f of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geq 0\, ; n \in \mathbb{N}). \tag{1. 2}$$

Let $f \in \mathcal{A}(n)$ given by (1. 1) and let $g \in \mathcal{A}(n)$ given by

$$g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k.$$

The Hadamard product(or convolution) of f and g is defined by

$$(f*g)(z) = z + \sum_{k=n+1}^{\infty} a_k b_k z^k = (g*f)(z) \ (z \in \mathbb{U}).$$

For complex parameters $\alpha_i, \beta_j \in \mathbb{C} - \{0, -1, -2, \ldots\}$ $(i = 1, 2, \ldots, l \,; j = 1, 2, \ldots, m)$ the generalized hypergeometric function ${}_lF_m(z)$ is defined by

$${}_lF_m(z) \equiv_l F_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_l)_k}{(\beta_1)_k \ldots (\beta_m)_k} \frac{z^k}{k!}$$

$$(l \leq m+1 \ , \ l, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \ , z \in \mathbb{U})$$

where $(\lambda)_k$ is the Pochhammer symbol defined by

$$(\lambda)_k := \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \begin{cases} 1, & k = 0 \\ \lambda(\lambda+1)\ldots(\lambda+k-1), & k \in \mathbb{N}. \end{cases}$$

Let $H_{l,m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) : \mathcal{A}(n) \to \mathcal{A}(n)$ be the linear operator defined by

$$H_{l,m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z) := z_l F_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z)$$

$$= z + \sum_{k=n+1}^{\infty} \Gamma_k a_k z^k \tag{1. 3}$$

where

$$\Gamma_k = \frac{(\alpha_1)_{k-1} \ldots (\alpha_l)_{k-1}}{(\beta_1)_{k-1} \ldots (\beta_m)_{k-1}} \frac{1}{(k-1)!} \quad (k \geq n+1). \tag{1. 4}$$

If $f \in \mathcal{T}(n)$ is given by (1. 2) then

$$H_{l,m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z) := z - \sum_{k=n+1}^{\infty} |\Gamma_k| a_k z^k. \tag{1. 5}$$

For simplicity, in the sequel, we shall write $H_{l,m}(\alpha_1, \beta_1) f(z)$ instead of $H_{l,m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z)$.

The linear operator $H_{l,m}(\alpha_1, \beta_1) f(z)$ is called the Dziok-Srivastava operator [5] and it contains, amongst its special cases, various other operators introduced and studied by Hohlov [7], Carlson-Shaffer [3], Bernardi [2], Libera [10], Livingston [12], Ruscheweyh [18], Srivastava-Owa ([15], [22]).

We say that a function $f \in \mathcal{A}(n)$ is in the class $S_{l,m}(\lambda, \beta, \gamma)$ if

$$\left| \frac{\dfrac{z\nu'(z)}{\nu(z)} - 1}{2\beta\gamma + \dfrac{z\nu'(z)}{\nu(z)} - 1} \right| < 1 \tag{1. 6}$$

$$(z \in \mathbb{U} \ , 0 < \beta \leq 1 \ , \gamma \in \mathbb{C} - \{0\})$$

where

$$\frac{z\nu'(z)}{\nu(z)} = \frac{z(H_{l,m}(\alpha_1, \beta_1) f(z))' + \lambda z^2 (H_{l,m}(\alpha_1, \beta_1) f(z))''}{(1-\lambda) H_{l,m}(\alpha_1, \beta_1) f(z) + \lambda z (H_{l,m}(\alpha_1, \beta_1) f(z))'} \tag{1. 7}$$

$$(z \in \mathbb{U} \ , 0 \leq \lambda \leq 1 \ , \alpha_i, \beta_j \in \mathbb{C} - \{0, -1, -2, \ldots\} \ , (i = 1, 2, \ldots, l; j = 1, 2, \ldots, m)).$$

A function f in the class $\mathcal{T}(n)$ is said to be in the class $TS_{l,m}(\lambda, \beta, \gamma)$ if it satisfies the following inequality:

$$\left| \frac{1}{\gamma} \left(\frac{z\nu'(z)}{\nu(z)} - 1 \right) \right| < \beta \tag{1. 8}$$

$$(z \in \mathbb{U}, \gamma \in \mathbb{C} - \{0\}, 0 \leq \lambda \leq 1, 0 < \beta \leq 1).$$

Finally, we say that a function $f \in \mathcal{T}(n)$ is in the class $TR_{l,m}(\lambda, \beta, \gamma)$ if

$$\left|\frac{1}{\gamma}\left(\nu'(z) - 1\right)\right| < \beta \tag{1. 9}$$

$$(z \in \mathbb{U}, \gamma \in \mathbb{C} - \{0\}, 0 \leq \lambda \leq 1, 0 < \beta \leq 1).$$

We note that there are some known subclasses of our classes of functions $S_{l,m}(\lambda, \beta, \gamma)$, $TS_{l,m}(\lambda, \beta, \gamma)$ and $TR_{l,m}(\lambda, \beta, \gamma)$.

Example1. If $l = 2$ and $m = 1$ with $\alpha_1 = 1 = \alpha_2$, $\beta_1 = 1$ then

$$TS_{2,1}(\lambda, \beta, \gamma) \equiv S_n(\gamma, \lambda, \beta).$$

Example2. If $l = 2$ and $m = 1$ with $\alpha_1 = 1 = \alpha_2$, $\beta_1 = 1$ then

$$TR_{2,1}(\lambda, \beta, \gamma) \equiv R_n(\gamma, \lambda, \beta).$$

The classes $S_n(\gamma, \lambda, \beta)$ and $R_n(\gamma, \lambda, \beta)$ were investigated in [1].

Example3. If $l = 2$ and $m = 1$ with $\alpha_1 = 1 = \alpha_2$, $\beta_1 = 1$, $\lambda = 0$, $\beta = |b|$, $\gamma = 1$ then

$$TS_{2,1}(0, |b|, 1) \equiv S_1^*(b),$$

where $b \in \mathbb{C} - \{0\}$.The class $S_1^*(b)$ was studied in [9].

In the present paper we obtain a sufficient condition, in terms of coefficient bounds, for a function to be in the class $S_{l,m}(\lambda, \beta, \gamma)$. We also determine an upper bound for the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for the class $S_{l,m}(\lambda, \beta, \gamma)$.

Furthermore, coefficient inequalities, radii of starlikeness and convexity, closure theorems, integral means inequalities and several inclusion relations associated with (n, δ)-neighborhoods for the classes $TS_{l,m}(\lambda, \beta, \gamma)$ and $TR_{l,m}(\lambda, \beta, \gamma)$ are obtained.

2. Coefficient estimates for the class $S_{l,m}(\lambda, \beta, \gamma)$

In this section we obtain a sufficient condition for a function $f \in \mathcal{A}(n)$ to be in the class $S_{l,m}(\lambda, \beta, \gamma)$ and we also determine an upper bound for the functional $|a_3 - \mu a_2^2|$.

Theorem 1. *Let $f \in \mathcal{A}(n)$ given by (1. 1). If*

$$\sum_{k=n+1}^{\infty} [1 + \lambda(k-1)](k + \beta|\gamma| - 1)|\Gamma_k||a_k| \leq \beta|\gamma| \quad (z \in \mathbb{U}) \tag{2. 1}$$

where Γ_k is given by (1. 4), then the function f is in the class $S_{l,m}(\lambda, \beta, \gamma)$.

Proof. Suppose that the inequality (2. 1) holds. We have for $z \in \mathbb{U}$,

$$|z\nu'(z) - \nu(z)| - |\nu(z)(2\beta\gamma - 1) + z\nu'(z)| = \left|\sum_{k=n+1}^{\infty} [1 + \lambda(k-1)](k-1)\Gamma_k a_k z^k\right| -$$

$$\left|2\beta\gamma z + \sum_{k=n+1}^{\infty} [1 + \lambda(k-1)](k + 2\beta\gamma - 1)\Gamma_k a_k z^k\right|$$

$\leq \sum_{k=n+1}^{\infty} [1 + \lambda(k-1)](k-1)|\Gamma_k||a_k||z|^k - 2\beta|\gamma||z| + \sum_{k=n+1}^{\infty} [1 + \lambda(k-1)](k +$

$2\beta|\gamma| - 1)|\Gamma_k||a_k||z|^k$

$\leq 2\left(\sum_{k=n+1}^{\infty} [1 + \lambda(k-1)](k + \beta|\gamma| - 1)|\Gamma_k||a_k| - \beta|\gamma|\right) \leq 0.$

which shows that f belongs to $S_{l,m}(\lambda, \beta, \gamma)$. Thus, the proof of the theorem is completed. □

In order to prove our next theorem we need the following lemma.

Lemma 2 ([8]). *Let $w(z)$ of the form*

$$w(z) = \sum_{k=1}^{\infty} c_k z^k \tag{2. 2}$$

be an analytic function in $\mathbb{U}$ such that $|w(z)| < 1$ $(z \in \mathbb{U})$. If μ is a complex number then

$$|c_2 - \mu c_1^2| \leq \max\{1, |\mu|\}. \tag{2. 3}$$

Theorem 3. *Let μ be a complex number. If the function $f \in \mathcal{A}(2)$ given by (1. 1) is in the class $S_{l,m}(\lambda, \beta, \gamma)$ then*

$$|a_3 - \mu a_2^2| \leq \frac{\beta|\gamma| \prod_{j=1}^{m} |\beta_j||\beta_j + 1|}{(2\lambda+1) \prod_{i=1}^{l} |\alpha_i||\alpha_i + 1|} \max\{1, |d|\}$$

where

$$d := \frac{4\beta\gamma\mu(2\lambda+1) \prod_{i=1}^{l}(\alpha_i + 1) \prod_{j=1}^{m} \beta_j}{(\lambda+1)^2 \prod_{i=1}^{l} \alpha_i \prod_{j=1}^{m} (\beta_j + 1)} - (2\beta\gamma + 1).$$

Proof. Let

$$w(z) := \frac{z\nu'(z) - \nu(z)}{(2\beta\gamma - 1)\nu(z) + z\nu'(z)}.$$

It follows that w is an analytic function in $\mathbb{U}$, $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$. Consider $w(z) = \sum_{k=1}^{\infty} c_k z^k$. Then

$$(c_1 z + c_2 z^2 + \ldots) = \frac{\frac{1}{2\beta\gamma} \sum_{k=2}^{\infty} (k-1)[1 + \lambda(k-1)]\Gamma_k a_k z^{k-1}}{1 + \frac{1}{2\beta\gamma} \sum_{k=2}^{\infty} [1 + \lambda(k-1)](2\beta\gamma + k - 1)\Gamma_k a_k z^{k-1}}$$

which implies

$$(c_1 z + c_2 z^2 + \ldots) \left(1 + \frac{1}{2\beta\gamma} \sum_{k=2}^{\infty} [1 + \lambda(k-1)](2\beta\gamma + k - 1)\Gamma_k a_k z^{k-1}\right)$$

$$= \frac{1}{2\beta\gamma} \sum_{k=2}^{\infty} (k-1)[1 + \lambda(k-1)]\Gamma_k a_k z^{k-1}. \tag{2. 4}$$

Equating the coefficients of z and z^2 in both sides of (2. 4) we obtain

$$a_2 = \frac{2\beta\gamma \prod_{j=1}^{m} \beta_j}{(\lambda+1)\prod_{i=1}^{l} \alpha_i} c_1$$

and

$$a_3 = \frac{\beta\gamma \prod_{j=1}^{m} \beta_j(\beta_j+1)}{(2\lambda+1)\prod_{i=1}^{l} \alpha_i(\alpha_i+1)} [c_2 + (2\beta\gamma+1)c_1^2].$$

Hence

$$a_3 - \mu a_2^2 = \frac{\beta\gamma \prod_{j=1}^{m} \beta_j(\beta_j+1)}{(2\lambda+1)\prod_{i=1}^{l} \alpha_i(\alpha_i+1)} (c_2 - dc_1^2) \tag{2. 5}$$

where

$$d := \frac{4\beta\gamma\mu(2\lambda+1)\prod_{i=1}^{l}(\alpha_i+1)\prod_{j=1}^{m}\beta_j}{(\lambda+1)^2\prod_{i=1}^{l}\alpha_i\prod_{j=1}^{m}(\beta_j+1)} - (2\beta\gamma+1).$$

It follows from (2. 5) that

$$|a_3 - \mu a_2^2| \leq \frac{\beta|\gamma| \prod_{j=1}^{m} |\beta_j||\beta_j+1|}{(2\lambda+1)\prod_{i=1}^{l} |\alpha_i||\alpha_i+1|} |c_2 - dc_1^2|.$$

In virtue of Lemma 2 we obtain

$$|a_3 - \mu a_2^2| \leq \frac{\beta|\gamma| \prod_{j=1}^{m} |\beta_j||\beta_j+1|}{(2\lambda+1)\prod_{i=1}^{l} |\alpha_i||\alpha_i+1|} \max\{1, |d|\}.$$

Thus, we completed the proof of the theorem. □

In the next sections we establish certain properties of the classes $TS_{l,m}(\lambda,\beta,\gamma)$ and $TR_{l,m}(\lambda,\beta,\gamma)$.

3. Coefficient inequalities

In this section we obtain necessary and sufficient conditions for a function to be in the class $TS_{l,m}(\lambda,\beta,\gamma)$ and $TR_{l,m}(\lambda,\beta,\gamma)$, respectively.

Theorem 4. *Let $f \in \mathcal{T}(n)$ given by (1. 2). Then f belongs to the class $TS_{l,m}(\lambda,\beta,\gamma)$ if and only if*

$$\sum_{k=n+1}^{\infty}[1+\lambda(k-1)](k+\beta|\gamma|-1)|\Gamma_k|a_k \leq \beta|\gamma| \tag{3. 1}$$

$$(\gamma \in \mathbb{C}-\{0\} \quad ,0 \leq \lambda \leq 1 \quad ,0<\beta \leq 1).$$

Proof. Suppose $f \in TS_{l,m}(\lambda,\beta,\gamma)$. By making use of (1. 8) we easily obtain

$$\operatorname{Re}\left(\frac{z\nu'(z)}{\nu(z)}-1\right) > -\beta|\gamma| \quad (z \in \mathbb{U}) \tag{3. 2}$$

which, in view of (1. 7), gives

$$\operatorname{Re}\left\{\frac{-\sum_{k=n+1}^{\infty}[1+\lambda(k-1)](k-1)|\Gamma_k|a_k z^{k-1}}{1-\sum_{k=n+1}^{\infty}[1+\lambda(k-1)]|\Gamma_k|a_k z^{k-1}}\right\} > -\beta|\gamma| \tag{3. 3}$$

Setting $z=r$ $(0 \leq r<1)$in (3. 3), we observe that the expression in the denominator on the left hand side of (3. 3) is positive for $r=0$ and also for all $r \in (0,1)$. Thus by letting $r \to 1^-$ through real values, (3. 3) leads us to the desired condition (3. 1) of the theorem.

Conversely, by applying the hypothesis (3. 1) and setting $|z|=1$, we find by using (1. 2) that

$$\left|\frac{z\nu'(z)}{\nu(z)}-1\right| = \left|\frac{\sum_{k=n+1}^{\infty}[1+\lambda(k-1)](k-1)|\Gamma_k|a_k z^{k}}{z-\sum_{k=n+1}^{\infty}[1+\lambda(k-1)]|\Gamma_k|a_k z^{k}}\right|$$

$$\leq \frac{\sum_{k=n+1}^{\infty}[1+\lambda(k-1)](k-1)|\Gamma_k|a_k|z|^{k-1}}{1-\sum_{k=n+1}^{\infty}[1+\lambda(k-1)]|\Gamma_k|a_k|z|^{k-1}}$$

$$\leq \frac{\beta|\gamma|\left(1-\sum_{k=n+1}^{\infty}[1+\lambda(k-1)]|\Gamma_k|a_k\right)}{1-\sum_{k=n+1}^{\infty}[1+\lambda(k-1)]|\Gamma_k|a_k} = \beta|\gamma|.$$

Hence, by the maximum modulus principle, we have $f \in TS_{l,m}(\lambda,\beta,\gamma)$, which evidently completes the proof of the theorem. □

Corollary 5. *If* $f \in TS_{l,m}(\lambda, \beta, \gamma)$ *then*

$$a_k \leq \frac{\beta|\gamma|}{\Phi(\lambda, \beta, \gamma, k)} \quad (k \geq n+1)$$

where

$$\Phi(\lambda, \beta, \gamma, k) = [1 + \lambda(k-1)](k + \beta|\gamma| - 1)|\Gamma_k| \tag{3. 4}$$
$$(0 \leq \lambda \leq 1 \ , \ \gamma \in \mathbb{C} - \{0\} \ , \ 0 < \beta \leq 1 \ , \ k \geq n+1).$$

Equality holds for the functions

$$f_k(z) = z - \frac{\beta|\gamma|}{\Phi(\lambda, \beta, \gamma, k)} z^k \quad (z \in \mathbb{U} \ , \ k \geq n+1).$$

By using the same arguments as in the proof of Theorem 4 we can establish the next theorem.

Theorem 6. *Let* $f \in \mathcal{T}(n)$ *given by (1. 2). Then* f *is in the class* $TR_{l,m}(\lambda, \beta, \gamma)$ *if and only if*

$$\sum_{k=n+1}^{\infty} k[1 + \lambda(k-1)]|\Gamma_k| a_k \leq \beta|\gamma| \tag{3. 5}$$
$$(\gamma \in \mathbb{C} - \{0\} \quad , 0 \leq \lambda \leq 1 \quad , 0 < \beta \leq 1)$$

Corollary 7. *If* $f \in TR_{l,m}(\lambda, \beta, \gamma)$ *then*

$$a_k \leq \frac{\beta|\gamma|}{\Psi(\lambda, \beta, \gamma, k)} \quad (k \geq n+1)$$

where

$$\Psi(\lambda, \beta, \gamma, k) = k[1 + \lambda(k-1)]|\Gamma_k| \tag{3. 6}$$
$$(0 \leq \lambda \leq 1 \ , \ \gamma \in \mathbb{C} - \{0\} \ , \ 0 < \beta \leq 1 \ , \ k \geq n+1).$$

Equality holds for the functions

$$f_k(z) = z - \frac{\beta|\gamma|}{\Psi(\lambda, \beta, \gamma, k)} z^k \quad (z \in \mathbb{U} \ , \ k \geq n+1).$$

4. Radii of close-to-convexity, starlikeness and convexity

We begin this section with the following theorem.

Theorem 8. *Let the function* f *defined by (1. 2) be in the class* $TS_{l,m}(\lambda, \beta, \gamma)$. *Then* f *is close-to-convex of order* δ $(0 \leq \delta < 1)$ *in* $|z| < r_1(\lambda, \beta, \gamma, \delta)$, *where*

$$r_1(\lambda, \beta, \gamma, \delta) = \inf_{k \geq n+1} \left[\frac{(1-\delta)[1 + \lambda(k-1)](k + \beta|\gamma| - 1)}{k\beta|\gamma|} |\Gamma_k| \right]^{\frac{1}{k-1}}$$

Proof. It is sufficient to prove that $|f'(z) - 1| \leq 1 - \delta$ $(0 \leq \delta < 1)$ for $z \in \mathbb{U}$ such that $|z| < r_1(\lambda, \beta, \gamma, \delta)$. We have

$$|f'(z) - 1| = \left| \sum_{k=n+1}^{\infty} k a_k z^{k-1} \right| \leq \sum_{k=n+1}^{\infty} k a_k |z|^{k-1}$$

Thus $|f'(z) - 1| \leq 1 - \delta$ if

$$\sum_{k=n+1}^{\infty} \left(\frac{k}{1-\delta} \right) a_k |z|^{k-1} \leq 1. \tag{4. 1}$$

By making use of (3. 1) we obtain

$$\sum_{k=n+1}^{\infty} \frac{[1+\lambda(k-1)](k+\beta|\gamma|-1)}{\beta|\gamma|}|\Gamma_k|a_k \leq 1.$$

Then the inequality (4. 1) will be true if

$$\left(\frac{k}{1-\delta}\right)|z|^{k-1} \leq \frac{[1+\lambda(k-1)](k+\beta|\gamma|-1)}{\beta|\gamma|}|\Gamma_k| \ (k \geq n+1)$$

or equivalently

$$|z| \leq \left[\frac{(1-\delta)[1+\lambda(k-1)](k+\beta|\gamma|-1)}{k\beta|\gamma|}|\Gamma_k|\right]^{\frac{1}{k-1}} \quad (k \geq n+1) \qquad (4.\ 2)$$

The theorem follows easily from (4. 2). □

Theorem 9. *Let the function f defined by (1. 2) be in the class $TS_{l,m}(\lambda, \beta, \gamma)$. Then f is starlike of order δ $(0 \leq \delta < 1)$ in $|z| < r_2(\lambda, \beta, \gamma, \delta)$, where*

$$r_2(\lambda, \beta, \gamma, \delta) = \inf_{k \geq n+1} \left[\frac{(1-\delta)[1+\lambda(k-1)](k+\beta|\gamma|-1)}{(k-\delta)\beta|\gamma|}|\Gamma_k|\right]^{\frac{1}{k-1}}$$

Proof. We must prove that $\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1-\delta$ $(0 \leq \delta < 1)$ for $z \in \mathbb{U}$ such that $|z| < r_2(\lambda, \beta, \gamma, \delta)$. We have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{-\sum_{k=n+1}^{\infty}(k-1)a_k z^{k-1}}{1-\sum_{k=n+1}^{\infty}a_k z^{k-1}}\right| \leq \frac{\sum_{k=n+1}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum_{k=n+1}^{\infty}a_k|z|^{k-1}}$$

Thus $\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1-\delta$ if

$$\sum_{k=n+1}^{\infty}\left(\frac{k-\delta}{1-\delta}\right)a_k|z|^{k-1} \leq 1. \qquad (4.\ 3)$$

In virtue of (3. 1) we have

$$\sum_{k=n+1}^{\infty} \frac{[1+\lambda(k-1)](k+\beta|\gamma|-1)}{\beta|\gamma|}|\Gamma_k|a_k \leq 1.$$

Hence, the inequality (4.3) will be true if

$$\left(\frac{k-\delta}{1-\delta}\right)|z|^{k-1} \leq \frac{[1+\lambda(k-1)](k+\beta|\gamma|-1)}{\beta|\gamma|}|\Gamma_k| \ (k \geq n+1)$$

or if

$$|z| \leq \left[\frac{(1-\delta)[1+\lambda(k-1)](k+\beta|\gamma|-1)}{(k-\delta)\beta|\gamma|}|\Gamma_k|\right]^{\frac{1}{k-1}} \quad (k \geq n+1).$$

Thus, the proof of our theorem is completed. □

Corollary 10. *Let the function f defined by (1. 2) be in the class $TS_{l,m}(\lambda,\beta,\gamma)$. Then f is convex of order δ $(0 \le \delta < 1)$ in $|z| < r_3(\lambda,\beta,\gamma,\delta)$, where*

$$r_3(\lambda,\beta,\gamma,\delta) = \inf_{k\ge n+1}\left[\frac{(1-\delta)[1+\lambda(k-1)](k+\beta|\gamma|-1)}{k(k-\delta)\beta|\gamma|}|\Gamma_k|\right]^{\frac{1}{k-1}}.$$

By using the same arguments as in the proofs of Theorems 8 and 9 we can obtain the radii of close-to-convexity, starlikeness and convexity for the class $TR_{l,m}(\lambda,\beta,\gamma)$.

Theorem 11. *Let f given by (1. 2) be in the class $TR_{l,m}(\lambda,\beta,\gamma)$. Then the function f is close-to-convex of order δ $(0 \le \delta < 1)$ in $|z| < \rho_1(\lambda,\beta,\gamma,\delta)$, where*

$$\rho_1(\lambda,\beta,\gamma,\delta) = \inf_{k\ge n+1}\left[\frac{(1-\delta)[1+\lambda(k-1)]}{\beta|\gamma|}|\Gamma_k|\right]^{\frac{1}{k-1}}.$$

Theorem 12. *Let f given by (1. 2) be in the class $TR_{l,m}(\lambda,\beta,\gamma)$. Then the function f is starlike of order δ $(0 \le \delta < 1)$ in $|z| < \rho_2(\lambda,\beta,\gamma,\delta)$, where*

$$\rho_2(\lambda,\beta,\gamma,\delta) = \inf_{k\ge n+1}\left[\frac{k(1-\delta)[1+\lambda(k-1)]}{(k-\delta)\beta|\gamma|}|\Gamma_k|\right]^{\frac{1}{k-1}}.$$

Corollary 13. *Let f given by (1. 2) be in the class $TR_{l,m}(\lambda,\beta,\gamma)$. Then the function f is convex of order δ $(0 \le \delta < 1)$ in $|z| < \rho_3(\lambda,\beta,\gamma,\delta)$, where*

$$\rho_3(\lambda,\beta,\gamma,\delta) = \inf_{k\ge n+1}\left[\frac{(1-\delta)[1+\lambda(k-1)]}{(k-\delta)\beta|\gamma|}|\Gamma_k|\right]^{\frac{1}{k-1}}.$$

5. Closure theorems

Let the functions $f_j \in \mathcal{T}(n)$ $(j = 1, 2, \ldots, p)$ defined by

$$f_j(z) = z - \sum_{k=n+1}^{\infty} a_{k,j} z^k \quad (z \in \mathbb{U}). \tag{5. 1}$$

We obtain the following results for the closure of functions in the classes $TS_{l,m}(\lambda,\beta,\gamma)$ and $TR_{l,m}(\lambda,\beta,\gamma)$.

Theorem 14. *Let the functions f_j $(j = 1, 2, \ldots, p)$ given by (5. 1) be in the class $TS_{l,m}(\lambda,\beta,\gamma)$ and let $c_j \ge 0$ $(j = 1, 2, \ldots, p)$ such that $\sum_{j=1}^{p} c_j = 1$. Then the function h defined by*

$$h(z) = \sum_{j=1}^{p} c_j f_j$$

is also in the class $TS_{l,m}(\lambda,\beta,\gamma)$.

Proof. In virtue of the definition of h, we can write

$$h(z) = \sum_{j=1}^{p} c_j \left[z - \sum_{k=n+1}^{\infty} a_{k,j} z^k\right]$$

$$= \left(\sum_{j=1}^{p} c_j\right) z - \sum_{k=n+1}^{\infty}\left(\sum_{j=1}^{p} c_j a_{k,j}\right) z^k$$

$$= z - \sum_{k=n+1}^{\infty} \left(\sum_{j=1}^{p} c_j a_{k,j} \right) z^k.$$

Since the functions f_j are in $TS_{l,m}(\lambda, \beta, \gamma)$, for every $j = 1, 2, \dots, p$ we have

$$\sum_{k=n+1}^{\infty} [1 + \lambda(k-1)](k + \beta|\gamma| - 1)|\Gamma_k| a_{k,j} \leq \beta|\gamma|.$$

Hence we get

$$\sum_{k=n+1}^{\infty} [1 + \lambda(k-1)](k + \beta|\gamma| - 1)|\Gamma_k| \left(\sum_{j=1}^{p} c_j a_{k,j} \right)$$

$$= \sum_{j=1}^{p} c_j \left(\sum_{k=n+1}^{\infty} [1 + \lambda(k-1)](k + \beta|\gamma| - 1)|\Gamma_k| a_{k,j} \right) \leq \sum_{j=1}^{p} c_j (\beta|\gamma|) = \beta|\gamma|$$

which implies that h is in the class $TS_{l,m}(\lambda, \beta, \gamma)$. Thus, the proof of the theorem is completed. □

Corollary 15. *The class $TS_{l,m}(\lambda, \beta, \gamma)$ is closed under convex linear combination.*

Proof. Assume that the functions f_j $(j = 1, 2)$ given by (5. 1) are in the class $TS_{l,m}(\lambda, \beta, \gamma)$. It is sufficient to show that the function h defined by

$$h(z) = \mu f_1(z) + (1 - \mu) f_2(z) \quad (0 \leq \mu \leq 1)$$

is in the class $TS_{l,m}(\lambda, \beta, \gamma)$.

By taking $p = 2$, $c_1 = \mu$ and $c_2 = 1 - \mu$ in Theorem 14 we obtain the corollary. □

Making use of the same arguments as in the proofs of Theorem 14 and Corollary 15, closure properties for the class $TR_{l,m}(\lambda, \beta, \gamma)$ can also be obtained.

6. Convolution and Integral Properties

In this section we shall prove that the classes $TS_{l,m}(\lambda, \beta, \gamma)$ and $TR_{l,m}(\lambda, \beta, \gamma)$ are closed under convolution and integral operator.

Theorem 16. *Let $g(z)$ of the form*

$$g(z) = z - \sum_{k=n+1}^{\infty} c_k z^k \quad (0 \leq c_k \leq 1 \ , \ k \geq n+1)$$

be analytic in $\mathbb{U}$. If the function f belongs to the class $TS_{l,m}(\lambda, \beta, \gamma)$ then the function $f * g$ is also in the class $TS_{l,m}(\lambda, \beta, \gamma)$.

Proof. Since $f \in TS_{l,m}(\lambda, \beta, \gamma)$ by (3. 1) we have

$$\sum_{k=n+1}^{\infty} [1 + \lambda(k-1)](k + \beta|\gamma| - 1)|\Gamma_k| a_k \leq \beta|\gamma|.$$

By making use of the last inequality and the fact that

$$(f * g)(z) = z - \sum_{k=n+1}^{\infty} a_k c_k z^k$$

we obtain

$$\sum_{k=n+1}^{\infty} [1+\lambda(k-1)](k+\beta|\gamma|-1)|\Gamma_k|a_k c_k$$

$$\leq \sum_{k=n+1}^{\infty} [1+\lambda(k-1)](k+\beta|\gamma|-1)|\Gamma_k|a_k \leq \beta|\gamma|$$

and hence, in virtue of Theorem 4, the result follows. □

Let $I_c : \mathcal{T}(n) \to \mathcal{T}(n)$ be the integral operator defined by

$$F(z) = I_c(f)(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt \ \ (c > -1 \ , \ z \in \mathbb{U}) \tag{6. 1}$$

We note that if $f \in \mathcal{T}(n)$ is given by (1.2) then

$$F(z) = z - \sum_{k=n+1}^{\infty} \frac{c+1}{c+k} a_k z^k. \tag{6. 2}$$

Theorem 17. *If the function f is in the class $TS_{l,m}(\lambda, \beta, \gamma)$ then the function F given by (6. 1) is also in $TS_{l,m}(\lambda, \beta, \gamma)$.*

Proof. From (6. 2) it results that $F(z) = (f * g)(z)$ $(z \in \mathbb{U})$, where

$$g(z) = z - \sum_{k=n+1}^{\infty} \frac{c+1}{c+k} z^k \ \text{ and } \ 0 \leq \frac{c+1}{c+k} \leq 1.$$

By Theorem 16, the proof is trivial. □

The proofs for the convolution and integral properties for the class $TR_{l,m}(\lambda, \beta, \gamma)$ are similar.

7. Integral Means Inequalities

In order to prove the results regarding integral means inequalities we need the concept of subordination between analytic functions and also a subordination theorem due to Littlewood [11].

Let f and g be two analytic functions in $\mathbb{U}$. The function g is said to be subordinate to f,denoted by $g \prec f$, if there exists a function $w(z)$ analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ such that $g(z) = f(w(z))$ $(z \in \mathbb{U})$.

Lemma 18 ([11]). *If f and g are two analytic functions in $\mathbb{U}$ such that $g \prec f$ then*

$$\int_0^{2\pi} |g(re^{i\theta})|^{\mu} d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^{\mu} d\theta \ \ (\mu > 0 \ , \ 0 < r < 1).$$

Theorem 19. *Suppose $f \in TS_{l,m}(\lambda, \beta, \gamma)$ and let the function $f_2(z)$ defined by*

$$f_2(z) = z - \frac{\beta|\gamma|}{\Phi(\lambda, \beta, \gamma, 2)} z^2 \ \ (z \in \mathbb{U})$$

where $\Phi(\lambda, \beta, \gamma, k)$ is defined by (3 . 4).If $\{\Phi(\lambda, \beta, \gamma, k)\}_{k=2}^{\infty}$ is a non-decreasing sequence, then

$$\int_0^{2\pi} |f(z)|^{\mu} d\theta \leq \int_0^{2\pi} |f_2(z)|^{\mu} d\theta \ \ (z = re^{i\theta} \ , \ 0 < r < 1 \ , \ \mu > 0).$$

Proof. Let $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$. For $z = re^{i\theta}$ $(0 < r < 1)$ and $\mu > 0$ we must show that

$$\int_0^{2\pi} |1 - \sum_{k=2}^{\infty} a_k z^{k-1}| d\theta \leq \int_0^{2\pi} |1 - \frac{\beta|\gamma|}{\Phi(\lambda,\beta,\gamma,2)} z| d\theta.$$

By applying Lemma 18 it would suffice to prove that

$$1 - \sum_{k=2}^{\infty} a_k z^{k-1} \prec 1 - \frac{\beta|\gamma|}{\Phi(\lambda,\beta,\gamma,2)} z \tag{7. 1}$$

Setting

$$1 - \sum_{k=2}^{\infty} a_k z^{k-1} = 1 - \frac{\beta|\gamma|}{\Phi(\lambda,\beta,\gamma,2)} w(z) \quad (z \in \mathbb{U})$$

we find that

$$w(z) = \sum_{k=2}^{\infty} \frac{\Phi(\lambda,\beta,\gamma,2)}{\beta|\gamma|} a_k z^{k-1}$$

which readily yields $w(0) = 0$. Since $\{\Phi(\lambda,\beta,\gamma,k)\}_{k=2}^{\infty}$ is a non-decreasing sequence, we have

$$\Phi(\lambda,\beta,\gamma,2) \leq \Phi(\lambda,\beta,\gamma,k) \quad (k \geq 2).$$

In virtue of (3. 1) we obtain

$$|w(z)| = |\sum_{k=2}^{\infty} \frac{\Phi(\lambda,\beta,\gamma,2)}{\beta|\gamma|} a_k z^{k-1}|$$

$$\leq |z| \sum_{k=2}^{\infty} \frac{\Phi(\lambda,\beta,\gamma,k)}{\beta|\gamma|} a_k \leq |z| < 1.$$

The last inequality shows that we have the subordination (7. 1), which evidently proves our theorem. □

The proof of the next theorem is the same with the proof of the Theorem (19).

Theorem 20. *Suppose $f \in TS_{l,m}(\lambda,\beta,\gamma)$ and let the function $f_2(z)$ defined by*

$$f_2(z) = z - \frac{\beta|\gamma|}{\Phi(\lambda,\beta,\gamma,2)} z^2 \quad (z \in \mathbb{U})$$

where $\Phi(\lambda,\beta,\gamma,k)$ is defined by (3. 4). If $\{\Phi(\lambda,\beta,\gamma,k)\}_{k=2}^{\infty}$ is a non-decreasing sequence, then

$$\int_0^{2\pi} |f'(z)|^\mu d\theta \leq \int_0^{2\pi} |f_2'(z)|^\mu d\theta \quad (z = re^{i\theta}\ ,\ 0 < r < 1\ ,\ \mu > 0).$$

In the same way we can obtain integral means inequalities for the class $TR_{l,m}(\lambda,\beta,\gamma)$.

Theorem 21. *Suppose $f \in TR_{l,m}(\lambda,\beta,\gamma)$ and let the function $f_2(z)$ defined by*

$$f_2(z) = z - \frac{\beta|\gamma|}{\Psi(\lambda,\beta,\gamma,2)} z^2 \quad (z \in \mathbb{U})$$

where $\Psi(\lambda,\beta,\gamma,k)$ is defined by (3 . 6).If $\{\Psi(\lambda,\beta,\gamma,k)\}_{k=2}^{\infty}$ is a non-decreasing sequence, then

$$\int_0^{2\pi} |f(z)|^\mu d\theta \leq \int_0^{2\pi} |f_2(z)|^\mu d\theta \quad (z = re^{i\theta}\ ,\ 0 < r < 1\ ,\ \mu > 0)$$

and

$$\int_0^{2\pi} |f'(z)|^\mu d\theta \leq \int_0^{2\pi} |f_2'(z)|^\mu d\theta \;\; (z = re^{i\theta} \;,\; 0 < r < 1 \;,\; \mu > 0).$$

8. Inclusion relationships involving the (n, δ)-neighborhoods

In this section we establish some inclusion relationships involving the (n, δ)-neighborhoods for each of the classes $TS_{l,m}(\lambda, \beta, \gamma)$ and $TR_{l,m}(\lambda, \beta, \gamma)$.

Following the earlier investigations by Goodman [6], Ruscheweyh [19], Silverman [20]and others ([14], [23]) we define the (n, δ)-neighborhood of a function $f \in \mathcal{T}(n)$ by

$$N_{n,\delta}(f) = \left\{ g \in \mathcal{T}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta \right\}. \quad (8.\ 1)$$

In particular, for identity function $e(z) = z$ $(z \in \mathbb{U})$ we immediately have

$$N_{n,\delta}(e) = \left\{ g \in \mathcal{T}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k|b_k| \leq \delta \right\}. \quad (8.\ 2)$$

Theorem 22. *If* $\{\Phi(\lambda, \beta, \gamma, k)\}_{k=2}^{\infty}$ *is a non-decreasing sequence and*

$$\delta := \frac{(n+1)\beta|\gamma|}{(\lambda n + 1)(n + \beta|\gamma|)|\Gamma_{n+1}|} \quad (8.\ 3)$$

then

$$TS_{l,m}(\lambda, \beta, \gamma) \subset N_{n,\delta}(e).$$

Proof. Let $f \in TS_{l,m}(\lambda, \beta, \gamma)$. Then in view of the assertion (3. 1) of Theorem 4 and the given condition

$$\Phi(\lambda, \beta, \gamma, n+1) \leq \Phi(\lambda, \beta, \gamma, k) \;\; (k \geq n+1)$$

we get

$$\Phi(\lambda, \beta, \gamma, n+1) \sum_{k=n+1}^{\infty} a_k \leq \beta|\gamma|$$

or

$$(\lambda n + 1)(n + \beta|\gamma|)|\Gamma_{n+1}| \sum_{k=n+1}^{\infty} a_k \leq \beta|\gamma|$$

which implies that

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{\beta|\gamma|}{(\lambda n + 1)(n + \beta|\gamma|)|\Gamma_{n+1}|}. \quad (8.\ 4)$$

Applying the assertion (3. 1) of Theorem 4 again, in conjuction with (8. 4) we find

$$(\lambda n + 1)|\Gamma_{n+1}| \sum_{k=n+1}^{\infty} k a_k \leq \beta|\gamma| + (1 - \beta|\gamma|)(\lambda n + 1)|\Gamma_{n+1}| \sum_{k=n+1}^{\infty} a_k$$

$$\leq \beta|\gamma| + (1 - \beta|\gamma|)(\lambda n + 1)|\Gamma_{n+1}| \frac{\beta|\gamma|}{(\lambda n + 1)(n + \beta|\gamma|)|\Gamma_{n+1}|}$$

$$= \frac{(n+1)\beta|\gamma|}{n + \beta|\gamma|}.$$

Hence

$$\sum_{k=n+1}^{\infty} ka_k \leq \frac{(n+1)\beta|\gamma|}{(\lambda n+1)(n+\beta|\gamma|)|\Gamma_{n+1}|} =: \delta$$

which in virtue of (8. 2), proves our theorem. □

Similarly, by applying the assertion (3. 5) of Theorem 6 instead of the assertion (3. 1) in Theorem 4 we can prove the following theorem.

Theorem 23. *If* $\{\Psi(\lambda,\beta,\gamma,k)\}_{k=2}^{\infty}$ *is a non-decreasing sequence and*

$$\delta := \frac{\beta|\gamma|}{(\lambda n+1)|\Gamma_{n+1}|}$$

then

$$TR_{l,m}(\lambda,\beta,\gamma) \subset N_{n,\delta}(e).$$

9. Neighborhoods for the classes $TS_{l,m}(\lambda,\beta,\gamma)$, $TR_{l,m}(\lambda,\beta,\gamma)$

In the sequence, we shall determine the neighborhood properties for each of the classes of functions

$$TS_{l,m}^{(\alpha)}(\lambda,\beta,\gamma) \text{ and } TR_{l,m}^{(\alpha)}(\lambda,\beta,\gamma)$$

which are defined as follows.

A function $f \in \mathcal{T}(n)$ is said to be in the class $TS_{l,m}^{(\alpha)}(\lambda,\beta,\gamma)$ if there exists a function $g \in TS_{l,m}(\lambda,\beta,\gamma)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1-\alpha \quad (z \in \mathbb{U}\ ,\ 0 \leq \alpha < 1) \tag{9. 1}$$

Analogously, a function $f \in \mathcal{T}(n)$ is said to be in the class $TR_{l,m}^{(\alpha)}(\lambda,\beta,\gamma)$ if there exists a function $g \in TR_{l,m}(\lambda,\beta,\gamma)$ such that the inequality (9. 1) holds true.

Theorem 24. *If* $g \in TS_{l,m}(\lambda,\beta,\gamma)$ *and*

$$\alpha := 1 - \frac{\delta(\lambda n+1)(n+\beta|\gamma|)|\Gamma_{n+1}|}{(n+1)[(\lambda n+1)(n+\beta|\gamma|)|\Gamma_{n+1}| - \beta|\gamma|]} \quad (\beta|\gamma| \geq 1) \tag{9. 2}$$

then $N_{n,\delta}(g) \subset TS_{l,m}^{(\alpha)}(\lambda,\beta,\gamma)$.

Proof. Suppose that $f \in N_{n,\delta}(g)$. Then, from definition (8. 1) we find that

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta$$

which readily implies the inequality

$$\sum_{k=n+1}^{\infty} |a_k - b_k| \leq \frac{\delta}{n+1} \quad (n \in \mathbb{N}).$$

Since $g \in TS_{l,m}(\lambda,\beta,\gamma)$, from (8. 4) we have

$$\sum_{k=n+1}^{\infty} b_k \leq \frac{\beta|\gamma|}{(\lambda n+1)(n+\beta|\gamma|)|\Gamma_{n+1}|}.$$

It follows that

$$\left|\frac{f(z)}{g(z)} - 1\right| < \frac{\sum\limits_{k=n+1}^{\infty} |a_k - b_k|}{1 - \sum\limits_{k=n+1}^{\infty} b_k}$$

$$\leq \frac{\delta}{n+1} \cdot \frac{(\lambda n + 1)(n + \beta|\gamma|)|\Gamma_{n+1}|}{(\lambda n + 1)(n + \beta|\gamma|)|\Gamma_{n+1}| - \beta|\gamma|} = 1 - \alpha$$

provided that α is given by (9. 2). Thus, the proof of our theorem is completed. □

The proof of Theorem 25 below is much akin to that of the Theorem 24.

Theorem 25. *If* $g \in TR_{l,m}(\lambda, \beta, \gamma)$ *and*

$$\alpha := 1 - \frac{\delta(\lambda n + 1)|\Gamma_{n+1}|}{(n+1)(\lambda n + 1)|\Gamma_{n+1}| - \beta|\gamma|} \quad (\beta|\gamma| \geq 1)$$

then $N_{n,\delta}(g) \subset TR_{l,m}^{(\alpha)}(\lambda, \beta, \gamma)$.

Remark 26. By taking $\Gamma_{n+1} = 1$ in Theorem 24 and also in Theorem 25 we obtain the inclusion relation of Altintaş et al. [1].

REFERENCES

[1] O. Altintaş, Ö. Özkan, H. M. Srivastava, *Neighborhoods of a class of analytic functions with negative coefficients*, Applied Math. Letters, **13**, 63-67(2000).

[2] S. D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc., **135**, 429-446(1969).

[3] B. C. Carlson, D. B. Shaffer, *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal., **15**, 737-745(1984).

[4] B. C. Carlson, *Special Functions of Applied Mathematics*, Academic Press, New York, (1977).

[5] J. Dziok, H. M. Srivastava, *Certain subclasses of analytic functions associated with the generalized hypergeometric functions*, Integ. Transf. Spec. Funct., **14**, 7-18(2003).

[6] A. W. Goodman, *Univalent functions and nonanalytic curves*, Proc. Amer. Math. Soc., **8**, 598-601(1957).

[7] Yu. E. Hohlov, *Operators and operations in the class of univalent functions*, Izv. Vyss. Ucebn. Zaved Matematika, **10**, 83-89(1978).

[8] F. R. Keogh, E. P. Merkes, *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Soc., **20**, 8-12(1969).

[9] A. Y. Lashin, *Starlike and convex functions of complex order involving a certain linear operator*, Indian J. Pure and Appl., **34**(7), 1101-1108(2003).

[10] R. J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc., **16**, 755-758(1965).

[11] J. E. Littlewood, *On inequalities in the theory of functions*, Proc. London Math. Soc., **23**, 481-519(1925).

[12] A. E. Livingston, *On the radius of univalence of certain analytic functions*, Proc. Amer. Math. Soc., **17**, 352-357(1966).

[13] S. Najafzadeh, S. R. Kulkarni, G. Murugusundaramoorthy, *Certain class of p-valent functions defined by Dziok-Srivastava linear operator*, General Mathematics, **14**(1), 65-76(2006).

[14] H. Orhan, M. Kamali, *Neighborhoods of a class of analytic functions with negative coefficients*, Acta Math. Paed. Neyireg, **21**(1), 55-61(2004).

[15] S. Owa, *On the distorsion theorems-I*, Kyungpook Math. J., **18**, 53-59(1978).

[16] J. K. Prajapat, R. K. Raina, H. M. Srivastava, *Inclusion and neighborhood properties for certain classes of multivalently analytic functions associated with the convolution structure*, J. Inequal. Pure and Appl. Math., **8**(1), Art. 7, 1-8(electronic)(2007).

[17] E. D. Rainville, *Special Functions*, Chelsea Publishing Company, New York, (1960).

[18] S. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc., **49**, 109-115(1975).

[19] S. Ruscheweyh, *Neighborhoods of univalent functions*, Proc. Amer. Math. Soc., **81**, 521-527(1981).

[20] H. Silverman, *Neighborhoods of classes of analytic functions*, Far. East J. Math. Sci., **3**(2), 165-169(1995).

[21] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc., **51**, 109-116(1975).

[22] H. M. Srivastava, S. Owa,*Some characterization and distorsion theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, linear operators and certain subclasses of analytic functions*, Nagoya Math. J., **106**, 1-28(1987).

[23] H. M. Srivastava, H. Orhan, *Coefficient inequalities and inclusion relations fore some families of analytic and multivalent functions*, Applied Math. Letters, **20**, 686-691(2007).

Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 42 (2010)pp. 41-50

On the Total Differential of Almost Quasiconformal Mappings

V.M. Miklyukov
Laboratory Superslow Processes
Volgograd State University
Universitetskii pr-t 100 400062, Volgograd
E-mail: miklyuk@mail.ru

Abstract. We give some sufficient conditions of existence of total differential at a point (which may be a boundary point)for almost quasiconformal mappings.

AMS (MOS) Subject Classification Codes: 30C65, 35J70, 58J05
Key Words: Almost quasiconformal mapping; quasiconformal mapping; elliptic PDE.

1. Auxiliary Concepts

1.1. Let D be a domain in $\mathbb{R}^n$ and $f : D \to \mathbb{R}^m$ be a vector function. The vector function $f : D \to \mathbb{R}^m$ has at a point $a \in D$ a *total differential*, if there exists a constant matrix

$$C = \{C_{ij}\}_{\substack{1\le i\le n\\ 1\le j\le m}}$$

such that

$$f(x) - f(a) = C \cdot (x-a) + o(|x-a|) \quad (x \to a, \quad x \in D). \tag{1. 1}$$

It is known, that a function f has a total differential at a point $a \in D$, if in a neighborhood of a there exist partial derivatives $\partial f_i/\, \partial x_j$ $(i = 1, \dots, n, j = 1, \dots, m)$, which are continuous at a. There are examples which show that the continuity of partial derivatives at a is not a necessary condition for the existence a total differential at a.

1.2. Let $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ be a mapping of the class $W^{1,n}_{\text{loc}}(D)$. We put

$$f'(x) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \cdots & \cdots \\ \dfrac{\partial f_m}{\partial x_1} \cdots & \dfrac{\partial f_m}{\partial x_n} \end{pmatrix} \textit{and, next, } \|f'(x)\| = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j}(x) \right)^2 \right)^{1/2}$$

We shall say, that a mapping $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ belongs to a class $\mathcal{W}^{1,n}_{\text{loc}}(D)$, if for an arbitrary subdomain $D' \subset\subset D$ there exists a constant $p > n$ which, in general, depends on D', such that $f \in W^{1,p}(D')$. A continuous mapping $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is *almost quasiconformal* in D with a measurable function $K(x) \ge 0$ and locally integrable function

$\delta(x) : D \to \mathbb{R}$, if $f \in \mathcal{W}^{1,n}_{\rm loc}(D)$ and almost everywhere on D the following property holds

$$\|f'(x)\|^n \le K(x)\ |J(x,f)| + \delta(x)\,, \tag{1. 2}$$

where

$$J(x,f) = \det\ (f'(x))\ .$$

The concept of almost quasiconformal maps belongs to Callender [6], however we note that the condition (1. 2) has in [6] a different form. Namely, in [6] it is assumed that $K(x) \equiv$ const, and instead of $|J(x,f)|$ it is written $J(x,f)$. Thus the class of maps considered here is essentially wider than the class considered by Callender in [6]. In particular, our definition permits to consider degenerate quasiconformal maps.

Under condition of preservation of the Jacobian sign and the assumptions

$$K \equiv \text{const} > 0, \quad \delta \equiv 0,$$

the supposition (1. 2) means that the mapping f is quasiregular [23, §3 Ch. I], [25, Sect. 14.1]. It should be noted that in the case of quasiregular maps it is assumed only that the vector-function f is continuous and belongs to $W^{1,n}_{\rm loc}(D)$, and the supposition $\mathcal{W}^{1,n}_{\rm loc}(D)$ holds automatically.

The assumption (1. 2) does not require that the sign of det $(f'(x))$ is constant. Thus, almost quasiconformal maps can change their orientation.

The following simple statement [12, Sect. 8.1] shows that the class of considered maps is very wide.

Proposition 1. *Let $f : D \to \mathbb{R}^n$ be a mapping and moreover*

$$f \in \text{ACL}(D) \quad \textit{and} \quad \operatorname{ess\,sup}_{x\in D}\|f'(x)\| \le q < \infty\,.$$

Then f is almost quasiconformal with $K = \epsilon\, n^{n/2}$ and $\delta = (1+\epsilon)\, q^n$, where $\epsilon =$ const $>$ 0 is arbitrary.

1.3. In the case, if $D \subset \mathbb{R}^2$ and $a \in \partial D$ is a multiple point of a boundary, the relation (1. 1) can depend on a direction of the approach to the point a from D and, consequently, the definition of the total differential must be more precise.

We define ends of a domain D using analogy with the Carathéodory theory of prime ends (see, for example, [13, §3]).

For an arbitrary set $U \subset D$ we put $[U] = \overline{U} \setminus \partial D$, where $\overline{U}$ is the closure with respect to $\mathbb{R}^n$. Let $\{U_k\}$, $k = 1, 2, \ldots$ be a family of subdomains $U_k \subset D$ with properties:

(i) for every $k = 1, 2, \ldots \quad [U_{k+1}] \subset U_k$,

(ii) $\bigcap\limits_{k=1}^{\infty} [U_k] = \emptyset$.

An arbitrary sequence $\{U_k\}$ with these properties is called a *chain* in D.

Let $\{U'_k\}$, $\{U''_k\}$ be two chains of subdomains of D. We say, that U'_k is *contained* in $\{U''_k\}$, if for every $m \ge 1$ there is a number $k(m)$ such that for all $k > k(m)$ the following property holds $U'_k \subset U''_m$. Two chains are called *equivalent*, if each of them is contained in the other one. The classes of equivalence ξ of chains are called *ends* of D.

To define an end ξ it is sufficient to set even one representative of the class of equivalence. If an end ξ is defined with a chain $\{U_k\}$, then we write $\xi \asymp \{U_k\}$.

A *body* of a end $\xi \asymp \{U_k\}$ is the set

$$|\xi| = \cap_{i=1}^{\infty} \overline{U_i}\,.$$

It is clear, that this set does not depend on the choice of a chain $\{U_k\}$.

Let $\{x_m\}_{m=1}^{\infty}$ be a sequence of points $x_m \in D$ which does not have condensation point in D. Such sequences are called *nonconvergent* in D.

Let $a_\xi \in |\xi|$ be an arbitrary point. A nonconvergent in D sequence of points $x_k \in D$ *converges to a point a_ξ with respect to the topology of ξ*, if $x_k \to a_\xi$ (with respect to the topology $\mathbb{R}^n$) and for some chain $\{U_k\} \in \xi$ the following property holds: for every $k = 1, 2, \ldots$ there is a number $m(k)$ such that $x_m \in U_k$ for arbitrary $m > m(k)$.

Let D be a domain in $\mathbb{R}^n$, ξ be an end of D, $a_\xi \in |\xi|$ be a point. We shall say, that a subdomain D' of D *adjoins* at a_ξ, if $a_\xi \in \partial D'$ and any sequence of points $x_k \in D'$, converging to a_ξ with respect to the topology $\mathbb{R}^n$, converges to this point with respect to the topology of ξ. We shall say, that a vector-function $f : D \to \mathbb{R}^m$ satisfies to the property

$$\lim_{x \to a_\xi} f(x) = A\,, \quad A = (A_1, \ldots, A_m)$$

if

$$f(x_k) \to A \quad \text{as} \quad x_k \to a_\xi$$

along every sequence of point $x_k \in D$, which converges to a_ξ with respect to the topology of ξ. The vector A is denoted by $f(a_\xi)$.

Suppose that a vector-function $f : D \to \mathbb{R}^m$ and a point a_ξ are such that $f(a_\xi)$ exists. We say, that f has a *total differential at a boundary point a_ξ*, if there exists a constant matrix $C = \{C_{ij}\}_{\substack{1\le i\le n \\ 1\le j\le m}}$, for which

$$f(x) - f(a_\xi) = C \cdot (x - a_\xi) + o(|x - a_\xi|) \quad (x \to a_\xi, \quad x \in D). \tag{1.3}$$

As in the case of an inner point, we shall say that

$$df(a_\xi) = C \cdot (x - a_\xi)$$

is a differential of f at a_ξ.

The differential of the vector-function at the boundary point need not be unique (see corresponding examples in [9]).

2. The weighted modulus

2.1. Recall the definition of the class ACL^p_σ. Let $D \subset \mathbb{R}^n$ be an open set. Fix i, $1 \le i \le n$, and denote by D_i^* an orthogonal projection of D onto the hyperplane $x_i = 0$. For an arbitrary locally summable in D function f we put

$$f_i^*(x_i', t, x_i'') \equiv f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)\,,$$

$$x_i' = (x_1, \ldots, x_{i-1}), \quad x_i'' = (x_{i+1}, \ldots, x_n)\,.$$

Next, let

$$D_i(x_i', x_i'') \equiv \{(x_i', t, x_i'') \in \mathbb{R}^n : (x_i', 0, x_i'') \in D_i^*\}\,.$$

A continuous function $f : D \to \mathbb{R}$ is called *absolutely continuous on lines* (or shortly, ACL), if for every $i = 1, \ldots, n$, the coordinate functions $f_i^*(x_i', t, x_i'')$ are absolutely continuous (with respect to the variable t) inside the union of linear intervals $D \cap D_i(x_i', x_i'')$ for $\mathcal{H}^{n-1}$-almost all points $(x_i', 0, x_i'') \in D_i^*$. (Here and below the symbol $d\mathcal{H}^p$ means an element of p-dimensional Hausdorff measure.)

Every ACL-function $f : D \to \mathbb{R}$ has partial derivatives $\partial f / \partial x_i$ $(i = 1, \ldots, n)$ almost everywhere in D. By the symbol $f' \equiv (\partial f_i / \partial x_j)$ we denote a formal derivative of f at points, where all partial derivatives exist. At points, in which the matrix f' is not defined, let us agree to take all $\partial f_i/\partial x_j = +\infty$ $(i = 1, \ldots, n; j = 1, \ldots, m)$.

Let $\sigma : D \to \mathbb{R}^1$ be a nonnegative measurable function, which is defined almost everywhere in D, and let $p \geq 1$ be a constant. The class $\mathrm{ACL}^p_\sigma(D)$ is the set of ACL-functions in D, for which

$$\int_D \|f'(x)\|^p \, \sigma(x) \, d\mathcal{H}^n < \infty .$$

In the case, if the weight function $\sigma \equiv 1$, we have the well known class $\mathrm{ACL}^p(D)$, which coincides with the set of continuous $W^1_p(D)$-functions [17, Theorems 5.6.2-3].

2.2. Let D be a domain in $\mathbb{R}^m$, $m > 1$, let $U \subset D$ be a countably $(\mathcal{H}^k, k)$-rectifiable set, $1 \leq k \leq m$, and let $\sigma : U \to \mathbb{R}^1$ be a nonnegative $\mathcal{H}^k$-measurable function. Fix a constant $p > 1$ and for an arbitrary family Γ of locally rectifiable arcs $\gamma \subset U$, we define a (p, σ)-modulus

$$\mathrm{mod}_{p,\sigma}(\Gamma; U) = \inf_\rho \frac{\int\limits_U \rho^p \, \sigma \, d\mathcal{H}^k}{\left(\inf\limits_{\gamma \in \Gamma} \int\limits_\gamma \rho \, d\mathcal{H}^1 \right)^p}, \tag{2. 4}$$

where the infimum is taken over all nonnegative, Borel measurable functions ρ in U. If $\Gamma = \emptyset$, then we put $\mathrm{mod}_{p,\sigma}(\Gamma; U) = \infty$.

In the case $U = D$ we have a standard definition of the weighted (p, σ)-modulus of the family Γ in $\mathbb{R}^n$ (see, for example, [18, Sect. 3.2]).

2.3. Let y and a be a pair of points such that $y \in D$ and either a is an interior point of D, or $a = a_\xi \in |\xi|$, where ξ is an end of the domain $D \subset \mathbb{R}^n$. We say that a simple Jordan arc γ, defined by a parametrization $x(\tau) : [0, 1) \to D$, *leads from y to a*, if $x(0) = y$ and

$$\lim_{\tau \to 1} x(\tau) = a \quad \text{as} \quad a \in D$$

and there is a sequence $\tau_k \to 1$, along which

$$\lim_{\tau_k \to 1} x(\tau_k) = a_\xi \quad \text{as} \quad a \in |\xi| .$$

We consider a family Γ of all locally rectifiable, simple Jordan arcs $\gamma \subset D$, leading from y to a. We put

$$\mathrm{mod}_{p,\sigma}\Gamma(y, a; D) = \mathrm{mod}_{p,\sigma}(\Gamma; D) . \tag{2. 5}$$

2.4. Let $D \subset \mathbb{R}^n$ be a domain and $a = a_\xi$ be its interior or boundary point. Fix a continuous vector-function $\nu : \overline{D} \to \mathbb{R}^k$, $1 \leq k < \infty$ and put $B^\nu(a, r) = \{x \in D : |\nu(x) - \nu(a)| < r\}$. By $B^\nu_D(a, r)$ we denote a connected component of $B^\nu(a, r)$, containing a if a is an interior point of D, and adjoining at a if $a \in |\xi|$.

By $S^\nu_D(a, r)$ we denote the relative boundary

$$S^\nu_D(a, r) = \partial B^\nu_D(a, r) \setminus \partial D .$$

In the case $\nu(x) \equiv x$ we shall use notations $B^n(a, r)$, $B^n_D(a, r)$ and $S^\nu_D(a, r)$, respectively.

Suppose that $\nu(x)$ is locally Lipschitz. Let $h(x) = |\nu(x) - \nu(a)|$ and let

$$0 < \operatorname{ess} \inf_{x \in D'} |\nabla h(x)| \leq \operatorname{ess} \sup_{x \in D'} |\nabla h(x)| < \infty \tag{2. 6}$$

on every subset $D' \Subset D$.

By Theorem 3.2.15 [16] (see also [18, Theorem 1.6.1]) for a.e. $t \in \mathbb{R}^1$ the sets $S^{\nu}_D(a,t)$ are countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable.

Fix a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set $S^{\nu}_D(a,t)$ and a measurable function $\sigma : S^{\nu}_D(a,t) \to \mathbb{R}^1$. Let U be a connected component of $S^{\nu}_D(a,t)$. For a pair of points a_1, $a_2 \in U$ let $\Gamma = \Gamma(a_1, a_2)$ stands for the family of all locally rectifiable arcs $\gamma \subset U$, joining points a_1 and a_2. We define a weighted modulus

$$\operatorname{mod}(a_1, a_2; \sigma) = \operatorname{mod}_{n,\sigma}\Gamma(a_1, a_2)\,. \tag{2. 7}$$

Next, let

$$\kappa(S^{\nu}_D(a,t), \sigma) = \inf_{U} \inf_{a_1, a_2 \in U} \operatorname{mod}(a_1, a_2; \sigma), \tag{2. 8}$$

where the first of infimums is taken over the collection $\{U\}$ of all connected components U of $S^{\nu}_D(a,t)$.

We put

$$\kappa^{\nu}(a,t) = \kappa(S^{\nu}_D(a,t), \sigma^*)\,, \quad \sigma^* = \frac{\sigma}{|\nabla h|},$$

where $\sigma : D \to \mathbb{R}^1$ is a nonnegative measurable function.

2.5. We shall need the following multidimensional version of known "Length and Area Principle" (see, for example, [19], [13]).

Lemma 2. [7] *Let D be a domain in $\mathbb{R}^n$, let $a = a_\xi \in \overline{D}$, let a vector-function $\nu : D \to \mathbb{R}^k$ satisfy (2. 6) and $\sigma(x) : D \to \mathbb{R}^1$ is a nonnegative, measurable function. Let $f : D \to \mathbb{R}^m$ be a vector-function of the class $\mathrm{ACL}^n_\sigma(D)$. Then for arbitrary t', $t'' \in h(D)$, $t' < t''$, the following inequality holds*

$$\int_{t'}^{t''} \Omega^n\left(f, S^{\nu}_D(a,t)\right) \kappa^{\nu}(a,t)\, dt \le \int_{D(t',t'')} \|f'(x)\|^n\, \sigma(x)\, d\mathcal{H}^n(x)\,. \tag{2. 9}$$

Here

$$D(t', t'') = \{x \in D : t' < |\nu(x) - \nu(a)| < t''\},$$

$$\Omega\left(f, S^{\nu}_D(a,t)\right) = \sup_{U} \operatorname{osc}\left(f, U\right)$$

and the infimum is taken over all connected components U of $S^{\nu}_D(a,t)$.

3. Main results

3.1. Let $D \subset \mathbb{R}^n$ be an open set. We say that a function $f : D \to \mathbb{R}^m$, $m \ge 1$, is *monotone*, if for every subdomain $U \subset D$ the following property holds

$$\operatorname{osc}\,(f, U) \le \operatorname{osc}\,(f, \partial' U)\,, \quad \partial' U = \partial U \setminus D\,.$$

Here and below by the symbol

$$\operatorname{osc}\,(\phi, E) = \sup_{x,y \in E} |\phi(x) - \phi(y)|$$

we denote the oscillation of a function ϕ on E.

Let $h(t) : [0, \infty) \to [0, \infty)$ be an upper semicontinuity function. We shall say, that $f : D \to \mathbb{R}^m$, $m \geq 1$, is *h-monotone*, if for every subdomain $U \subset D$ we have

$$h\left(\text{osc}\,(f, U)\right) \leq \text{osc}\,(f, \partial' U)\,,$$

and is *α-monotone*, $0 < \alpha \equiv \text{const} < \infty$, if f is h-monotone with $h(t) = t^\alpha$.

Some examples of α-monotone functions were given in [8].

Fix a continuous vector-function $\nu : \overline{D} \to \mathbb{R}^k$. We say, that a vector-function $f : D \to \mathbb{R}^m$, $m \geq 1$, is *weakly (h, ν)-monotone close to a point $a = a_\xi$* (interior or boundary), if

$$\limsup_{r \to 0} \frac{h\left(\text{osc}\,(f, B^\nu_D(a, r))\right)}{\text{osc}\,(f, S^\nu_D(a, r))} < \infty\,, \tag{3. 10}$$

and *weakly (α, ν)-monotone close to a point a*, if f is weakly (h, ν)-monotone close to a for $h(t) = t^\alpha$.

It is clear, that every monotone in the Lebesgue sense function is weakly (α, ν)-monotone, $\alpha = 1$, close to every point.

3.2 For an arbitrary continuous mapping $y = \varphi(x) : D \subset \mathbb{R}^n \to \mathbb{R}^n$ and for a set $A \subset D$ by $N(y; \varphi, A)$, we shall denote the number of preimages of a point $y \in \mathbb{R}^n$ in A. Next we put

$$n(x; \varphi, A) = N(y; \varphi, A)\,, \quad \text{where} \quad y = \varphi(x)\,.$$

3.3. The following statement is the main result of this paper

Theorem 3. *Suppose that a vector-function $f : D \to \mathbb{R}^n$ is an almost quasiconformal mapping of a domain $D \subset \mathbb{R}^n$ in the sense (1. 2), for which*

$$\int\limits_D \frac{\delta(x)\, dx}{K(x)} < \infty\,. \tag{3. 11}$$

Then for every subdomain $A \subset D$ the following inequality holds

$$\int\limits_A \frac{\|f'(x)\|^n\, dx}{K(x)\, n(x; f, A)} \leq \text{mes}_n(f(A)) + \int\limits_A \frac{\delta(x)\, dx}{K(x)\, n(x; f, A)}\,. \tag{3. 12}$$

On the other hand, let $a = a_\xi \in \overline{D}$ be an interior or boundary point of the domain and let $\nu : \overline{D} \to \mathbb{R}^k$ be a continuous vector-function satisfying (2. 6). If

i) for some $p > n$ and some constant matrix $C = (C_{ij})^n_{i,j=1}$ the following assumption holds

$$\frac{\limsup\limits_{\substack{y \to a \\ y \in D}} \displaystyle\int\limits_{B^\nu_D(a, r(a,y))} \frac{\|f'(x) - C\|^p\, dx}{K(x)\, n(x; f, B^\nu_D(a, r(a, y)))}}{r^p\, \text{mod}_{p, \sigma_r} \Gamma(y, a; B^\nu_D(a, r(a, y)))} = 0\,, \tag{3. 13}$$

where

$$r(a, y) = \inf\{t > 0 : y \in B^\nu_D(a, t)\}, \quad \sigma_r(x) = \frac{1}{K(x)\, n(x; f, B^\nu_D(a, r))}\,, \tag{3. 14}$$

or

ii) the vector-function $f(x) - C \cdot x$ is weakly (α, ν)-monotone close to a and there is a constant $\lambda > 1$, for which

$$\limsup_{\substack{y \to a \\ y \in D}} \int\limits_{B^\nu_D(a, \lambda r(a,y))} \frac{\|f'(x) - C\|^n \, dx}{K(x)\, n(x; f, B^\nu_D(a, r(a,y)))} \Bigg/ r^{n\alpha}(a,y) \int\limits_{r(a,y)}^{\lambda r(a,y)} \kappa^\nu(a,t) \frac{dt}{t} = 0, \tag{3. 15}$$

then f has at the point $a = a_\xi$ the total differential $C \cdot dx$.

3.4. Consider some particular cases of Theorem. Let $w = f(z) : D \subset \mathbb{C}^1 \to \mathbb{C}^1$ be a generalized solution of a Beltrami equation

$$f_{\bar z} = \mu(z)\, f_z\,, \tag{3. 16}$$

where $\mu(z)$ is a measurable complex-valued function, and by symbols

$$f_z = \frac{1}{2}(f_x - i\, f_y)\,, \quad f_{\bar z} = \frac{1}{2}(f_x + i\, f_y)$$

we denote formal derivatives.

Note that in contrast to the traditional case (see, for example, [10, Ch. V], [11, Ch. 1]) we do not assume here, that $|\mu(z)| < 1$.

We have

$$J(z, f) = |f_z|^2 - |f_{\bar z}|^2 = (1 - |\mu(z)|^2)|f_z|^2$$

and

$$\|f'\|^2 = |f_z|^2 + |f_{\bar z}|^2 = (1 + |\mu(z)|^2)|f_z|^2\,.$$

Thus,

$$\|f'\|^2 \le \frac{1 + |\mu|^2}{|1 - |\mu|^2|}\, |J(z, f)|$$

and (1. 2) holds with

$$K(z) = \frac{1 + |\mu(z)|^2}{|1 - |\mu(z)|^2|}\,, \quad \delta(z) \equiv 0\,.$$

The assumption (3. 11) holds always.

Here we have also

$$\sigma_r(z) = \frac{1}{K(x)\, n(x; f, B^\nu_D(a, r))} = \frac{|1 - |\mu(z)|^2|}{(1 + |\mu(z)|^2)\, n(x; f, B^\nu_D(a, r))}\,.$$

For schlicht maps $n(x; f, B^\nu_D(a, r)) \equiv 1$ and

$$\sigma_r(z) = \frac{|1 - |\mu(z)|^2|}{1 + |\mu(z)|^2}\,.$$

Theorem connects the differentiability of f at a singular point $a = a_\xi$ with the behavior of the characteristic $\mu(z)$ close to its neighborhood. In the case, if a is an inner point, see [1], [2], [3, Ch. VI], [4, Ch. 11]. In the case, if $a = a_\xi$ is a boundary point and $\mu(z) \equiv 0$, see related results in [5, Ch. 11].

In the case, if the matrix C is orthogonal, Theorem gives conditions, under which maps $f : D \to \mathbb{R}^n$ are conformal at $a = a_\xi$.

For space quasiregular maps, near questions were being considered in [24, Ch. VI].

4. Proof of Theorem

Let $\varphi : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping. This mapping φ is called *absolutely continuous*, if for every $\varepsilon > 0$ there is $\delta > 0$ such that for an arbitrary measurable set $E \subset D$, $\text{mes}_n E < \delta$, we have $\text{mes}_n \varphi(E) < \varepsilon$. In particular, every absolutely continuous mapping possesses the Lusin N-property.

Lemma 4. [20] *If a mapping φ is continuous and belongs to the class $\mathcal{W}^{1,n}_{\text{loc}}(D)$, then φ is absolutely continuous on every subdomain $D' \Subset D$.*

Applying Lemma 2, we conclude that the following statement holds (see, for example, [22]).

Lemma 5. *If a mapping φ is continuous and belongs to $\mathcal{W}^{1,n}_{\text{loc}}(D)$, then for an arbitrary integrable in $\varphi(D)$ function $u(y)$, the function $(u \circ \varphi)(x)|J(x,\varphi)|$ is integrable in D, and moreover*

$$\int\limits_{\varphi(D)} u(y)\, N(y;\varphi,A)\, dy = \int\limits_{A} (u\circ\varphi)(x)\, |J(x,\varphi)|\, dx\,. \tag{4. 17}$$

In particular, if we observe that

$$J(x,\varphi)\, J(y,\varphi^{-1}) = 1\,, \quad y = \varphi(x)\,,$$

and set

$$u(y) = \frac{1}{N(y;\varphi,A)}\,,$$

then using (4. 17) we have

$$\text{mes}_n(\varphi(A)) = \int\limits_{\varphi(A)} dy = \int\limits_{A} \frac{|J(x,\varphi)|}{n(x;\varphi,A)}\, dx\,.$$

From this, using (1. 2) we conclude that

$$\int\limits_{A} \frac{\|f'(x)\|^n - \delta(x)}{K(x)\, n(x;f,A)}\, dx \le \text{mes}_n(f(A))\,,$$

and, thus, we obtain (3. 12).

Consider the family of locally rectifiable arcs $\Gamma(y,a;B^{\nu}_D(a,|y-a|))$, lying in $B^{\nu}_D(a,|y-a|)$ and joining the point $y \in B^{\nu}_D(a,|y-a|)$ with the point a. Choose in (2. 4) the function $\rho(x) = \|f'(x) - C\|$. We find

$$\text{mod}_{p,\sigma}\Gamma(y,a;B^{\nu}_D(a,|y-a|)) \le \frac{\displaystyle\int\limits_{B^{\nu}_D(a,|y-a|)} \|f'(x)-C\|^p \sigma(x)\, d\mathcal{H}^n}{\displaystyle\inf_{\gamma\in\Gamma(y,a;B^{\nu}_D(a,|y-a|))} \left(\int\limits_{\gamma} \|f'(x)-C\|\,|dx|\right)^p}\,. \tag{4. 18}$$

If γ is an arc of the family $\Gamma(y,a;B^{\nu}_D(a,|y-a|))$, then

$$|f(y) - f(a) - C\cdot(y-a)| \le \int\limits_{\gamma} \|f'(x) - C\|\, d\mathcal{H}^1\,.$$

Thus using (4. 18), for every point $y \in D$ we have

$$|f(y) - f(a) - C \cdot (y-a)|^p \leq \frac{\int\limits_{B^{\nu}_D(a,|y-a|)} \|f'(x) - C\|^p \sigma(x)\, d\mathcal{H}^n}{\mathrm{mod}_{p,\sigma}\Gamma(y,a;B^{\nu}_D(a,|y-a|))} \tag{4. 19}$$

By virtue of (4. 19) the assumption (3. 13) implies realization (1. 1) (respectively, (1. 3)) and, thus, the existence in the case $i)$ of the total differential at $a = a_\xi$.

We first prove the statement in the case $ii)$. Fix a point $y \in D$ and consider the subdomain

$$B^{\nu}_D(a,r)\,, \quad a = a_\xi\,, \quad r = r(a,y)\,,$$

adjoining at the and ξ and containing y in its closure. We put $f^*(x) = f(x) - f(a) - C \cdot (x-a)$. Applying Lemma 1 by virtue of (2. 9), we have

$$\int\limits_{r(a,y)}^{\lambda r(a,y)} \Omega^n\left(f^*, S^{\nu}_D(a,t)\right) \kappa^{\nu}(a,t)\, dt \leq \int\limits_{D(r(a,y),\lambda r(a,y))} \|(f^*)'(x)\|^n\, \sigma_{\lambda r(a,y)}(x)\, d\mathcal{H}^n(x)\,,$$

where $\sigma_{\lambda r(a,y)}(x)$ is defined in (3. 14).

The mapping $f^*(x)$ is weakly (α,ν)-monotone close to a and by virtue of (3. 10) for every t, $r(a,y) < t < \lambda\, r(a,y)$, and some constant $A < \infty$ the following estimates hold

$$|f^*(y) - f^*(a)|^\alpha \leq \mathrm{osc}^\alpha(f^*, B^{\nu}_D(a,t)) \leq A\,\Omega\,(f^*, S^{\nu}_D(a,t))\,.$$

From this we obtain

$$|f^*(y) - f^*(a)|^{\alpha n} \int\limits_{r(a,y)}^{\lambda r(a,y)} \kappa^{\nu}(a,t)\, dt \leq$$

$$\leq A^n \int\limits_{D(r(a,y),\lambda r(a,y))} \|f'(x) - C\|^n\, \sigma_{\lambda r(a,y)}(x)\, d\mathcal{H}^n(x)\,.$$

The assumption (3. 15) implies (1. 1) (and, respectively, (1. 3)). Theorem is proved. □

Acknowledgements

The author is indebted to reviewers and students of VolGU Ol'ga and Oleg Yakshew for a number of useful remarks and corrections.

References

[1] O. Teichmüller, Untersuchungen über konforme und quasikonforme Abbildung, Deutsche Math., v. 3, 1938, 621-678.

[2] P.P. Belinskii, Behaviour of quasiconformal mapping at a isolated singular point, Uch. zap. L'vov un-ta, v. 29, ser. mech.-math., n. 6, 1954, 58-70 (in Russian).

[3] H. Wittich, Neuere untersuchungen über eindeutige analytische funktionen, Springer - Verlag, Berlin - Göttingen - Heidelberg, 1955.

[4] Ju.Ju. Trochimchuk, Differentiation, inner maps and criterions of analyticity, Kiew, In-t mathem. NAN Ukrainy, 2007 (in Russian).

[5] Ch. Pommerenke, Boundary behaviour of conformal maps, Springer-Verlag, Berlin - Heidelberg - New York etc., 1992.

[6] E.D. Callender, Hölder-continuity of N-dimensional quasiconformal mappings, Pacific J. Math., v. 1, 1960, 49-515.

[7] V.M. Miklyukov, A modification of "Lenght and Area" principle on abstract surfaces, Uzbek Mathematical Journal, 2009, n. 1, 68-79 (in Russian).

[8] V.M. Miklyukov, Some conditions for the existence of the total differential, Sib. math. j., v. 51, n. 4, 2010, 639-647.

[9] V.M. Miklyukov, Approximation of function close to singularity, Notes of seminar "Superslow processes", n. 4, Volgograd: Izd-vo VolGU, 2009 (in Russian).

[10] Lars V. Ahlfors, Lectures on Quasiconformal Mappings; with addit. chapters by C.J. Earle ... [et al.]. - 2nd. ed., University lecture series; v. 38, American Mathem. Soc. Providence, Rhode Island, 2006.

[11] S.L. Krushkal', Quasiconformal mappings and Riemann surfaces, V.H. Winston & Sons, Washington, D.C., 1979.

[12] V.M. Miklyukov, Geometrical analysis: differential forms, almost solutions, almost-quasiconformal maps, Volgograd: Izd-vo VolGU, 2007 (in Russian).

[13] G.D. Suvorov, Families of plane topological maps, Novosibirsk: Izd-vo SO AN SSSR, 1965 (in Russian).

[14] S. Saks, Theory of the integral, Warsaw, 1939.

[15] L.C. Evans and R.F. Gariepy, Measure theory and fine properties of functions. Studies in advanced Mathematics, CRC Press, Boca Raton – New York – London – Tokyo, 1992.

[16] H. Federer, Geometric measure theory, Springer-Verlag, Berlin, 1969.

[17] A. Kufner, O. John, and S. Fučik, Function spaces, Noordhoff International Publishing, Leyden, 1977.

[18] V.M. Miklyukov, Introduction to nonsmooth analysis, 2-nd edition, Volgograd: Izd-vo VolGU, 2008 (in Russian).

[19] J. Lelong-Ferrand, Représentation conforme et transformations a intégrale de Dirichlet bornée, Gauthier—Villars, Paris, 1955.

[20] O. Martio and J. Malý, Lusin's condition (N) and mappings of the class $W^{1,n}$, J. Reine Angew. Math., b. 485, 1995, 19-36.

[21] G.D. Suvorov, Generalized "lenght and area" principle in the theory of maps, Kiew: Izd-vo 'Naukova Dumka', 1985 (in Russian).

[22] S.K. Vodop'yanov, Differentiability of maps of Carno groups of sobolev classes, Mathem. sb., v. 194, n. 6, 2003, 67-86 (in Russian).

[23] Ju.G. Reshetnyak, Space mappings with bounded distortion, Novosibirsk: Izd-vo "Nauka", 1982 (in Russian).

[24] Ju.G. Reshetnyak, Theorems of stability in geometry and analysis, Novosibirsk: Izd-vo "Nauka", 1982 (in Russian).

[25] J. Heinonen, T. Kilpeläinen and O. Martio, Nonlinear potential theory of degenerate elliptic equations, Oxford: Clarendon Press. 1993.

Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 42 (2010) pp. 51-56

Common Fixed Point Theorem for Weakly Compatible Maps Satisfying E.A. Property in Intuitionistic Fuzzy Metric Spaces

Saurabh Manro and S.S. Bhatia
School of Mathematics and Computer Applications
Thapar University, Patiala-147 004
India

Sanjay kumar
Deenbandhu Chhotu Ram University of Science and Technology
Murthal (Sonepat)
India

Abstract. In this paper, we use the notion of E.A. property in intuitionistic fuzzy metric space and prove a common fixed point theorem for weakly compatible mappings using this property.

AMS (MOS) Subject Classification Codes: 47H10, 54H25
Key Words: Intuitionistic fuzzy metric space, E.A property, Weakly compatible maps.

1. Introduction

Atanassove[3] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. In 2004, Park[9] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms. Recently, in 2006, Alaca et al.[2] using the idea of Intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norm and continuous t- conorms as a generalization of fuzzy metric space due to Kramosil and Michalek[7]. In 2006, Turkoglu[12] proved Jungck's[6] common fixed point theorem in the setting of intuitionistic fuzzy metric spaces for commuting mappings. In this paper, we use the notion of E.A. property in intuitionistic fuzzy metric space and prove a common fixed point theorem for weakly compatible mappings using this property.

2. Preliminaries

The concepts of triangular norms (t -norm) and triangular conorms (t -conorm) are known as the axiomatic skelton that we use, are characterization of fuzzy intersections and union respectively. These concepts were originally introduced by Menger [8] in study of statistical metric spaces.

Definition 1. A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if * is satisfies the following conditions:

(i) * is commutative and associative;
(ii)* is continuous;
(iii) $a * 1 = a$ for all a in $[0, 1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 2. A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$is continuous t-conorm if $\diamond$ is satisfies the following conditions:

(i) $\diamond$ is commutative and associative;
(ii) $\diamond$ is continuous;
(iii) $a \diamond 0 = a$ for all a $\in [0, 1]$;
(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Alaca et al.[2] defined the notion of intuitionistic fuzzy metric space as follows :

Definition 3. A 5-tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, * is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions:

(i) $M(x, y, t) + N(x, y, t) \leq 1$ for all x, y in X and $t > 0$;
(ii) $M(x, y, 0) = 0$ for all x, y in X;
(iii) $M(x, y, t) = 1$ for all x, y in X and $t > 0$ if and only if $x = y$;
(iv) $M(x, y, t) = M(y, x, t)$ for all x, y in X and $t > 0$;
(v) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all x, y, z in X and $s, t > 0$;
(vi) for all x, y in X, $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous;
(vii) $\lim_{t \to \infty} M(x, y, t) = 1$ for all x, y in X and $t > 0$;
(viii) $N(x, y, 0) = 1$ for all x, y in X;
(ix) $N(x, y, t) = 0$ for all x, y in X and $t > 0$ if and only if $x = y$;
(x) $N(x, y, t) = N(y, x, t)$ for all x, y in X and $t > 0$;
(xi) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$ for all x, y, z in X and $s, t > 0$;
(xii) for all x, y in X, $N(x, y, .) : [0, \infty) \to [0, 1]$ is right continuous;
(xiii) $\lim_{t \to \infty} N(x, y, t) = 0$ for all x, y in X.
Then (M, N) is called an intuitionistic fuzzy metric. The functions M(x, y, t) and N(x, y,t) denote the degree of nearness and the degree of non-nearness between x and y w.r.t. t respectively.

Remark 4. Every fuzzy metric space (X, M, *) is an intuitionistic fuzzy metric space of the form $(X, M, 1 - M, *, \diamond)$ such that t-norm * and t-conorm $\diamond$ are associated as $x \diamond y = 1 - ((1 - x) * (1 - y))$ for all x, y in X.

Remark 5. [2] In intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$, M(x, y, *) is non-decreasing and $N(x, y, \diamond)$ is non-increasing for all x, y in X.

Definition 6. [2] Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. Then
(a) a sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for all $t > 0$ and $p > 0$, $\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1$ and $\lim_{n \to \infty} N(x_{n+p}, x_n, t) = 0$.
(b) a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for all $t > 0$, $\lim_{n \to \infty} M(x_n, x, t) = 1$ and $\lim_{n \to \infty} N(x_n, x, t) = 0$.

(c) $(X, M, N, *, \diamond)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Example 1. Let X = $\{1/n : n = 1, 2, 3, ...\} \cup \{0\}$ and let * be the continuous t-norm and $\diamond$ be the continuous t-conorm defined by a * b = ab and $a \diamond b = min\{1, a + b\}$ respectively, for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$ and x, y in X, define (M, N) by M(x, y, t) = $\frac{t}{t+|x-y|}$ if $t > 0$, M(x, y, t)= 0 if $t = 0$, and N(x, y, t) = $\frac{|x-y|}{t+|x-y|}$ if $t > 0$, N(x, y, t)= 1 if $t = 0$. Clearly, $(X, M, N, *, \diamond)$ is complete intuitionistic fuzzy metric space.

Definition 7. [12] A pair of self mappings (f , g) of a intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be commuting if
M(fgx , gfx , t) = 1 and N(fgx , gfx , t) = 0 for all x in X .

Recently, Amari and Moutawakil[1] introduced a generalization of non compatible maps as E.A. property.

Definition 8. [1] Let A and S be two self-maps of a metric space (X,d). The pair (A,S) is said to satisfy E.A. property, if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some t in X.

Example 2. Let X = $[0, +\infty)$. Define S, T : X → X by Tx = $\frac{x}{4}$ and Sx = $\frac{3x}{4}$, for all x in X. Consider the sequence $\{x_n\} = \{1/n\}$. Clearly $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 0$. Then S and T satisfy E.A. property.

Example 3. Let X = $[2, +\infty)$. Define S, T : X→X by Tx = x+1 and Sx = 2x+1 for all x in X. Suppose that the E.A. property holds. Then, there exists in X, a sequence $\{x_n\}$ satisfying $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z in X. Therefore, $\lim_{n\to\infty} x_n = z - 1$ and $\lim_{n\to\infty} x_n = \frac{z-1}{2}$. Thus, z = 1, which is a contradiction, is said to satisfy the E.A. property if there exist a sequence $\{x_n\}$ in X such that since 1 is not contained in X. Hence S and T do not satisfy E.A. property.

Definition 9. A pair of self mappings (f , g) of a intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to satisfy the E.A. property if there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} M(fx_n, gx_n, t) = 1$ and $\lim_{n\to\infty} N(fx_n, gx_n, t) = 0$.

Example 4. Let X = $[0, \infty)$. Let us consider $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space as in example 1. Define T, S : X → $[0, \infty)$ by Tx = $\frac{x}{5}$ and Sx = $\frac{2x}{5}$. Now, $\lim_{n\to\infty} M(Sx_n, Tx_n, t) = 1$ and $\lim_{n\to\infty} N(Sx_n, Tx_n, t) = 0$. Clearly S and T satisfies E.A. property.
Jungck [6] introduced the notion of weakly compatible maps as follows:

Definition 10. A pair of self mappings (f , g) of a intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be weakly compatible if they commute at the coincidence points i.e. Tu = Su for some u in X , then TSu = STu.

Alaca [2] proved the following result:

Lemma 11. *Let* $(X, M, N, *, \diamond)$ *be intuitionistic fuzzy metric space and for all x , y in* X, $t > 0$ *and if for a number* $k \in (0,1)$, $M(x,y,kt) \geq M(x,y,t)$ *and* $N(x,y,kt) \leq N(x,y,t)$ *then x = y.*

3. Weakly compatible maps and E.A. property

Turkoglu et al.[12] proved the following Theorem:

Theorem 12. *Let A, B, S and T be self maps of complete intuitionistic fuzzy metric space* $(X, M, N, *, \diamond)$ *with continuous t-norm and continuous t- conorm defined by* $a * a \geq a$ *and* $(1-a) \diamond (1-a) \leq (1-a)$ *for all a in* $[0,1]$, *satisfying the following conditions:*

(1) $A(X) \subset T(X), B(X) \subset S(X)$,
(2) S and T are continuous,
(3) The pairs (A , S) and (B , T) are compatible maps,
(4) for all x, y in X, k in $(0,1), t > 0$
$M(Ax, By, kt) \geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) * M(By, Sx, 2t) * M(Ax, Ty, t)$ *and*
$N(Ax, By, kt) \leq N(Sx, Ty, t) \diamond N(Ax, Sx, t) \diamond N(By, Ty, t) \diamond N(By, Sx, 2t) \diamond N(Ax, Ty, t)$,
Then A, B, S and T have a unique common fixed point in X.

Now,we generalize theorem 12 for weakly compatible maps using E.A property. Our theorem generalise theorem 12 in the following way:

(a) relaxing the continuity requirement of maps and,
(b) relaxing the completeness of the space X.

Theorem 13. *Let A, B, S and T be self maps of intuitionistic fuzzy metric space* $(X, M, N, *, \diamond)$ *with continuous t-norm and continuous t- conorm defined by* $a * b = min\{a, b\}$ *and* $a \diamond b = max\{a, b\}$ *for all a, b in* $[0,1]$ *satisfying the following conditions:*

(i) for all x, y in X, k in $(0,1), t > 0$
(ii) (A, S) and (B, T) are weakly compatible,
(iii) (A, S) or (B, T) satisfies E.A. property,
(iv) $A(X) \subset T(X)$, $B(X) \subset S(X)$,
If one of A, B, S and T is a complete subspace of X then A, B, S and T have unique common fixed point in X.

Proof: Suppose the pair (B, T) satisfies the E.A. property. Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = p$ for some p in X. Since $B(X) \subset S(X)$, there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n = p$. Hence $\lim_{n\to\infty} Sy_n = p$. We shall show that $\lim_{n\to\infty} Ay_n = p$. From (i), we have
$M(Ay_n, Bx_n, kt) \geq M(Sy_n, Tx_n, t) * M(Ay_n, Sy_n, t) * M(By_n, Ty_n, t) *$
$M(Bx_n, Sy_n, 2t) * M(Ay_n, Tx_n, t)$ and

$N(Ay_n, Bx_n, kt) \leq N(Sy_n, Tx_n, t) \diamond N(Ay_n, Sy_n, t) \diamond N(Bx_n, Tx_n, t) \diamond$
$N(Bx_n, Sy_n, 2t) \diamond N(Ay_n, Tx_n, t)$,
Taking limit as $n \to \infty$, we get $M(Ay_n, p, kt) \geq M(Ay_n, p, t)$ and $N(Ay_n, p, kt) \leq N(Ay_n, p, t)$.
Using Lemma 11, we have $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sx_n = p$. Suppose that S(X) is a complete subspace of X. Then p = Su for some u in X. Subsequently, we have $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = p = Su$. Now, we shall show that Au = Su. From (4), we have
$M(Au, Bx_n, kt) \geq M(Su, Tx_n, t) * M(Au, Su, t) * M(Bx_n, Tx_n, t) * M(Bx_n, Su, 2t) * M(Au, Tx_n, t)$ and
$N(Au, Bx_n, kt) \leq N(Su, Tx_n, t) \diamond N(Au, Su, t) \diamond N(By_n, Tx_n, t) \diamond N(Bx_n, Su, 2t) \diamond N(Au, Tx_n, t)$,
Taking limit as $n \to \infty$ we get $M(Au, Su, kt) \geq M(Au, Su, t)$ and $N(Au, Su, kt) \leq N(Au, Su, t)$.
Now by using Lemma 11, we have Au = Su. Therefore (A, S) have coincidence point. The weak compatibility of A and S implies that ASu = SAu and thus AAu = ASu = SAu = SSu. As $A(X) \subset T(X)$, there exists v in X such that Au = Tv. We claim that Tv = Bv. From (i), we have M(Au, Bv, kt) $\geq M(Su, Tv, t) * M(Au, Su, t) * M(Bv, Tv, t) * M(Bv, Su, 2t) * M(Au, Tv, t)$ and $N(Au, Bv, kt) \leq N(Su, Tv, t) \diamond N(Au, Su, t) \diamond N(Bv, Tv, t) \diamond N(Bv, Su, 2t) \diamond N(Au, Tv, t)$, Now by using Lemma 11, we have Au = Bv. Hence, Tv = Bv. Thus we have Au = Su = Tv = Bv. The weak compatibility of B and T implies that BTv = TBv = TTv = BBv. Finally, we show that Au is the common fixed point of A, B, S and T. From (i), we have
$M(Au, AAu, kt) = M(AAu, Bv, kt) \geq M(SAu, Tv, t) * M(AAu, SAu, t) * M(Bv, Tv, t) * M(Bv, SAu, 2t) * M(AAu, Tv, t)$ and $N(Au, AAu, kt) = N(AAu, Bv, kt)$
$\leq N(SAu, Tv, t) \diamond N(AAu, SAu, t) \diamond N(Bv, Tv, t) \diamond N(Bv, SAu, 2t) \diamond N(AAu, Tv, t)$,
we have $M(Au, AAu, kt) = M(AAu, Bv, kt) \geq M(AAu, Bv, t)$
and $N(Au, AAu, kt) = N(AAu, Bv, kt) \leq N(AAu, Bv, t)$.
Now, the use of Lemma 11 gives AAu = Bv =Au and thus, AAu = Au. Therefore, Au = AAu = SAu is the common fixed point of A and S. Similarly, we prove that Bv is the common fixed point of B and T. Since Au = Bv, Au is common fixed point of A, B, S, and T. The proof is similar when T(X) is assumed to be a complete subspace of X. The cases in which A(X) or B(X) is a complete subspace of X are similar to the cases in which T(X) or S(X), respectively is complete subspace of X as $A(X) \subset T(X)$ and $B(X) \subset S(X)$. Finally now we show that the common fixed point is unique. If possible, let x_0 and y_0 be two common fixed points of A, B, S, and T. Then by condition (i),
$M(x_0, y_0, kt) = M(Tx_0, By_0, kt)$
$\geq M(Sx_0, Ty_0, t) * M(Ax_0, Sx_0, t) * M(By_0, Ty_0, t) * M(By_0, Sx_0, 2t) * M(Ax_0, Ty_0, t)$
and $N(x_0, y_0, kt) = N(Tx_0, By_0, kt)$
$\leq N(Sx_0, Ty_0, t) \diamond N(Ax_0, Sx_0, t) \diamond N(By_0, Ty_0, t) \diamond N(By_0, Sx_0, 2t) \diamond N(Ax_0, Ty_0, t)$,
By fixed point property and using intuitionistic fuzzy metric space, we get
$M(x_0, y_0, kt) \geq M(x_0, y_0, t)$ and $N(x_0, y_0, kt) \leq N(x_0, y_0, t)$.
This implies, by using Lemma 11 that $x_0 = y_0$. Therefore, the mappings A, B, S, and T have a unique common fixed point.

Example 5. Let $(X, M, N, *, \diamond)$ be a intuitionistic fuzzy metric space with X = $[0, 1]$, t-norm * and t-conorm $\diamond$ defined by $a * b = min\{a, b\}$ and $a \diamond b = max\{a, b\}$ where a, b in $[0, 1]$, respectively. Let (M, N) is the intuitionistic fuzzy set on $X^2 \times (0, \infty)$, defined

by M(x, y, t)= $(\exp(\frac{|x-y|}{t}))^{-1}$ for all $t > 0$, M(x, y, t)= 0 when $t = 0$ and N(x, y, t) =$[(\exp(\frac{|x-y|}{t}))-1][(\exp(\frac{|x-y|}{t}))^{-1}]$, N(x, y, t) = 1 when $t = 0$. Then it is well known that $(X, M, N, *, \diamond)$ is a intuitionistic fuzzy metric space. Let us define self maps A, B, S, and T on X such that Ax = $\frac{x}{64}$, Tx = $\frac{x}{2}$, Bx = $\frac{x}{32}$, Sx = $\frac{x}{4}$ then for $k \in [1/16, 1)$
$M(Ax, By, kt) = (\exp(\frac{|x/64-y/32|}{kt}))^{-1}$
$\geq(\exp(\frac{|x/4-y/2|}{t}))^{-1}$ = M(Sx, Tx, t)
$\geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(Bx, Ty, t) * M(By, Sx, 2t) * M(Ax, Ty, t)$ and
$N(Ax, By, kt)=[(\exp(\frac{|x/64-y/32|}{t}))-1][(\exp(\frac{|x/64-y/32|}{t}))^{-1}]$
$\leq[(\exp(\frac{|x/4-y/2|}{t}))-1][(\exp(\frac{|x/4-y/2|}{t}))^{-1}] = N(Sx, Ty, t)$
$\leq$ N(Sx, Ty, t) $\diamond$ N(Ax, Sx, t) $\diamond$ N(Bx, Ty, t) $\diamond$ N(By, Sx, 2t) $\diamond$ N(Ax, Ty, t).
Clearly,
(a) condition (i) of above theorem holds,
(b) for sequence $\{x_n\} = \{1/n\}$, pairs (A, S) and (B,T) satisfies E.A. property,
(c) $A(X) \subset T(X), B(X) \subset S(X)$,
(d) one of A(X), B(X),S(X) or T(X) is complete subsets of X,
(e) the pairs (A , S) and (B , T) are weakly compatible at x = 0 which is the coincident point of the maps A , B, S and T.
Thus all the conditions of Theorem 13 are satisfied and also x = 0 is the unique common fixed point of A, B, S and T.

REFERENCES

[1] M. Aamri, D. El Moutawakil: some new common fixed point theorems under strict contractive conditions, *J. Math. Anal. Appl.*,270, 181-188 (2002).
[2] C. Alaca, D. Turkoglu, and C. Yildiz, Fixed points in Intuitionistic fuzzy metric spaces,*Chaos, Solitons and Fractals*, 29, 1073-1078 (2006).
[3] K. Atanassov, Intuitionistic Fuzzy sets, *Fuzzy sets and system*, 20, 87-96(1986).
[4] D.Coker, An introduction to Intuitionistic Fuzzy topological spaces, *Fuzzy Sets and System*, 88, 81- 89 (1997).
[5] A. Grorge and P. Veeramani, On some results in fuzzy metric spaces, *Fuzzy Sets and Systems*, 64, 395- 399 (1994).
[6] G. Jungck,Commuting mappings and fixed points, *Amer. Math. Monthly*, 83, 261-263 (1976).
[7] I. Kramosil and J. Michalek, Fuzzy metric and Statistical metric spaces,*Kybernetica*, 11, 326-334 (1975).
[8] K. Menger, Statistical metrices, *Proc. Nat. Acad. Sci. (USA)*,28 (1942).
[9] J. H. Park, Intuitionistic fuzzy metric spaces, *Chaos, Solitons and Fractals*, 22, 1039-1046 (2004).
[10] J. S. Park, Y. C. Kwun, and J. H. Park, A fixed point theorem in the Intuitionistic fuzzy metric spaces, *Far East J. Math. Sci.*, 16, 137-149 (2005).
[11] R. Saadati and J. H. Park, On the Intuitionistic fuzzy topological spaces, *Chaos, Solitons and Fractals* 27, 331-344 (2006).
[12] D. Turkoglu, C. Alaca and C. Yildiz, Common fixed point theorems of compatible maps in intuitionistic fuzzy metric spaces, *Southeast asian Bulletin of Mathematics*, 32, 21-33 (2008).

Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 42 (2010) pp. 57-65

On γ-Semi-Continuous Functions

Sabir Hussain
Department of Mathematics
Islamia University Bahawalpur, Pakistan.
Present Address: Department of Mathematics,
Yanbu University, P. O. Box 31387,
Yanbu Alsinaiyah , Saudi Arabia.
Email: sabiriub@yahoo.com.

Bashir Ahmad
Department of Mathematics
King Abdul Aziz University, P. O. Box 80203,
Jeddah 21589, Saudi Arabia.
Email: drbashir9@gmail.com.

Takashi Noiri
2949-1 Shiokita-cha, Hinagu, Yatsushiro-shi,
Kumamoto-ken, 869-5142, Japan.
Email: t.noiri@nifty.com.

Abstract. In this paper, we continue studying the properties of γ-semi-continuous and γ-semi-open functions introduced in [5].

AMS (MOS) Subject Classification Codes: Primary 54A05, 54A10, 54D10.
Key Words: γ-closed (open), γ-closure , γ^*-semi-closed (open), γ^*-semi-closure, γ-regular, γ^*-semi-interior, γ-semi-continuous function, γ-semi-open (closed) function.

1. Introduction

N. Levine [11] introduced the notion of semi-open sets in topological spaces. A. Csaszar [7,8] defined generalized open sets in generalized topological spaces. In 1975, Maheshwari and Prasad [12] introduced concepts of semi-T_1-spaces and semi-R_0-spaces. In 1979, S. Kasahara [10] defined an operation α on topological spaces. Carpintero, et. al [6] introduced the notion of α-semi-open sets as a generalization of semi-open sets. B. Ahmad and F.U. Rehman [1, 14] introduced the notions of γ-interior, γ-boundary and γ-exterior points in topological spaces. They also studied properties and characterizations of (γ ,β)-continuous mappings introduced by H. Ogata [13]. In [2-4], B. Ahmad and S. Hussain introduced the concept of γ_0-compact, γ^*-regular, γ-normal spaces and explored their many interesting properties. They initiated and discussed the concepts of γ^*-semi-open sets , γ^*-semi-closed sets, γ^*-semi-closure, γ^*-semi-interior points in topological spaces [5,9]. In [9], they introduced Λ_s^γ-set and Λ^{s^γ}-set by using γ^*-semi-open sets. Moreover, they also introduced γ-semi-continuous function and γ-semi-open (closed) functions in topological spaces and established several interesting properties.

In this paper, we continue studying the properties of γ-semi-continuous functions and γ-semi-open function introduced by B. Ahmad and S. Hussain [5].

Hereafter, we shall write space in place of topological space in the sequel.

2. PRELIMINARIES

We recall some definitions and results used in this paper to make it self-contained.

Definition 1. [13] Let (X,τ) be a space. An operation $\gamma : \tau \rightarrow$ P(X) is a function from τ to the power set of X such that V $\subseteq V^\gamma$, for each V $\in \tau$, where V^γ denotes the value of γ at V. The operations defined by γ(G) = G, γ(G) = cl(G) and γ(G) = intcl(G) are examples of operation γ.

Definition 2. [13] Let A be a subset of a space X. A point x $\in$ A is said to be γ-interior point of A, if there exists an open nbd N of x such that $N^\gamma \subseteq$ A. The set of all such points is denoted by int_γ(A). Thus

$$int_\gamma\,(A) = \{x \in A : x \in N \in \tau \text{ and } N^\gamma \subseteq A\} \subseteq A.$$

Note that A is γ-open [13] iff A =int_γ(A). A set A is called γ- closed [13] iff X-A is γ-open.

Definition 3. [10] A point x $\in$ X is called a γ-closure point of A $\subseteq$ X, if $U^\gamma \cap A \neq \phi$, for each open nbd U of x. The set of all γ-closure points of A is called γ-closure of A and is denoted by cl_γ(A). A subset A of X is called γ-closed, if $cl_\gamma(A)$ $\subseteq$A. Note that cl_γ(A) is contained in every γ-closed superset of A.

Definition 4. [14] The γ-exterior of A, written $ext_\gamma(A)$ is defined as the γ-interior of $(X - A)$. That is, $int_\gamma(A) = ext_\gamma(X - A)$.

Definition 5. [14] The γ-boundary of A, written $bd_\gamma(A)$ is defined as the set of points which do not belong to γ-interior or the γ-exterior of A.

Definition 6. [13] An operation γ on τ is said be regular, if for any open nbds U,V of x $\in$ X, there exists an open nbd W of x such that $U^\gamma \cap V^\gamma \supseteq W^\gamma$.

Definition 7. [13] An operation γ on τ is said to be open, if for every open nbd U of each $x \in X$, there exists γ-open set B such that $x \in B$ and $U^\gamma \subseteq B$.

Definition 8. [2] Let $A \subseteq X$. A point $x \in X$ is said to be γ-limit point of A, if $U \cap \{A - \{x\}\} \neq \phi$, where U is a γ-open set in X. The set of all γ-limit points of A denoted A^d_γ is called γ-derived set.

Definition 9. [9] A subset A of a space (X,τ) is said to be a γ^*-semi-open set, if there exists a γ -open set O such that $O \subseteq A \subseteq cl_\gamma(O)$. The set of all γ^*-semi-open sets is denoted by $SO_{\gamma^*}(X)$.

Definition 10. [5] A function $f : (X, \tau) \rightarrow (Y, \tau)$ is said to be γ-semi-continuous, if for any γ-open B of Y, $f^{-1}(B)$ is γ^*-semi-open in X.

Definition 11. [5] A function $f : (X, \tau) \rightarrow (Y, \tau)$ is said to be γ-semi-open (closed), if for each γ-open (closed) set U in X, $f(U)$ is γ^*-semi-open (closed) in Y.

Definition 12. [5] A set A in a space X is said to be γ^*-semi-closed, if there exists a γ-closed set F such that $int_\gamma(F) \subseteq A \subseteq F$.

Proposition 13. [5] *A subset A of X is γ^*-semi-closed if and only if $X - A$ is γ^*-semi-open.*

Definition 14. A subset A of X is said to be γ-semi-nbd of a point $x \in X$, if there exists a γ^*-semi-open set U such that $x \in U \subseteq A$.

Definition 15. [9] Let A be a subset of space X . The intersection of all γ^*-semi-closed sets containing A is called γ^*-semi-closure of A and is denoted by $scl_{\gamma^*}(A)$. Note that A is γ^*-semi-closed if and only if $scl_{\gamma^*}(A) = A$.

Definition 16. [5] Let A be a subset of a space X . The union of all γ^*-semi-open sets of X contained in A is called γ^*-semi-interior of A and is denoted by $sint_{\gamma^*}(A)$.

Lemma 17. *Let A be a subset of a space X. Then $x \in scl_{\gamma^*}(A)$ if and only if for any γ-semi-nbd N_x of x in X, $A \cap N_x \neq \phi$.*

Proof. Let $x \in scl_{\gamma^*}(A)$ .Suppose on the contrary, there exists a γ-semi-nbd N_x of x in X such that $A \cap N_x \neq \phi$. Then there exists $U \in SO_{\gamma^*}(A)$ such that $x \in U \subseteq N_x$. Therefore, $U \cap A = \phi$, so that $A \subseteq X - U$. Clearly $X - U$ is γ^*-semi-closed in X and hence $scl_{\gamma^*}(A) \subseteq X - U$. Since $x \notin X - U$, we obtain $x \notin scl_{\gamma^*}(A)$. This is contradiction to the hypothesis. This proves the necessity.

Conversely, suppose that every γ-semi-nbd of x in X meets A. If $x \notin scl_{\gamma^*}(A)$, then by definition there exists a γ^*-semi-closed F of X such that $A \subseteq F$ and $x \notin F$. Therefore we have $x \in X - F \in SO_{\gamma^*}(X)$. Hence $X - F$ is γ-semi-nbd of x in X. But $(X - F) \cap A = \phi$. This is contradiction to the hypothesis. Thus $x \in scl_{\gamma^*}(A)$. □

3. γ-Semi-Open Functions

Theorem 18. *Let $f : X \to Y$ be a function from a space X into a space Y and γ is an open, monotone and regular operation. Then the following statements are equivalent:*
(1) f is γ-semi-open.
(2) $f(int_\gamma(A)) \subseteq sin_{\gamma^*}(f((A))$ for each subset A of X.
(3) For each $x \in X$ and each γ-open-nbd U of x, there exists a γ-semi-nbd V of $f(x)$ such that $V \subseteq f(U)$.

Proof. (1) $\Rightarrow$ (2). Suppose that f is γ-semi-open, and let A be an arbitrary subset of X. Since $f(int_\gamma(A))$ is γ^*-semi-open and $f(int_\gamma(A)) \subseteq f(A)$, then $f(int_\gamma(A)) \subseteq sin_{\gamma^*}(f((A))$.

(2) $\Rightarrow$ (3). Let U be an arbitrary γ-open-nbd of $x \in X$. Then there exists γ-open set O such that $x \in O \subseteq U$. By hypothesis, we have $f(O) = f(int_\gamma(O)) \subseteq sin_{\gamma^*}(f((O))$ and hence $f(O) \subseteq sin_{\gamma^*}(f((O))$. Therefore it follows that $f(O)$ is γ-semi-open-nbd in Y such that $f(x) \in f(O) \subseteq f(U)$. This proves (3).

(3) $\Rightarrow$ (1). Let U be an arbitrary γ-open set in X. For each $y \in f(U)$, by hypothesis there exists a γ-semi-nbd V_y of y in Y such that $V_y \subseteq f(U)$. Since V_y is a γ-semi-nbd of y, there exists a γ^*-semi-open set A_y in Y such that $y \in A_y \subseteq V_y$. Therefore $f(U) = \bigcup\{A_y : y \in f(U)\}$ is a γ^*-semi-open in Y, since is γ regular [9]. This shows that f is a γ-semi-open function. □

Theorem 19. *A bijective function $f : X \to Y$ is γ-semi-open if and only if $f^{-1}(scl_{\gamma^*}(B)) \subseteq cl_\gamma(f^{-1}(B))$ for every subset B of Y, where γ is an open operation.*

Proof. Let B be an arbitrary subset of Y. Put

$$U = X - cl_\gamma(f^{-1}(B)) \tag{3. 1}$$

Clearly U is a γ-open set in X. Then by hypothesis, $f(U)$ is a γ^*-semi-open set in Y, or $Y - f(U)$ is γ^*-semi-closed set in Y. Since f is onto, from (3. 1), it follows $B \subseteq Y - f(U)$. Thus we have $scl_{\gamma^*}(B) \subseteq Y - f(U)$. Since f is one-one, we have $f^{-1}(scl_{\gamma^*}(B)) \subseteq f^{-1}(Y) - f^{-1}f(U) = X - f^{-1}f(U) \subseteq X - U = cl_\gamma(f^{-1}(B))$. This proves the necessity.

Conversely, let U be an arbitrary -open set in X. Put $B = Y - f(U)$. Since f is bijective, therefore by hypothesis, $f(U) \cap scl_{\gamma^*}(B) = f(U \cap f^{-1}(scl_{\gamma^*}(B))) \subseteq f(U \cap cl_\gamma(f^{-1}(B)))$. Since U is γ-open, therefore by Lemma 2(3) [14], we have $U \cap cl_\gamma(f^{-1}(B)) \subseteq cl_\gamma(U \cap f^{-1}(B))$. Moreover, it is obvious that $U \cap f^{-1}(B) = \phi$. Thus we have $f(U) \cap scl_{\gamma^*}(B) = \phi$ and hence $scl_{\gamma^*}(B) \subseteq Y - f(U) = B$. Therefore B is a γ^*-semi-closed in Y and hence $f(U)$ is a γ^*-semi-open set in Y. This proves that f is a γ-semi-open mapping. □

Definition 20. [13] A function $f : (X, \tau, \gamma) \to (Y, \delta, \beta)$ is said to be (γ, β)-continuous, if for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U such that $x \in U$ and $f(U^\gamma) \subseteq V^\beta$, where γ and β are operations on τ and δ respectively.

Definition 21. [13] A function $f : (X, \tau, \gamma) \to (Y, \delta, \beta)$ is said to be (γ, β)-open (closed), if for any γ-open (closed) set A of X, $f(A)$ is γ-open (closed) in Y.

Theorem 22. [1] *Let $f : (X, \tau, \gamma) \to (Y, \delta, \beta)$ be a function and β be an open operation on Y. Then f is (γ, β)-continuous if and only if for each β-open set V in Y, $f^{-1}(V)$ is γ-open in X.*

Theorem 23. [1] *Let $f : (X, \tau, \gamma) \to (Y, \delta, \beta)$ be a function and β be an open operation on Y. Then the following are equivalent:*
(1) f is (γ, β)-open .
(2) $f^{-1}(cl_\beta(B)) \subseteq cl_\gamma(f^{-1}(B))$.
(3) $f^{-1}(bd_\beta(B)) \subseteq bd_\gamma(f^{-1}(B))$ for any subset B of Y.

Theorem 24. *If a function $f : (X, \tau, \gamma) \to (Y, \delta, \beta)$ is a (γ, β)-open and a (γ, β)-continuous, then the inverse image $f^{-1}(B)$ of each β^*-semi-open set B in Y is γ^*-semi-open in X, where β is an open operation on Y.*

Proof. Let B be an arbitrary β^*-semi-open set in Y. Then there exists β-open set V in Y such that $V \subseteq B \subseteq cl_\beta(V)$. Since f is (γ, β)-open, using Theorem 23, we have $f^{-1}(V) \subseteq f^{-1}(B) \subseteq f^{-1}(cl_\beta(V)) \subseteq cl_\gamma(f^{-1}(V))$. Since f is (γ, β)-continuous and V is β-open in Y, by Theorem 22, $f^{-1}(V)$ is γ-open in X. This shows that $f^{-1}(B)$ is γ^*-semi-open set in X. □

Theorem 25. *Let X, Y and Z be three spaces and let $f : X \to Y$ be a function, $g : Y \to Z$ be an injective function and $gof : X \to Z$ is a γ-semi-open function. Then we have:*
(1) If f is (γ, β)-continuous and surjective, then g is γ-semi-open.
(2) If g is (β, α)-open, (β, α)-continuous and injective, then f is γ-semi-open, where β is open operation on Y.

Proof. (1) Let V be a β-open set in Y. Then $f^{-1}(V)$ is γ-open in X, because f is (γ, β)-continuous. Since gof is γ-semi-open and f is surjective, therefore

$g(V) = (gof)(f^{-1}(V))$ is α^*-semi-open in Z. This shows that g is a γ-semi-open function.
(2) Since g is injective, therefore for $A \subseteq X$, $f(A) = g^{-1}(g(f(A)))$. Let U be a γ-open set in X, then $gof(U)$ is α^*-semi-open. Thus by Theorem 24, $g^{-1}(g(f(U))) = f(U)$ is β^*-semi-open in Y. This shows that f is a γ-semi-open function. □

Let $B \subseteq X$, $\gamma : \tau \to P(X)$ be an operation. We define $\gamma_B : \tau_B \to P(X)$ as $\gamma_B(U \cap B) = \gamma(U) \cap B$. From here γ_B is an operation and satisfies that $cl_{\gamma_B}(U \cap B) \subseteq cl_\gamma(U \cap B) \subseteq cl_\gamma(U) \cap cl_\gamma(B)$. Using this fact we prove the following:

Theorem 26. *Let X be a space and B a γ^*-semi-open set in X containing a subset A of X. If A is γ^*-semi-open in the subspace B, then A is γ^*-semi-open in X, where γ is a regular operation.*

Proof. Let A be γ_B^*-semi-open in the subspace B. Then there exists a γ_B-open set U_B in B such that $U_B \subseteq A \subseteq cl_{\gamma_B}(U_B)$. Since U_B is γ_B-open in B, there exists a γ-open set U in X such that $U_B = U \cap B$[4]. Thus we have $U \cap B \subseteq A \subseteq cl_{\gamma_B}(U \cap B) \subseteq cl_\gamma(U \cap B) = cl_\gamma(A) \cap cl_\gamma(B)$. Since B is γ^*-semi-open set in X and U is γ-open in X, therefore $U \cap B$ is γ-open in X. Consequently, A is a γ^*-semi-open set in X. □

Theorem 27. *Let X and Y be spaces. If a bijective function $f : X \to Y$ is a γ-semi-open, then for each γ-open set $V (\neq \phi)$ in Y $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is γ-semi-open, where γ is a regular operation.*

Proof. Let U_V be an arbitrary $\gamma_{f^{-1}(V)}$-open set in $f^{-1}(V)$. Then there exists a γ-open set U in X such that $U_V = U \cap f^{-1}(V)$. Now we have $[f|_{f^{-1}(V)}](U_V) = f(U \cap f^{-1}(V)) = f(U) \cap V$. Since $f(U)$ is γ^*-semi-open and V is γ-open, $f(U) \cap V$ is γ^*-semi-open. Hence $[f|_{f^{-1}(V)}](U_V)$ is also γ_V^*-semi-open in V. This shows that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is a γ-semi-open mapping. □

Theorem 28. *A bijective function $f : X \to Y$ is γ-semi-open if and only if for any subset V of Y and for any γ-closed set F of X containing $f^{-1}(V)$, there exists a γ^*-semi-closed set G of Y containing V such that $f^{-1}(G) \subseteq F$.*

Proof. Let $V \subseteq Y$ and F be a γ-closed set of X containing $f^{-1}(V)$. Put $G = Y - f(X - F)$. Since f is γ-semi-open, so G is γ^*-semi-closed sets in Y . As f is bijective, it follows from $f^{-1}(V) \subseteq F$ that $V \subseteq G$. Calculations give $f^{-1}(G) \subseteq F$.

Conversely, suppose U is γ-open set. Put $V = Y - f(U)$. Then $X - U$ is γ-closed set in X containing $f^{-1}(V)$. By hypothesis, there exists a γ^*-semi-closed set G of Y such that $V \subseteq G$ and $f^{-1}(G) \subseteq (X - U)$. On the other hand, it follows from $V \subseteq G$ that $f(U) = (Y - V) \subseteq (Y - G)$. Therefore, we obtain $f(U) = (Y - G) \in SO_{\gamma^*}(Y)$. This shows that f is γ-semi-open. □

Lemma 29. [5]*The following properties of a subset A of X are equivalent:*
(1) A is γ^*-semi-closed.
(2) $int_\gamma(cl_\gamma(A)) \subseteq A$.
(3) $X - A$ is γ^*-semi-open.

Theorem 30. *If $f : X \to Y$ is (γ, β)-open and (γ, β)-continuous mapping, then the inverse image $f^{-1}(B)$ of each β^*-semi-closed B in Y is γ^*-semi-closed in X, where β is an open operation on Y.*

Proof. This follows from Theorem 24 and Lemma 29. □

Theorem 31. *Let $f : X \to Y$ be surjective and $g : Y \to Z$ be an injective function and let $gof : X \to Z$ be a γ-semi-closed function. Then*
(1) If f is (γ, β)-continuous and surjective, then g is β-semi-closed.
(2) If g is (β, α)-open, (β, α)-continuous and injective, then f is γ-semi-closed, where β is an open operation on Y.

Proof. (1) Suppose H is an arbitrary β-closed set in Y. Then $f^{-1}(H)$ is γ-closed in X because f is (γ, β)-continuous. Since gof is γ-semi-closed and f is surjective, $gof(f^{-1}(H)) \subseteq g(f(f^{-1}(H))) = g(H)$, is α^*-semi-closed in Z. This implies that g is β-semi-open function. This proves (1).
(2) Since g is injective so for every subset A of X, $f(A) = g^{-1}(g(f(A)))$. Let F be an arbitrary γ-closed set in X. Then $gof(F)$ is γ^*-semi-closed. It follows immediately from Theorem 30 that $f(F)$ is γ^*-semi-closed set in Y. This implies that f is γ-semi-closed. □

4. γ-Semi-Closed Functions

Theorem 32. *Let γ be an open and monotone operation. A function $f : X \to Y$ is γ-semi-closed if and only if $f(cl_\gamma(A)) \supseteq int_\gamma(cl_\gamma(f(A)))$ for every subset A of X.*

Proof. Suppose f is a γ-semi-closed mapping and A is an arbitrary subset of X. Then $f(cl_\gamma(A))$ is γ^*-semi-closed in Y. Then by Lemma 29, we obtain
$f(cl_\gamma(A)) \supseteq int_\gamma(cl_\gamma(f(cl_\gamma(A)))) \supseteq int_\gamma(cl_\gamma(f(A)))$. This implies that $f(cl_\gamma(A)) \supseteq int_\gamma(cl_\gamma(f(A)))$.

Conversely, suppose that F is an arbitrary γ-closed set in X. Then by hypothesis, we have $int_\gamma(cl_\gamma(f(F))) \subseteq f(cl_\gamma(F)) = f(F)$. By Lemma 29, $f(F)$ is γ^*-semi-closed in Y. This implies that f is γ-semi-closed. □

Recall [9] that the intersection of all γ^*-semi-closed sets containing A is called γ-semi-closure of A and is denoted by $scl_{\gamma^*}(A)$. Clearly A is γ^*-semi-closed if and only if $scl_{\gamma^*}(A) = A$.

Theorem 33. *Let γ be an open and monotone operation. A function $f : X \to Y$ is γ-semi-closed if and only if $scl_{\gamma^*}(A) \subseteq f(cl_\gamma(A))$ for every subset A of X.*

Proof. Suppose f is a γ-semi-closed mapping and A is an arbitrary subset of X. Then $f(cl_\gamma(A))$ is γ^*-semi-closed. Since $f(A) \subseteq f(cl_\gamma(A))$, we obtain
$scl_{\gamma^*}(f(A)) \subseteq f(cl_\gamma(A))$. This implies $scl_{\gamma^*}(f(A)) \subseteq f(cl_\gamma(A))$. □

Sufficiency follows from Theorem 32.

Theorem 34. *A surjective function $f : X \to Y$ is γ-semi-closed if and only if for each subset B in Y and each γ-open set U in X containing $f^{-1}(B)$, there exists a γ^*-semi-open set V in Y containing B such that $f^{-1}(V) \subseteq U$, where γ is a monotone and regular operation.*

Proof. Suppose B is an arbitrary subset in Y and U is an arbitrary γ-open set in X containing $f^{-1}(B)$. We put

$$V = Y - f(X - U) \tag{4. 2}$$

Then V is γ^*-semi-open set in Y. Since $f^{-1}(B) \subseteq U$, calculations give $B \subseteq V$. Moreover, by 4. 2 , we have $f^{-1}(V) = f^{-1}(Y) - f^{-1}(f(X - U)) = X - f^{-1}(f(X - U)) \subseteq X - (X - U) = U$.

Conversely, suppose that F is an arbitrary γ-closed set in X. Let y be an arbitrary point in $Y - f(F)$, then $f^{-1}(y) \subseteq X - f^{-1}(f(F)) \subseteq X - F$, and $X - F$ is γ-open in

X. Hence by the hypothesis, there exists a γ^*-semi-open set V_y containing y such that $f^{-1}(V_y) \subseteq X - F$. This implies that $v \in V_y \subseteq Y - f(F)$. We obtain that $Y - f(F) = \bigcup\{V_y : y \in Y - f(F)\}$ is γ^*-semi-open in Y, since union of any collection of γ^*-semi-open sets is γ^*-semi-open. Therefore $f(F)$ is γ^*-semi-closed. □

5. γ-Semi-Continuous Functions

Theorem 35. *Let $f : X \to Y$ be a function and γ is an open operation. Then the following are equivalent:*
(1) f is γ-semi-continuous.
(2) $int_\gamma(cl_\gamma(f^{-1}(B))) \subseteq f^{-1}(cl_\gamma(B))$ for each subset B of Y.
(3) $f(int_\gamma(cl_\gamma(A))) \subseteq cl_\gamma(f(A))$ for each subset A of X.

Proof. (1) $\Rightarrow$ (2). Let B be an arbitrary subset of Y. Then by (1), $f^{-1}(cl_\gamma(B))$ is a γ^*-semi-closed set of X. Since $B \subseteq cl_\gamma(B)$, by Lemma 29, we get $f^{-1}(cl_\gamma(B)) \supseteq int_\gamma(cl_\gamma(f^{-1}(cl_\gamma(B)))) \supseteq int_\gamma(cl_\gamma(f^{-1}(B)))$implies that
$int_\gamma(cl_\gamma(f^{-1}(B))) \subseteq f^{-1}(cl_\gamma(B))$.

(2) $\Rightarrow$ (3). Let A be an arbitrary subset of X. Put $B = f(A)$. Then $A \subseteq f^{-1}(B)$. Therefore by hypothesis, we have $int_\gamma(cl_\gamma(A)) \subseteq int_\gamma(cl_\gamma(f^{-1}(B))) \subseteq f^{-1}(cl_\gamma(B))$. Consequently, we have $f(int_\gamma(cl_\gamma(A))) \subseteq ff^{-1}(cl_\gamma(B)) \subseteq cl_\gamma(B) = cl_\gamma(f(A))$. This gives (3).

(3) $\Rightarrow$ (1). Let F be an arbitrary γ-closed set of Y. Put $A = f^{-1}(F)$, then $f(A) \subseteq F$. Therefore by hypothesis, we have

$$f(int_\gamma(cl_\gamma(A))) \subseteq cl_\gamma(f(A)) \subseteq cl_\gamma(F) = F \tag{5. 3}$$

By 5. 3 , we have $int_\gamma(cl_\gamma(A)) \subseteq f^{-1}f(int_\gamma(cl_\gamma(A))) \subseteq f^{-1}(cl_\gamma(f(A))) \subseteq f^{-1}(cl_\gamma(F)) = f^{-1}(F)$, or $int_\gamma(cl_\gamma(A)) \subseteq f^{-1}(F)$,. By Lemma 29, $f^{-1}(F)$ is a γ^*-semi-closed set in X. This implies that f is γ-semi-continuous. □

Definition 36. Let X be a space $A \subseteq X$ and $p \in X$. Then p is a γ^*-semi-limit point of A, for all γ^*-semi-open set U containing p such that $U \cap (A - \{p\}) \neq \phi$. The set of all γ^*-semi-limit point of A is said to be γ^*-semi-derived set of A and is denoted by $sd_{\gamma^*}(A)$.

Clearly if $A \subseteq B$ then

$$sd_{\gamma^*}(A) \subseteq sd_{\gamma^*}(B) \tag{5. 4}$$

Remark 37. From the definition, it follows that p is a γ^*-semi-limit point of A if and only if $p \in scl_{\gamma^*}(A - \{p\})$.

Theorem 38. *The γ^*-semi-derived set, sd_{γ^*}, has the following properties:*
(1) $scl_{\gamma^*}(A) = A \cup sd_{\gamma^*}(A)$.
(2) $sd_{\gamma^*}(A \cup B) = sd_{\gamma^*}(A) \cup sd_{\gamma^*}(B)$. In general
(3) $\bigcup_i sd_{\gamma^*}(A_i) = sd_{\gamma^*}(\bigcup_i(A_i))$.
(4) $sd_{\gamma^*}(sd_{\gamma^*}(A)) \subseteq sd_{\gamma^*}(A)$.
(5) $scl_{\gamma^*}(sd_{\gamma^*}(A)) = sd_{\gamma^*}(A)$.

Proof. (1) Let $x \in scl_{\gamma^*}(A)$. Then $x \in C$, for every γ^*-semi-closed superset C of A. Now
(i) If $x \in A$, then $x \in A \cup sd_{\gamma^*}(A)$.
(ii) If $x \notin A$, then we prove that $x \in scl_{\gamma^*}(A)$.
To prove (ii), suppose U is γ^*-semi-open set containing x. Then $U \cap A \neq \phi$, for otherwise, $A \subseteq X - U = C$, where C is a γ^*-semi-closed superset of A not containing x. This contradicts the fact that x belongs to every γ^*-semi-closed superset C of A. Therefore $x \in sd_{\gamma^*}(A)$ gives $x \in A \cup sd_{\gamma^*}(A)$.

Conversely, suppose that $x \in A \cup sd_{\gamma^*}(A)$, we show that $x \in scl_{\gamma^*}(A)$. If $x \in A$ then $x \in scl_{\gamma^*}(A)$. If $x \in sd_{\gamma^*}(A)$, then we show that x is in every γ^*-semi-closed superset of A. We suppose otherwise that there is γ^*-semi-closed superset C of A not containing x. Then $x \in X - C = U$(say), which is γ^*-semi-open and $U \cap A = \phi$. This implies that $x \notin sd_{\gamma^*}(A)$. This contradiction proves that $x \in scl_{\gamma^*}(A)$. Consequently $scl_{\gamma^*}(A) = A \cup sd_{\gamma^*}(A)$. This proves (1).

(2) $sd_{\gamma^*}(A \cup B) \subseteq sd_{\gamma^*}(A) \cup sd_{\gamma^*}(B)$.
Let $x \in sd_{\gamma^*}(A \cup B)$. Then $x \in scl_{\gamma^*}((A \cup B) - \{x\})$ or $x \in scl_{\gamma^*}((A - \{x\}) \cup (B - \{x\})$ implies $x \in scl_{\gamma^*}(A - \{x\})$ or $x \in scl_{\gamma^*}(B - \{x\})$. This gives $x \in sd_{\gamma^*}(A)$ or $x \in sd_{\gamma^*}(B)$. Therefore $x \in sd_{\gamma^*}(A) \cup sd_{\gamma^*}(B)$. This proves $sd_{\gamma^*}(A \cup B) \subseteq sd_{\gamma^*}(A) \cup sd_{\gamma^*}(B)$

Converse follows directly by using the property 5. 4 .

(3) The proof is immediate by property 5. 4 .

(4) Suppose that $x \notin sd_{\gamma^*}(A)$. Then $x \notin scl_{\gamma^*}(A - \{x\})$. This implies that there is γ^*-semi-open set U such that $x \in U$ and $U \cap (A - \{x\}) = \phi$. We prove that $x \notin sd_{\gamma^*}(sd_{\gamma^*}(A))$. Suppose on the contrary that $x \in sd_{\gamma^*}(sd_{\gamma^*}(A))$. Then $x \in scl_{\gamma^*}(sd_{\gamma^*}(A) - \{x\})$. Since $x \in U$, we have $U \cap (sd_{\gamma^*}(A) - \{x\}) \neq \phi$. Therefore there is a $q \neq x$ such that $q \in U \cap (sd_{\gamma^*}(A))$. It follows that $q \in (U - \{x\}) \cap (sd_{\gamma^*}(A) - \{x\})$. Hence $(U - \{x\}) \cap (sd_{\gamma^*}(A) - \{x\}) \neq \phi$, a contradiction to the fact that $(U \cap (sd_{\gamma^*}(A) - \{x\}) = \phi$. This implies that $x \notin sd_{\gamma^*}(sd_{\gamma^*}(A))$ and so $sd_{\gamma^*}(sd_{\gamma^*}(A)) \subseteq sd_{\gamma^*}(A)$. This proves (4).

(5) This is a consequence of (1), (2) and (4). □

Theorem 39. [5] *Let $f : X \to Y$ be a function. Then the following are equivalent:*
(1) $f : X \to Y$ is γ-semi-continuous.
(2) $scl_{\gamma^*}(f^{-1}(A)) \subseteq f^{-1}(cl_\gamma(A))$ for each subset A of Y.

Theorem 40. *Let $f : X \to Y$ be a function and γ is an open operation. Then the following are equivalent:*
(1) $f : X \to Y$ is γ-semi-continuous.
(2) $f(sd_{\gamma^*}(A)) \subseteq cl_\gamma(f(A))$ for any subset A of X.

Proof. (1) $\Rightarrow$ (2). Suppose that f is γ-semi-continuous. Let A be any set in X. Since $cl_\gamma(f(A))$ is γ-closed in Y. $f^{-1}(cl_\gamma(A))$ is γ^*-semi-closed in X. $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(cl_\gamma(f(A)))$ gives $scl_{\gamma^*}(A) \subseteq scl_{\gamma^*}(f^{-1}(cl_\gamma(f(A)))) = f^{-1}(cl_\gamma(f(A)))$. Therefore $f(sd_{\gamma^*}(A)) \subseteq f(scl_{\gamma^*}(A)) \subseteq ff^{-1}(cl_\gamma(f(A))) \subseteq cl_\gamma(f(A))$. Consequently, $f(sd_{\gamma^*}(A)) \subseteq cl_\gamma(f(A))$.

(2) $\Rightarrow$ (1). Suppose that $f(sd_{\gamma^*}(A)) \subseteq cl_\gamma(f(A))$, for $A \subseteq X$. Let B be any γ-closed subset of Y. We show that $f^{-1}(B)$ is γ^*-semi-closed in X. By hypothesis, $f(sd_{\gamma^*}(f^{-1}(B))) \subseteq cl_\gamma(f(f^{-1}(B))) \subseteq cl_\gamma(B) = B$ or $f(sd_{\gamma^*}(f^{-1}(B))) \subseteq B$ gives $sd_{\gamma^*}(f^{-1}(B)) \subseteq f^{-1}f(sd_{\gamma^*}(f^{-1}(B))) \subseteq f^{-1}(B)$ or $sd_{\gamma^*}(f^{-1}(B)) \subseteq f^{-1}(B)$ implies $f^{-1}(B)$ is γ^*-semi-closed in X. Thus f is γ-semi-continuous. □

Theorem 41. [5] *Let $f : X \to Y$ be a function and $x \in X$. Then f is γ-semi-continuous if and only if for each γ-open set B containing f(x), there exists $A \in SO_{\gamma^*}(X)$ such that $x \in A$ and $f(A) \subseteq B$, where γ is a regular operation.*

We use Theorem 41 and prove the following:

Theorem 42. *Let $f : X \to Y$ be an injective function. If f is γ-semi-continuous then $f(sd_{\gamma^*}(A)) \subseteq (f(A))^d_\gamma$ for every $A \subseteq X$, where γ is a regular operation.*

Proof. Suppose that f is γ-semi-continuous. Let $A \subseteq X, a \in sd_{\gamma^*}(X)$ and V be a γ-open-nbd of $f(a)$. Since f is γ-semi-continuous then by Theorem 41, there exists a γ-semi-open-nbd U of a such that $f(U) \subseteq V$. But $a \in sd_{\gamma^*}(A)$, therefore there exists an element $a_1 \in U \cap A$ such that $a \neq a_1$; then $f(a_1) \in f(A)$ and since f is an injection $f(a) \neq f(a_1)$. Thus every γ-open-nbd V of $f(a)$ contains an element $f(a_1)$ of $f(A)$ different from $f(a)$. Consequently $f(a) \in (f(A))^d_\gamma$. We have therefore, $f(sd_{\gamma^*}(A)) \subseteq (f(A))^d_\gamma$. □

The following theorem follows from Theorem 40:

Theorem 43. *Let $f : X \to Y$ be a function. If for every $A \subseteq X$, $f(sd_{\gamma^*}(A)) \subseteq (f(A))^d_\gamma$, then f is γ-semi-continuous, where γ is an open operation.*

Theorem 44. *A function $f : X \to Y$ is γ-semi-continuous if and only if $f^{-1}(int_\gamma(B)) \subseteq sint_{\gamma^*}(f^{-1}(B))))$, for each $B \subseteq Y$, where γ is a regular operation.*

Proof. For any $B \subseteq Y$, $int_\gamma(B) = Y - cl_\gamma(Y - B)$ [14]. This implies $f^{-1}(int_\gamma(B)) = f^{-1}(Y - cl_\gamma(Y - B)) = X - f^{-1}(cl_\gamma(Y - B))$. Since f is γ-semi-continuous, by Theorem 39 we have $scl_{\gamma^*}(f^{-1}(Y - B)) \subseteq f^{-1}(cl_\gamma(Y - B))$. Hence $f^{-1}(int_\gamma(B)) \subseteq X - scl_{\gamma^*}(f^{-1}(Y - B))$. Thus $f^{-1}(int_\gamma(B)) \subseteq X - scl_{\gamma^*}(X - f^{-1}(B))$. Hence $f^{-1}(int_\gamma(B)) \subseteq X - scl_{\gamma^*}(X - f^{-1}(B)) = sint_{\gamma^*}(f^{-1}(B))))$.

Conversely, let B be an arbitrary γ-open set in Y. Then $int_\gamma(B) = B$. By hypothesis $f^{-1}(B) = f^{-1}(int_\gamma(B)) \subseteq sint_{\gamma^*}(f^{-1}(B))$ implies $f^{-1}(B) \subseteq sint_{\gamma^*}(f^{-1}(B))$. But $sint_{\gamma^*}(f^{-1}(B)) \subseteq f^{-1}(B)$. Therefore, $f^{-1}(B) = sint_{\gamma^*}(f^{-1}(B))$. Thus $f^{-1}(B)$ is γ^*-semi-open. Consequently, f is γ-semi-continuous. □

References

[1] B. Ahmad and F. U. Rehman: *Operations on Topological Spaces-II, Math. Today,* 11 (1993),13-20.

[2] B. Ahmad and S. Hussain: *Properties of γ-Operations on Topological Spaces, Aligarh Bull. Math.,* 22 (1) (2003),45-51.

[3] B. Ahmad and S. Hussain: *γ- Convergence in Topological Spaces, Southeast Asian Bull. Math.,* 29 (5)(2005), 835-842.

[4] B. Ahmad and S. Hussain: *γ^*- regular and γ-normal Spaces, Math. Today,* 22 (1)(2006), 37-44.

[5] B. Ahmad and S. Hussain: *γ^*-Semi-Open Sets in Topological Spaces, Southeast Asian Bull. Math.,* 34(6)(2010),(In Press).

[6] C. Carpintero, E. Rosas, J. Vielma: *Operadores Asociados a uns Topologia Γ Sobre un Conjunto X y Nociones Conexas, Divulgaciones Mathematicas,* 6 (2)(1998), 139-144.

[7] A. Csaszar : *Generalized Open Sets, Acta Math. Hungar.,* 75(1997), 65-87.

[8] A. Csaszar : *Generalized Topology, Generalized Continuity, Acta Math. Hungar.,* 96(2002),351-357.

[9] S. Hussain, B. Ahmad and T. Noiri : *γ^*-Semi-Open Sets in Topological Spaces , Asian Eur. J. Math.,* 3(3)(2010), 427-433. DOI: 10.1142/S1793557110000337.

[10] S. Kasahara: *Operation- Compact Spaces, Math. Japon.,* 24(1979), 97-105.

[11] N. Levine: *Semi-Open Sets and Semi Continuity in Topological Spaces, Amer. Math. Monthly,* 70(1963), 36-41.

[12] S. N. Meheshwari and R. Prasad: *On R_0-spaces, Portugal Math.,* 34(1975), 213-217.

[13] H. Ogata : *Operations on Topological Spaces and Associated Topology , Math. Japon.,* 36 (I) (1991), 175-184.

[14] F.U. Rehman and B. Ahmad: *Operations on Topological Spaces-I, Math. Today,* 10(1992), 29-36.

Instructions for Author

1. The journal is meant for publication of research papers and review articles covering state of the art in a particular area of mathematical science.

2. Manuscripts should be in suitable form for publication. As far as possible, the use of complicated notations should be avoided. Illustrations and diagrams should be submitted on separate sheets and not included in the text. They should be of good quality.

3. The contributors are required to provide a soft copy (in pdf form) of the paper composed in Latex/Tex at the following addresses
   shahid_siddiqi@math.pu.edu.pk
   shahidsiddiqimath@yahoo.co.uk
   or the same may be sent in triplicate to the Chief Editor, PUJM, Department of Mathematics, University of the Punjab, Quaid-e-Azam Campus, Lahore 54590, Pakistan.
   If the paper is accepted, the source file of the paper in Latex/Tex AMS format will be required.

4. Reference should be given at the end of the paper and be referred by numbers in serial order on square brackets, e.g. [3]. Reference should be typed as follows:
   Reference to Paper:
   [3] C.S. Hoo, *BCI-algebras with conditions*, Math. Japonica 32, No. 5 (1987) 749-756.
   Reference to Book:
   B. Mitchel: *Theory of categories*, New York: Academic Press, 1965.

5. The decision to accept or reject a paper for publication in the Journal rests fully with the Editorial Board.

6. Authors, whose papers will be published in the Journal, will receive free reprints of their papers and a copy of the issue containing their contributions.

7. The Journal which is published annually will be supplied free of cost in exchange with other Journals of Mathematics.

CONTENTS

P.U. PRESS